This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

**Usage guidelines**

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

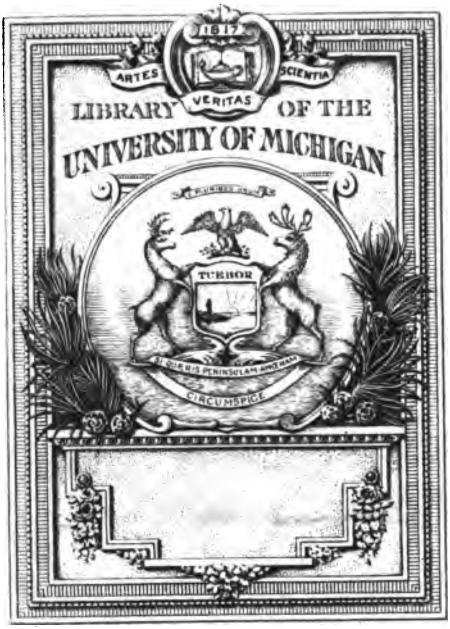
+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

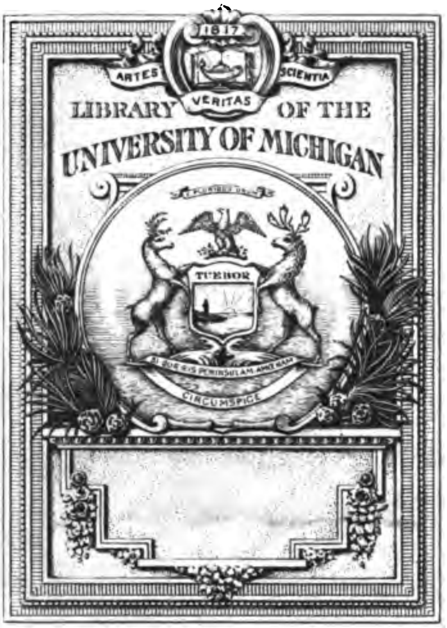
**About Google Book Search**

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at http://books.google.com/



LIBRARY OF THE
UNIVERSITY OF MICHIGAN

QA
529
.S37
pt. 3/4


1817
ARTES
SCIENTIA
VERITAS
LIBRARY OF THE
UNIVERSITY OF MICHIGAN
E PLURIBUS UNUM
TUEBOR
SI QUAERIS PENINSULAM AMOENAM
CIRCUMSPICE

QA
529
.S37
pt. 3/4

Recuerdo á mi apreciable y queri-
do amigo, Francisco de P. Zárate
Sn México Agto 24/78.
H Campillo

QA
529
.C37
pt.3

# HIGHER GEOMETRY

AND

# TRIGONOMETRY:

BEING THE

THIRD PART OF A SERIES

ON

ELEMENTARY AND HIGHER

GEOMETRY, TRIGONOMETRY, AND MENSURATION,

CONTAINING MANY VALUABLE DISCOVERIES AND IMPROVEMENTS IN MATHEMATICAL
SCIENCE, ESPECIALLY IN RELATION TO THE QUADRATURE OF THE CIRCLE,
AND SOME OTHER CURVES, AS WELL AS THE CUBATURE OF CERTAIN
CURVILINEAR SOLIDS; DESIGNED AS A TEXT-BOOK FOR COLLEGIATE
AND ACADEMIC INSTRUCTION, AND AS A PRACTICAL
COMPENDIUM OF MENSURATION.

BY NATHAN SCHOLFIELD.

---

NEW YORK:
PUBLISHED BY COLLINS, BROTHER & CO.
No. 254 Pearl Street.
• • • • •
1845.

QA
529
S37
pt. 3

Entered according to Act of Congress, in the year 1845, by
NATHAN SCHOLFIELD,
In the Clerk's Office of the District Court of Connecticut.

G. W. WOOD, PRINTER, 29 GOLD ST., NEW YORK.

Hist of Sci
Ranzeula
10-9-41
44074

# PREFACE.

This part of the series consists of spherical geometry, taken mostly from Brewster's translation of Legendre's work. Analytical plane and spherical trigonometry, based on the subject, as found in Rutherford's edition of Hutton's Mathematics, being originally abridged from the larger works of Cagnoli, and others; but, in this work, much improved and enlarged. To which are added many practical exercises on the subject, by way of application. In this treatise will be found many curious and highly useful problems in trigonometrical surveying, and topographical operations, not before published. The properties of the circle are introduced advantageously into trigonometrical problems—hence we are enabled, by geometrical construction, and trigonometrical analysis, to determine many otherwise extremely difficult problems, in a manner at once simple, elegant, and satisfactory. The application of algebra to geometry, is discussed in such manner as to combine the principles of the two sciences. The properties of the *parabolic*, *elliptical*, and *hyperbolic curves*, being such as are formed by the sections of a cone, and hence are usually denominated *conic sections*, are also discussed. This subject is, with some alterations and additions, taken from Rutherford's edition of Hutton's Mathematics. It is the design of the author, to preserve an unbroken connection from pure elementary to the higher Geometry and mensuration; and with this object in view the present volume, being the third part of the series, is prepared.

The well established reputation and the high respectability of the authors from whom our selections have been made, renders it unnecessary for us to discuss their merits in order to secure a favorable reception of this. It will only be necessary for us, in the following pages, to preserve the same degree of accuracy and perspicuity in our digressions as characterise those works, and we shall have nothing to fear from the criticisms of scientific amateurs and mathematicians.

# CONTENTS.

SPHERICAL GEOMETRY.

PAGE.
Definitions and General Propositions, - - - - - - - 9

ANALYTICAL PLANE TRIGONOMETRY.

Chap. I.—Definitions and Illustration of Principles, - - - 27
Chap. II.—General Formulæ, - - - - - - - - 45
Chap. III.—Formulæ for the Solution of Triangles, - - - 66
Chap. IV.—Construction of Trigonometrical Tables, - - - 73
Chap. V.—Logarithms, - - - - - - - - 77
Chap. VI.—Solution of Right Angled Triangles, - - - 93
Chap. VII.—Solution of Oblique Angled Triangles, - - - 98
Chap. VIII.—On the use of Subsidiary Angles, - - - - 107
Chap. IX.—On the Solution of Geometrical Problems by Trigonometry, - - - - - - - . - 110
Chap. X.—Problems in Trigonometrical Surveying, &c. - - 113

SPHERICAL TRIGONOMETRY.

Chap. I.—General Principles and Illustrations, - - - - 131
Chap. II.—Solution of Right Angled Spherical Triangles, - - 144
Chap. III.—Solution of Oblique Angled Spherical Triangles, - 150
Chap. IV.—On the use of Subsidiary Angles, - - - - 154

## APPLICATION OF ALGEBRA TO GEOMETRY.

PAGE.
Construction of Algebraical Quantities, - - - - - - 158
Geometrical Questions, the modes of forming Equations therefrom, and their Solutions, - - - - - - - - 165
Determination of Algebraic Expressions for Surfaces and Solids, 180

## CONIC SECTIONS.

The Parabola and its Properties, - - - - - - 189
The Ellipse and its Properties, - - - - - - - - 197
The Hyperbola and its Properties, - - - - - - 215

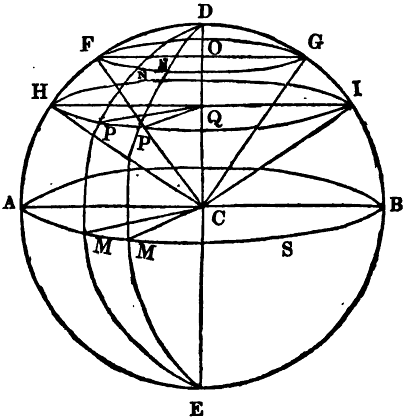
# SPHERICAL GEOMETRY.

## SPHERICAL GEOMETRY.

### DEFINITIONS.

1. The *sphere* is a solid terminated by a curve surface, all the points of which are equally distant from a point within, called the *centre.*

The sphere may be conceived to be generated by the revolution of a semicircle DAE about its diameter DE; for the surface described in this movement, by the curve DAE, will have all its points equally distant from its centre C.



2. The *radius of a sphere* is a straight line, drawn from the centre to any point of the surface; the *diameter,* or *axis,* is a line passing through this centre, and terminated on both sides by the surface.

All the radii of a sphere are equal; all the diameters are equal, and each double of the radius.

3. It will be shown (Prop. I.) that every section of the sphere, made by a plane, is a circle. This granted, a *great circle* is a section which passes through the centre; a *small circle,* one which does not pass through the centre.

4. A *plane* is *tangent* to a sphere, when their surfaces have but one point in common.

5. The *pole of a circle* of a sphere is a point in the surface equally distant from all the points in the circumference of this circle.

6. A *spherical triangle* is a portion of the surface of a sphere, bounded by three arcs of great circles.

Those arcs, named the *sides* of the triangle, are always supposed to be each less than a semi-circumference. The angles which their planes form with each other, are the angles of the triangle.

7. A spherical triangle takes the name of *right-angled, isosceles, equilateral,* in the same cases as a rectilineal triangle.

8. A *spherical polygon* is a portion of the surface of a sphere, terminated by several arcs of great circles.

9. A *lune* is that portion of the surface of a sphere which is included between two great semicircles, meeting in a common diameter.

10. A *spherical wedge,* or *ungula,* is that portion of the solid sphere which is included between the same great semi-circles, and has the lune for its base.

11. A *spherical pyramid* is a portion of the solid sphere included between the planes of a solid angle, whose vertex is the centre. The *base* of the pyramid is the spherical polygon intercepted by the same planes.

12. A *zone* is the portion of the surface of the sphere included between two parallel planes, which form its *bases.* One of those planes may be tangent to the sphere; in which case, the zone has only a single base.

13. A *spherical segment* is the portion of the solid sphere included between two parallel planes which form its bases.

One of these planes may be tangent to the sphere; in which case, the segment has only a single base.

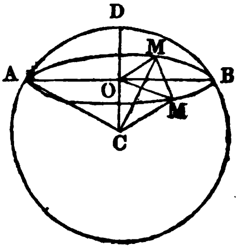
14. The *altitude of a zone* or *of a segment* is the distance between the two parallel planes, which form the bases of the zone or segment.

15. Whilst the semicircle DAE (Def. 1.) revolving round its diameter DE, describes the sphere, any circular sector, as DCF or FCH, describes a solid, which is named a *spherical sector.*

16. The symbal $\because$ which occurs in this volume, is used to denote *because;* when applied in algebraic notation.

### PROPOSITION I. THEOREM.

*Every section of a sphere, made by a plane, is a circle.*



Let AMB be a section, made by a plane, in the sphere, whose centre is C. From the point C, draw CO perpendicular to the plane AMB; and different lines CM, CM, to different points of the curve AMB, which terminates the section.

The oblique lines CM, CM, CA, are equal, being radii of the sphere; hence (Prop. VI. B. I. *El. S. Geom.*) they are equally distant from the perpendicular CO; therefore all the lines OM, MO, OB, are equal. Consequently, the section AMB is a circle, whose centre is O.

*Cor.* 1. If the section passes through the centre of the sphere, its radius will be the radius of the sphere; hence, all great circles are equal.

*Cor.* 2. Two great circles always bisect each other; for their common intersection, passing through the centre, is a diameter.

*Cor.* 3. Every great circle divides the sphere and its surface into two equal parts; for, if the two hemispheres were separated, and afterwards placed on the common base, with their convexities turned the same way, the two surfaces would exactly coincide, no point of the one being nearer the centre than any point of the other.

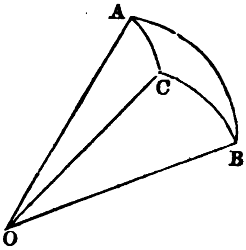
*Cor.* 4. The centre of a small circle, and that of the sphere, are in the same straight line, perpendicular to the plane of the small circle.

*Cor.* 5. Small circles are the less the further they lie from the centre of the sphere; for, the greater CO is, the less is the chord AB, the diameter of the small circle AMB.

*Cor.* 6. An arc of a great circle may always be made to pass through any two given points of the surface of the sphere; for the two given points, and the centre of the sphere, make three points, which determine the position of a plane. But if the two given points were at the extremities of a diameter, these two points and the centre would then lie in one straight line, and an infinite number of great circles might be made to pass through the two given points.

### PROPOSITION II. THEOREM.

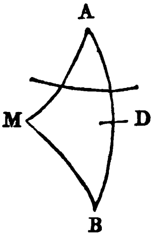
*In every spherical triangle, any side is less than the sum of the other two.*



Let O be the centre of the sphere, and ACB the triangle: draw the radii OA, OB, OC. Imagine the planes AOB, AOC, COB, to be drawn; those planes will form a solid angle at the centre O; and the angles AOB, AOC, COB, will be measured by AB, AC, BC, the sides of the spherical triangle. But each of the three plane angles forming a solid angle is less than the sum of the other two (Prop. XXI., B. I. *El. S. Geom.*): hence any side of the triangle ABC is less than the sum of the other two.

### PROPOSITION III. THEOREM.

*The shortest distance from one point to another, on the surface of a sphere, is the arc of the great circle which joins the two given points.*



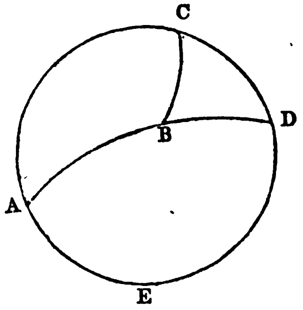
Let ADB be the arc of the great circle which joins the points A and B; and without this line, if possible, let M be a point of the shortest path between A and B. Through the point M, draw MA, MB, arcs of great circles; and take BD=MB.

By the last Proposition, the arc ADB is shorter than AM+MB; take BD=BM respectively from both; there will remain AD<AM. Now, the distance of B from M, whether it be the same with the arc BM, or with any other line, is equal to the distance of B from D; for, by making the plane of the great circle BM to revolve about the diameter which passes through B, the point M may be brought into the position of the point D; and the shortest line between M and B, whatever it may be, will then be identical with that between D and B: hence the two paths from A to B, one passing through M, the other through D, have an equal part in each, the part from M to B equal to the part from D to B. The first part is the shorter by hypothesis; hence the distance

from A to M must be shorter than the distance from A to D, which is absurd; the arc AM being proved greater than AD. Hence no point of the shortest line from A to B can lie out of the arc ADB; hence this arc is itself the shortest distance between its two extremities.

### PROPOSITION IV. THEOREM.

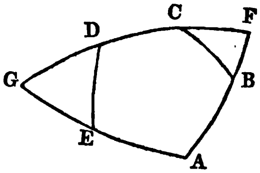
*The sum of the three sides of a spherical triangle is less than the circumference of a great circle.*



Let ABC be any spherical triangle; produce the sides AB, AC, till they meet again in D. The arcs ABD, ACD, will be semicircumferences, since (Prop. I. *Cor.* 2.) two great circles always bisect each other. But in the triangle BCD, we have (Prop. II.) the side BC<BD+CD; add AB+AC to both; we shall have AB+AC+BC<ABD+ACD:—that is to say, less than a circumference.

### PROPOSITION V. THEOREM.

*The sum of all the sides of any spherical polygon is less than the circumference of a great circle.*

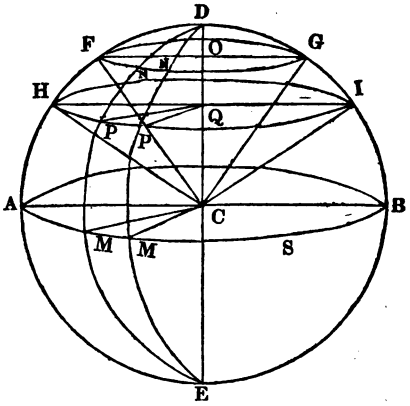


Take the pentagon ABCDE, for example. Produce the sides AB, DC, till they meet in F; then, since BC is less than BF+CF, the perimeter of the pentagon ABCDE will be less than that of the quadrilateral AEDF. Again produce the sides AE, FD, till they meet in G; we shall have ED<EG+DG; hence the perimeter of the quadrilateral AEDF is less than that of the triangle AFG; which last is itself less than the circumference of a great circle: hence, for a still stronger reason, the perimeter of the polygon ABCDE is less than this same circumference.

*Scholium.* This proposition is fundamentally the same as Prop. XXII. B. I. *El. S. Geom.*; for O being the centre of the sphere, a solid angle may be conceived as formed at O, by the plane angles AOB, BOC, COD, &c.; and the sum of these angles must be less than four right angles; which is exactly the proposition we have been engaged with. The demonstration here given is different from that of Prop. XXII. B. I. *El. S. Geom.*; both, however, suppose that the polygon ABCDE is convex, or that no side produced will cut the figure.

### PROPOSITION VI. THEOREM.

*The poles of a great circle of the sphere, are the extremities of that diameter of the sphere which is perpendicular to this circle; and these extremities are also the poles of all small circles parallel to it.*



Let ED be perpendicular to the great circle AMB; then will E and D be its poles; as also the poles of the parallel small circles HPP, FNG.

For, DC being perpendicular to the plane AMB, is perpendicular to all the straight lines CA, CM, CB, &c., drawn through its foot in this plane; hence all the arcs DA, DM, DB, &c., are quarters of the circumference. So likewise are all the arcs EA, EM, EB, &c.; hence the points D and E are each equally distant from all the points of the circumference AMB; hence (Def. 5.) they are the poles of that circumference.

Again, the radius DC, perpendicular to the plane AMB, is perpendicular to its parallel FNG; hence (Prop. I. *Cor.* 4.) it passes through O the centre of the circle FNG; hence, if the oblique lines DF, DN, DG be drawn, these oblique lines will diverge equally from the perpendicular DO, and will themselves be equal. But, the chords being equal, the arcs are equal; hence the point D is the pole of the small circle FNG; and, for like reasons, the point E is the other pole.

*Cor.* 1. Every arc DM, drawn from a point in the arc of a great circle AMB to its pole, is a quarter of the circumference, which, for the sake of brevity, is usually named a *quadrans*, or quadrant: and this quadrant at the same time makes a right angle with the arc AM. For (Prop. XVII. B. I. *El. S. Geom.*) the line DC being perpendicular to the plane AMC, every plane DMC passing through the line DC, is perpendicular to the plane AMC; hence the angle of these planes, or (Def. 6.) the angle AMD, is a right angle.

*Cor.* 2. To find the pole of a given arc AM, draw the indefinite arc MD perpendicular to AM; take MD equal to a quadrant; the point D will be one of the poles of the arc AM: or thus, at the two points A and M, draw the arcs AD and MD perpendicular to AM; their point of intersection, D, will be the pole required.

*Cor.* 3. Conversely, if the distance of the point D from each of the points A and M, is equal to a quadrant, the point D will be the pole of the arc AM, and also the angles DAM, AMD, will be right.

For, let C be the centre of the sphere, and draw the radii CA, CD, CM. Since the angles ACD, MCD are right, the line CD is perpendicular to the two straight lines CA, CM; hence it is perpendicular to their plane (Prop. V. B. I. *El. S. Geom.*); hence the point D is the pole of the arc AM; and consequently the angles DAM, AMD are right.

*Scholium.* The properties of these poles enable us to describe arcs of a circle on the surface of a sphere, with the same facility as on a plane surface. It is evident, for instance, that by turning the arc DF, or any other line extending to the same distance, round the point D, the extremity F will describe the small circle FNG; and, by turning the quadrant DFA round the point D, its extremity A will describe the arc of the great circle AM.

If the arc AM were required to be produced, and nothing were given but the points A and M through which it was to pass, we should first have to determine the pole D, by the intersection of two arcs described from the points A and M as centres, with a distance equal to a quadrant. The pole D being found, we might describe the arc AM and its prolongation, from D as a centre, and with the same distance as before.

In fine, if it is required from a given point P to let fall a perpendicular on the given arc AM, produce this arc to S, till the distance PS be equal to a quadrant; then, from the pole S, and with the same distance, describe the arc PM, which will be the perpendicular required.

### PROPOSITION VII. THEOREM.

*Every plane perpendicular to a radius at its extremity is tangent to the sphere.*

Let FAG (see the next diagram) be a plane perpendicular to the radius OA, at its extremity A. Any point M in this plane being assumed, and OM, AM being joined, the angle OAM will be right, and hence the distance OM will be greater than OA. Hence the point M lies without the sphere; and as the same can be shown for every other point of the plane FAG, this plane can have no point but A common to it and the surface of the sphere; hence (Def. 4.) it is tangent.

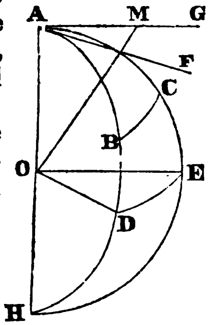
*Scholium.* In the same way, it may be shown that two spheres have but one point in common, and therefore touch each other, when the distance between their centres is equal to the sum, or the difference of their radii; in which case, the centres and the point of contact lie in the same straight line.

### PROPOSITION VIII. THEOREM.

*The angle formed by two arcs of great circles, is equal to the angle formed by the tangents of these two arcs at their points of intersection, and is measured by the arc described from this point of intersection, as a pole, and limited by the sides, produced if necessary.*

Let the angle BAC be formed by the two arcs AB, AC; then it will be equal to the angle FAG formed by the tangents AF, AG, and be measured by the arc DE, described about A as a pole.

For the tangent AF, drawn in the plane of the arc AB, is perpendicular to the radius AO; and the tangent AG, drawn in the plane of the arc AC, is perpendicular to the same radius AO. Hence, (Prop. XX. B. I. *El. S. Geom.*) the angle FAG is equal to the angle contained by the planes OAB, OAC; which is that of the arcs AB, AC, and is named BAC.



In like manner, if the arcs AD and AE are both quadrants, the line OD, OE will be perpendicular to OA, and the angle DOE will still be equal to the angle of the planes AOD, AOE; hence the arc DE is the measure of the angle contained by these planes, or of the angle CAB.

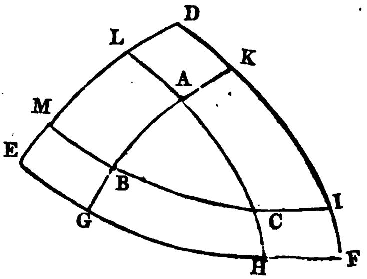
*Cor.* The angles of spherical triangles may be compared together, by means of the arcs of great circles described from their vertices as poles and included between their sides : hence it is easy to make an angle of this kind equal to a given angle.

*Scholium.* Vertical angles, such as ACO and BCN (see diagram to Prop. XXI.) are equal; for either of them is still the angle formed by the two planes ACB, OCN.

It is further evident, that, in the intersection of two arcs ACB, OCN, the two adjacent angles ACO, OCB taken together, are equal to two right angles.

### PROPOSITION IX. THEOREM.

*If from the vertices of the three angles of a spherical triangle, as poles, three arcs be described forming a second triangle, the vertices of the angles of this second triangle will be respectively poles of the sides of the first.*



From the vertices A, B, C, as poles, let the arcs EF, FD, ED be described, forming on the surface of the sphere, the triangle DFE; then will the points D, E, and F be respectively poles of the sides BC, AC, AB.

For, the point A being the pole of the arc EF, the distance AE is a quadrant; the point C being the pole of the arc DE, the distance CE is likewise a quadrant: hence the point E is removed the length of a quadrant from each of the points A and C; hence (Prop VI. Cor. 3.) is the pole of the arc AC. It might be shown, by the same method, that D is the pole of the arc BC, and F that of the arc AB.

*Cor.* Hence the triangle ABC may be described by means of DEF, as DEF is described by means of ABC.

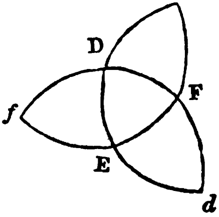
### PROPOSITION X. THEOREM.

*The same supposition continuing as in the last proposition, each angle in the one of the triangles, will be measured by the semicircumference* minus *the side lying opposite to it in the other triangle.*

Produce the sides (see the preceding diagram) AB, AC, if necessary, till they meet EF, in G and H. The point A being the pole of the arc GH, the angle A will be measured by that arc. But the arc EH is a quadrant, and likewise GF, E being the pole of AH, and F of AG; hence $EH+GF$ is equal to the semicircumference. Now, $EH+GF$ is the same as $EF+GH$; hence the arc GH, which measures the angle A, is equal to a semicircumference *minus* the side EF. In like manner, the angle B will be measured by $\frac{1}{2}$ *circ.*—DF: the angle C, by $\frac{1}{2}$ *circ.*—DE.

And this property must be reciprocal in the two triangles, since each of them is described in a similar manner by means of the other. Thus we shall find the angles D, E, F, of the triangle DEF to be measured respectively by $\frac{1}{2}$ *circ.*—BC, $\frac{1}{2}$ *circ.*—AC. $\frac{1}{2}$ *circ.*—AB. Thus the angle D, for example, is measured by the arc MI; but $MI+BC=MC+BI=\frac{1}{2}$ *circ.*; hence the arc MI, the measure of D, is equal to $\frac{1}{2}$ *circ.*—BC; and so of all the rest.

*Scholium.* It must further be observed, that besides the triangle DEF, three others might be formed by the intersection of the three arcs DE, EF, DF. But the proposition immediately before us is applicable only to the central triangle, which is distinguished from the other three by the circumstance (see diagram to Prop. IX.) that the two angles A and D lie on the same side of BC, the two B and E on the same side of AC, and the two C and F on the same side of AB.

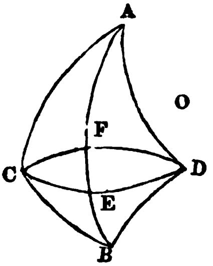


Various names have been given to the triangles ABC, DEF; we shall call them *polar triangles.*

### PROPOSITION XI. THEOREM.

*If around the vertices of the two angles of a given spherical triangle, as poles, the circumference of two circles be described which shall pass through the third angle of the triangle; if then, through the other point in which those circumferences intersect, and the two first angles of the triangle, the arcs of great circles be drawn, the triangle thus formed will have all its parts equal to those of the first triangle.*

Let ABC be the given triangle, CED, DFC the arc described about A and B as poles; then will the triangle ADB have all its parts equal to those of ABC.



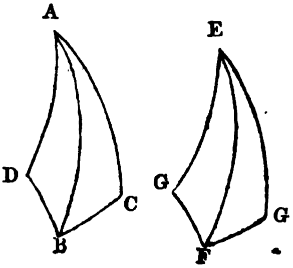
For, by construction, the side AD=AC, DB=BC, and AB is common; hence those two triangles have their sides equal, each to each. We are now to show, that the angles opposite these equal sides are also equal.

If the centre of the sphere is supposed to be at O, a solid angle may be conceived as formed at O by the three plane angles AOB, AOC, BOC; likewise another solid angle may be conceived as formed by the three plane angles AOB, AOD, BOD. And because the sides of the triangle ABC are equal to those of the triangle ADB, the plane angles forming the one of these solid angles, must be equal to the plane angles forming the other, each to each. But in this case we have shown (Prop. XXIII. *El. S. Geom.*) that the planes, in which the equal angles lie, are equally inclined to each other; hence, all the angles of the spherical triangle DAB are respectively equal to those of the triangle CAB, namely, DAB=BAC, DBA=ABC, and ADB=ACB; hence, the sides and the angles of the triangle ADB are equal to the sides and the angles of the triangle ACB.

*Scholium.* The equality of those triangles is not, however, an absolute equality, or one of superposition; for it would be impossible to apply them to each other exactly, unless they were isosceles. The equality meant here is what we have already named an equality by *symmetry*; therefore, we shall call the triangles ACB, ADB, *symmetrical triangles.*

### PROPOSITION XII. THEOREM.

*Two triangles on the same sphere, or on equal spheres, are equal in all their parts, when they have each an equal angle included between equal sides.*



Suppose the side AB=EF, the side AC=EG, and the angle BAC =FEG; the triangle EFG may be placed on the triangle ABC, or on ABD symmetrical with ABC, just as two rectilineal triangles are placed upon each other, when they have an equal angle included between equal sides. Hence all the parts of the triangle EFG will be equal to all the parts of the triangle ABC; that is, besides the three parts equal by hypothesis, we shall have the side BC=FG, the angle ABC=EFG, and the angle ACB=EGF.

### PROPOSITION XIII. THEOREM.

*Two triangles on the same sphere, or on equal spheres, are equal in all their parts, when two angles and the included side of the one are respectively equal to two angles and the included side of the other.*

For, one of those triangles, or the triangle symmetrical with it, may be placed on the other, as is done in the corresponding case of rectilineal triangles, (Prop. IX. B. II. *El. Geom.*)

### PROPOSITION XIV. THEOREM.

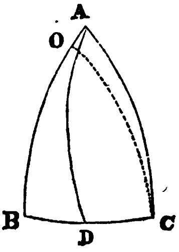
*If two triangles on the same sphere, or on equal spheres have all their sides respectively equal, their angles will likewise be all respectively equal, the equal angles lying opposite the equal sides.*

This truth is evident from Proposition XI, where it is shown that, with three given sides AB, AC, BC, (see the diagram,) there can only be two triangles ACB, ABD, differing as to the position of their parts, and equal as to the magnitude of those parts. Hence, those two triangles, having all their sides respectively equal in both, must either be absolutely equal, or at least *symmetrically* so; in both of which cases their corresponding angles must be equal, and lie opposite to equal sides.

### PROPOSITION XV. THEOREM.

*In every isosceles spherical triangle, the angles opposite the equal sides are equal; and conversely, if two angles of a spherical triangle are equal, the triangle is isosceles.*

*First.* Suppose the side AB=AC; we shall have the angle C=B. For, if the arc be drawn from the vertex A to the middle point D of the base, the two triangles ABD, ACD will have all the sides of the one respectively equal to the corresponding sides of the other, namely, AD common, BD=DC, and AB=AC: hence, by the last Proposition, their angles will be equal; therefore B=C.



*Secondly.* Suppose the angle B=C; we

shall have the side AC=AB. For, if not, let AB be the greater of the two; take BO=AC, and join OC. The two sides BO, BC are equal to the two AC, BC; the angle OBC, contained by the first two is equal to ACB contained by the second two. Hence (Prop. XII.) the two triangles BOC, ACB have all their other parts equal; hence the angle OCB=ABC: but by hypothesis, the angle ABC=ACB; hence we have OCB=ACB, which is absurd; hence it is absurd to suppose AB different from AC; hence the sides AB, AC, opposite to the equal angles B and C, are equal.

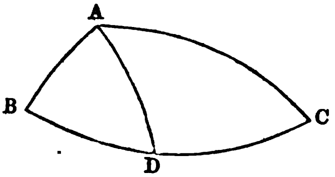
*Scholium.* The same demonstration proves the angle BAD =DAC, and the angle BDA=ADC. Hence the two last are right angles; hence *the arc drawn from the vertex of an isosceles spherical triangle to the middle of the base, is at right angles to the base, and bisects the vertical angle.*

### PROPOSITION XVI. THEOREM.

*In any spherical triangle, the greater side is opposite the greater angle; and conversely, the greater angle is opposite the greater side.*

Let the angle A be greater than the angle B, then will BC be greater than AC; and conversely, if BC is greater than AC, then will the angle A be greater than B.

*First.* Suppose the angle A>B; make the angle BAD= B: then (Prop. XV.) we shall have AD=DB; but AD+DC is greater than AC; hence, putting DB in place of AD, we shall have DB+DC, or BC>AC.



*Secondly.* If we suppose BC>AC, the angle BAC will be greater than ABC. For, if BAC were equal to ABC, we should have BC=AC; if BAC were less than ABC, we should then, as has just been shown, find BC<AC. Both these conclusions are false: hence the angle BAC is greater than ABC.

### PROPOSITION XVII. THEOREM.

*If two triangles on the same sphere, or on equal spheres, are mutually equiangular, they will also be mutually equilateral.*

Let A and B be the two given triangles; P and Q their polar triangles. Since the angles are equal in the triangles A and B, the sides will be equal in their polar triangles P and Q, (Prop. X. :) but since the triangles P and Q are mutually equilateral, they must also (Prop. XIV.) be mutually equiangular; and, lastly, the angles being equal in the triangles P and Q, it follows (Prop. X.) that the sides are equal in their polar triangles A and B. Hence the mutually equiangular triangles A and B are at the same time mutually equilateral.

*Scholium.* This proposition is not applicable to rectilineal triangles; in which equality among the angles indicates only proportionality among the sides. Nor is it difficult to account for the difference observable, in this respect, between spherical and rectilineal triangles. In the proposition now before us, as well as in Propositions XII, XIII, XIV, which treat of the comparison of triangles, it is expressly required that the arcs be traced on the same sphere, or on equal spheres. Now similar arcs are to each other as their radii; hence, on equal spheres, two triangles cannot be similar without being equal. Therefore it is not strange that equality among the angles should produce equality among the sides.

The case would be different, if the triangles were drawn upon unequal spheres; there, the angles being equal, the triangles would be similar, and the homologous sides would be to each other as the radii of their spheres.

### PROPOSITION XVIII. THEOREM.

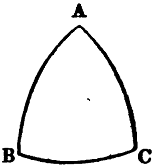
*The sum of all the angles in any spherical triangle is less than six right angles, and greater than two.*

For, in the first place, every angle of a spherical triangle is less than two right angles (see the following Scholium): hence the sum of all the three is less than six right angles.

Secondly, the measure of each angle of a spherical triangle (Prop. X.) is equal to the semicircumference *minus* the corresponding side of the polar triangle; hence the sum of all the three, is measured by three semicircumferences *minus* the sum of all the sides of the polar triangle. Now (Prop. IV.), this latter sum is less than a circumference; therefore, taking it away from three semicircumferences, the remainder will be greater than one semicircumference, which is the measure of two right angles; hence, in the second place, the sum of all the angles in a spherical triangle is greater than two right angles.

*Cor.* 1. The sum of all the angles of a spherical triangle is not constant, like that of all the angles of a rectilineal triangle; it varies between two right angles and six, without ever arriving at either of these limits. Two given angles therefore do not serve to determine the third.

*Cor.* 2. A spherical triangle may have two, or even three angles, right, two or three obtuse.



If the triangle ABC is *bi-rectangular*, in other words, has two right angles B and C, the vertex A will (Prop. X.) be the pole of the base BC; and the sides AB, AC will be quadrants.

If the angle A is also right, the triangle ABC will be *tri-rectangular;* its angles will all be right, and its sides quadrants. The tri-rectangular triangle is contained eight times in the surface of the sphere.

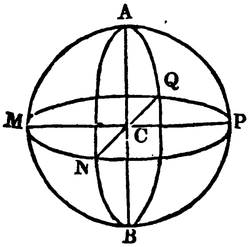
*Scholium.* In all the preceding observations, we have supposed, in conformity with (Def. 6,) that our spherical triangles have always each of their sides less than a semicircumference; from which it follows that any one of their angles is always less than two right angles. For (see diagram to Prop. IV.) if the side AB is less than a semicircumference, and AC is so likewise, both those arcs will require to be produced before they can meet in D. Now the two angles ABC, CBD taken together, are equal to two right angles; hence the angle ABC itself, is less than two right angles.

We may observe, however, that some spherical triangles do exist, in which certain of the sides are greater than a semicircumference, and certain of the angles greater than two right angles. Thus, if the side AC is produced so as to form a whole circumference ACE, the part which remains after substracting the triangle ABC from the hemisphere, is a new triangle also designated by ABC, and having AB, BC, AEDC for its sides. Here, it is plain, the side AEDC is greater than the semicircumference AED; and, at the same time, the angle B opposite to it exceeds two right angles, by the quantity CBD.

The triangles whose sides and angles are so large, have been excluded from our Definition; but the only reason was, that the solution of them, or the determination of their parts, is always reducible to the solution of such triangles as are comprehended by the Definition. Indeed, it is evident enough, that if the sides and angles of the triangle ABC are known, it will be easy to discover the angles and sides of the triangle which bears the same name, and is the difference between a hemisphere and the former triangle.

### PROPOSITION XIX. THEOREM.

*The surface of a lune is to the surface of the sphere, as the angle of this lune, is to four right angles, or as the arc which measures that angle, is to the circumference.*



Let AMNB be a lune; then will its surface be to the surface of the sphere as the angle NCM to four right angles, or as the arc NM to the circumference of a great circle.

Suppose, in the first place, the arc MN to be the circumference MNPQ as some one rational number is to another, as 5 to 48, for example. The circumference MNPQ being divided into 48 equal parts, MN will contain 5 of them; and if the pole A were joined with the several points of division, by as many quadrants, we should in the hemisphere AMNPQ have 48 triangles, all equal, because all their parts are equal. Hence the whole sphere must contain 96 of those partial triangles, the lune AMBNA will contain 10 of them; hence the lune is to the sphere as 10 is to 96, or as 5 to 48, in other words, as the arc MN is to the circumference.

If the arc MN is not commensurable with the circumference, we may still show, by a mode of reasoning frequently exemplified already, that in this case also, the lune is to the sphere as MN is to the circumference.

*Cor.* 1. Two lunes are to each other as their respective angles.

*Cor.* 2. It was shown above (Prop. XVIII. Cor. 2.) that the whole surface of the sphere is equal to eight tri-rectangular triangles; hence, if the area for one such triangle is taken for unity, the surface of the sphere will be represented by 8. This granted, the surface of the lune, whose angle is A, will be expressed by 2A (the angle A being always estimated from the right angle assumed as unity:) since 2A : 8 : : A : 4. Thus we have here two different unities; one for angles, being the right angle; the other for surfaces being the tri-rectangular spherical triangle, or the triangle whose angles are all right, and whose sides are quadrants.

Or if the area of one such triangle is represented by T, the surface of the whole sphere will be expressed by 8T, and the surface of the lune whose angle is A, will be expressed by 2A×T. for

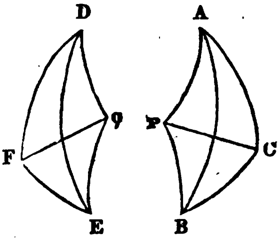
$$4 : A : : 8T : 2A \times T$$

in which expression, A represents such a part of unity, as the angle of the lune is of one right angle.

*Scholium.* The spherical ungula, bounded by the planes AMB, ANB, is to the whole solid sphere, as the angle A is to four right angles. For, the lunes being equal, the spherical ungulas will also be equal; hence two spherical ungulas are to each other, as the angles formed by the planes which bound them.

### PROPOSITION XX. THEOREM.

*Two symmetrical spherical triangles are equal in surface.*



Let ABC, DEF be two symmetrical triangles, that is to say, two triangles having their sides AB=DE, AC=DF, CB=EF, and yet incapable of coinciding with each other: we are to show that the surface ABC is equal to the surface DEF.

Let P be the pole of the small circle passing through the three points A, B, C; from this point draw (Prop. VI.) the equal arcs, PA, PB, PC; at the point F, make the angle DFQ=ACP, the arc FQ=CP; and join DQ, EQ.

The sides DF, FQ are equal to the sides AC, CP; the angle DFQ=ACP: hence (Prop. XII.) the two triangles DFQ, ACP are equal in all their parts; hence the side DQ=AP, and the angle DQF=APC.

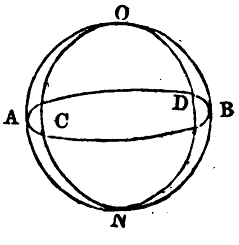
In the proposed triangles DFE, ABC, the angles DFE, ACB opposite to the equal sides DE, AB, being equal (Prop. XIII.) if the angles DFQ, ACP, which are equal by construction, be taken away from them, there will remain the angle QFE, equal to PCB. Also the sides QF, FE are equal to the sides PC, CB; hence the two triangles FQE, CPB are equal in all their parts; hence the side QE=PB, and the angle FQE =CPB.

Now, the triangles DFQ, ACP, which have their sides respectively equal, are at the same time isosceles, and capable of coinciding, when applied to each other; for having placed PA on its equal QF, the side PC will fall on its equal QD, and thus the two triangles will exactly coincide; hence they are equal; and the surface DQF=APC. For a like reason, the surface FQE=CPB, and the surface DQE=APB; hence we have DQF+FQE—DQE=APC+CPB—APB, or DFE= ABC; hence the two symmetrical triangles ABC, DEF are equal in surface.

*Scholium.* The poles P and Q might lie within the triangles ABC, DEF: in which case it would be requisite to add the three triangles DQF, FQE, DQE together, in order to make up the triangle DEF; and in like manner to add the three triangles APC, CPB, APB together, in order to make up the triangle ABC: in all other respects, the demonstration and the result would still be the same.

### PROPOSITION XXI. THEOREM.

*If the circumferences of two great circles intersect each other on the surface of a hemisphere, the sum of the opposite triangles thus formed, is equivalent to the surface of a lune whose angle is equal to the angle formed by the circles.*



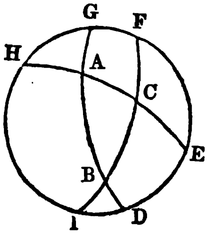
Let the circumferences AOB, COD, intersect on the hemisphere OACBD; then will the opposite triangles AOC, BOD be equal to the lune whose angle is BOD.

For, producing the arcs OB, OD on the other hemisphere, till they meet in N, the arc OBN will be a semi-circumference, and AOB one also; and taking OB from both, we shall have BN=AO. For a like reason, we have DN=CO, and BD=AC. Hence the two triangles AOC, BDN have their three sides respectively equal; besides, they are so placed as to be symmetrical; hence (Prop. XIX. Sch.) they are equal in surface, and the sum of the triangles AOC, BOD is equivalent to the lune OBNDO, whose angle is BOD.

*Scholium.* It is likewise evident that the two spherical pyramids, which have the triangles AOC, BOD for bases, are together equivalent to the spherical ungula whose angle is BOD.

### PROPOSITION XXII. THEOREM.

*The surface of a spherical triangle is measured by the excess of the sum of its three angles above two right angles, multiplied by the tri-rectangular triangle.*



Let ABC be the proposed triangle: produce its sides till they meet the great circle DEFG, drawn at pleasure without the triangle. By the last Theorem, the two triangles ADE, AGH, are together equivalent to the lune whose angle is A, and which is measured by 2A.T (Prop. XIX. Cor. 2.) Hence we have ADE+AGH=2A.T; and, for a like reason, BGF+BID=2B.T, and CIH+CFE=2C.T. But the sum of these six triangles exceeds the hemisphere by twice the triangle ABC, and the hemisphere is represented by 4T; therefore, twice the triangle ABC is equal to 2A.T+2B.T+2C.T—4T; and consequently, once ABC=(A+B+C—2)T; hence every spherical triangle is measured by the sum of all its angles *minus* two right angles, multiplied by the tri-rectangular triangle.

*Cor.* 1. However many right angles there may be in the sum of the three angles minus two right angles, just so many tri-rectangular triangles, or eighths of the sphere, will the proposed triangle contain. If the angles, for example, are each equal to $\frac{4}{3}$ of a right angle, the three angles will amount to four right angles, and the sum of the angles minus two right angles will be represented by 4—2, or 2; therefore the surface of the triangle will be equal to two tri-rectangular triangles, or to the fourth part of the whole surface of the sphere.

*Scholium.* While the spherical triangle ABC is compared with the tri-rectangular triangle, the spherical pyramid, which has ABC for its base, is compared with the tri-rectangular pyramid, and a similar proportion is found to subsist between them. The solid angle at the vertex of the pyramid, is in like manner compared with the solid angle at the vertex of the tri-rectangular pyramid. These comparisons are founded on the coincidence of the corresponding parts. If the bases of the pyramids coincide, the pyramids themselves will evidently coincide, and likewise the solid angles at their vertices. From this, some consequences are deduced.

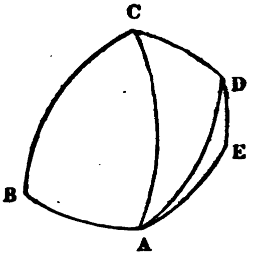
*First.* Two triangular spherical pyramids are to each other as their bases; and, since a polygonal pyramid may always be divided into a certain number of triangular ones, it follows that any two spherical pyramids are to each other as the polygons which form their bases.

*Second.* The solid angles at the vertices of these pyramids are also as their bases: hence, for comparing any two solid angles, we have merely to place their vertices at the centres of two equal spheres, and the solid angles will be to each other as the spherical polygons intercepted between their planes or faces.

The vertical angle of the tri-rectangular pyramid is formed by three planes, at right angles to each other. This angle, which may be called a *right solid angle*, will serve as a very natural unit of measure for all other solid angles. If, for example, the area of the triangle is $\frac{3}{4}$ of the tri-rectangular triangle, then the corresponding solid angle will also be $\frac{3}{4}$ of the right solid angle.

### PROPOSITION XXIII. THEOREM.

*The surface of a spherical polygon is measured by the sum of all its angles,* minus *two right angles multiplied by the number of sides in the polygon* less *two, into the tri-rectangular triangle.*



From one of the vertices A, let diagonals AC, AD, be drawn to all the other vertices; the polygon ABCDE will be divided into as many triangles, *minus* two, as it has sides. But the surface of each triangle is measured by the sum of all its angles *minus* two right angles, into the tri-rectangular triangle; and the sum of the angles in all the triangles is evidently the same as that of all the angles of the polygon: hence, the surface of the polygon is equal to the sum of all its angles, diminished by twice as many right angles as it has sides, less two, into the tri-rectangular triangle.

*Scholium.* Let $s$ be the sum of all the angles in a spherical polygon, $n$ the number of its sides, and T the tri-rectangular triangle; the right angle being taken for unity, the surface of the polygon will be measured by

$$\left(s-2(n-2,)\right)\,T, \text{ or } (s-2\,n+4)\;T$$

# ANALYTICAL PLANE TRIGONOMETRY.

## CHAPTER I.

Plane Trigonometry is the science which treats of the relations of the sides and angles of plane triangles.

In every triangle there are six parts: three sides and three angles; which have such relations to each other that the value of one depends on the value of the others; and if a sufficient number of these are known the others may thereby be determined.

The sides of triangles consist of absolute magnitude, but the angles are only the relations of those sides to each other in position or direction, without regard to their magnitudes.

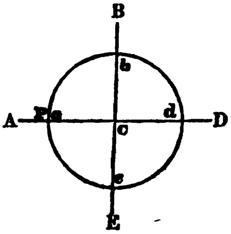
Angles have no absolute measure in terms of the sides; but are, nevertheless, susceptible of measure; for if two lines meet each other the space included between them within a given distance from their point of contact is proportional to their mutual inclination, and hence (Prop. XVIII. Cor. B. III. *El. Geom.*) the arc of the circumference of a circle intercepted by two lines drawn from its centre, may be regarded as the measure of the angle or inclination of those lines, and therefore the arc of the circumference may be regarded as the measure of angular magnitude.

For this purpose the circumference of the circle is supposed to be divided into 360 equal parts, called degrees, and each of those degrees is divided into 60 equal parts called minutes, and each minute into 60 equal parts called seconds; and so on, to thirds, fourths, &c.

These divisions are designated by the following characters, $^{\circ}$ $'$ $''$ $'''$ &c. Thus the expression $30^{\circ}\ 20'\ 12''\ 22'''$, represents an arc or an angle of 30 degrees 20 minutes 12 seconds 22 thirds.

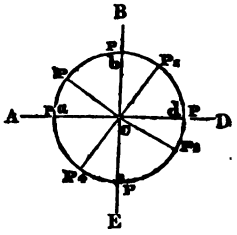
The circumference of any circle may in this manner be applied as the measure of angles, without regard to its magnitude or the length of its radius; hence a degree is not a magnitude of any definite length, but is a certain portion of the whole circumference of any circle, for it is evident that the 360th part of the circumference of a large circle is greater than the same part of a smaller one, but the number of degrees in the small circumference is the same as in the large one.

The fourth part of the circumference of a circle is called a quadrant and contains 90 degrees: hence 90 degrees is the measure of the right angle.



Thus, if we draw two straight lines AD, BE, so as to cross each other at right angles, and from their point of intersection, C, we discribe a circle with any radius so as to cut those lines in any points, as *a*, *b*, *d*, *e*, the circumference of the circle will thus be divided into four equal arcs, *ab*, *bd*, *de*, *ea*, each of which measures or subtends a right angle at the centre C, of the circle.

If a line CP be made to revolve round a fixed point C as the centre of a circle, and so as to pass successively through every point of the circumference, commencing in the point *a*, then, while it is in the position C*a*, or while it coincides with the line C*a*, those two lines form but one, and intercept no arc on the circumference of the circle, and hence form no angle with each other; but when the line C*a*P comes into the position CP, it forms with AC an acute angle at C, which is measured by the arc *a*P, and when it comes into the position C*b*P, it then forms a right angle ACP with the line AC, which angle is measured by the quadrant *ab*. Now let it come into the position $CP_2$, and the angle which it forms with CA, will be measured by the arc $aP_2$, which is greater than a right angle, and hence is an obtuse angle.



Let it now come into the position C*d*P; it then coincides with the right line C*d*, which is a portion of the line AC produced, since the line CP, in this position, coincides with the line AD, it can be said to form with it no angle; yet the space passed over by the line CP, from the position C*a*P, is equal to two quadrants, or two right angles equal to 180 degrees, and for trigonometrical investigation the lines C*b*P and CA are said to subtend the angle measured by the arc *abd*.

After passing the point *d*, and coming into the position $CP_3$, it forms with AC, and on the upper side of it the angle $P_3CA$

measured by the arc *ae*$P_3$, but having passed over the arc *abd*$P_3$, is said to contain, with the line CA, the angle ACP$_3$ on the upper side of those lines measured by the arc *abd*$P_3$, greater than two right angles. When it comes in the position C*e*P it is said to subtend, with the line AC from the same side of it, the angle measured by the arc *abde*, or three quadrants, equal to three right angles.

When in the position CP$_4$, it is said to contain with CA, and on the same side of it, an angle greater than three right angles.

Finally, when the line CP has completed an entire revolution, having returned to its original position, CA, it will have formed an angle with it equal to four right angles.

If the line CP continues to revolve, it is manifest that the angle will increase, and may with this view form with CA, angles greater than four, than five, or than any given number of right angles.

*ab* is called the first quadrant of the circle, *bd* the second, *de* the third, and *ea* the fourth quadrant.

It must be borne in mind, that the line CP cannot, geometrically, be said to contain with another line, AC, an angle greater, nor quite equal to, two right angles, but in view of its supposed motion round one of its extremities, C, as a centre, it is said to contain, with the line AC, all the angular space through which it has passed in its revolution.

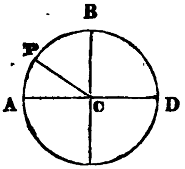
Thus, let the line CP have performed one complete revolution, from the position C*a*P to the same position again, the angle which it forms with the line CA, though absolutely nothing, is in view of its supposed motion measured by the quadrants $ab+bd+de+ea$, each of which quadrants are readily recognized as being contained by their several lines of division, when by removing those lines of division of the circumference, those several angles are all converted into one containing the whole circumference; hence, in view of this motion or relation of the two lines, they are said to contain an angle measured by the whole circumference.

## DEFINITIONS AND ILLUSTRATIONS.

The following symbols are sometimes used.

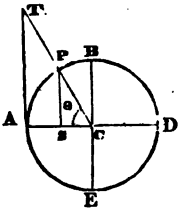
1. The complement of an arc or of an angle, is what remains after taking that arc or that angle from 90 degrees. Thus, if $\theta$ be any arc or angle, the complement is $90°-\theta$

2. Supplement of an arc or an angle, is what remains after taking that arc or angle from two right angles, or 180 degrees. Thus, if $\theta$ be any arc or angle, the supplement of $\theta$ is $180°-\theta$.



If AP be any arc, and ACP be any angle, $\theta$ measured by that arc, then the complement of the angle $\theta$ is the angle PCB measured by the arc PB, and the supplement of the angle $\theta$ is the angle PCD measured by the arc PBD, and if BCP is any angle $\theta$ measured by the arc PB, then PCA is the complement of $\theta$, and if PCD is any angle $\theta$, then will PCA be its supplement.

3. To represent the ratios of the sides and angles of triangles, right lines are drawn in and about a circle called sines, tangents, secants, &c.



Draw two right lines AD,BE cutting each other at right angles in the point C, with the centre C and any distance as radius, describe a circle cutting the lines in the points A, B, D, E.

Draw the radius CP forming with CA any angle $ACP=\theta$. From P draw PS perpendicular on CA. From A draw AT a tangent to the circumference at A. Produce CP to meet AT in T.

4. Then the ratio of PS to the radius CA of the circle, is called the *sine* of the angle PCA.

Or, $$\frac{PS}{CA}=\sin.\ \theta.$$

5. The ratio of AT to the radius CA of the circle, is called the *tangent* of the angle TCA or PCA.

Hence, $$\frac{AT}{CA}=\tan.\ \theta.$$

6. And the ratio of CT to the radius is the *secant* of the angle PCA.

Or, $$\frac{CT}{CA}=\sec.\ \theta.$$

7. The ratio of AS to the radius of the circle, is called the *versed sine* of the angle PCA.

Or, $$\frac{AS}{CA}=\text{v. sin. }\theta.$$

8. The sine of the complement of an angle, is called the sine complement, or *cosine* of that angle.

Thus, sin. $(90°-\theta)$=cos. $\theta$, hence cos. $(90°-\theta)$=sin. $\theta$.

9. The tangent of the complement of any given angle, is called the *cotangent* of that angle.

Thus, tan. $(90°-\theta)$=cot. $\theta$, hence cot. $(90°-\theta)$=tan. $\theta$

10. The secant of the complement of any given angle, is called the *cosecant* of that angle.

Or, sec. $(90°-\theta)$=cosec. $\theta$, hence cosec. $(90°-\theta)$=sec. $\theta$.

11. The versed sine of the complement of any angle, is called the *co-versed sine* of that angle.

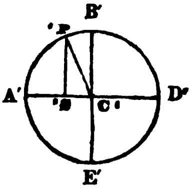
Or, v. sin. $(90°-\theta)$=co-v. sin. $\theta$,

and hence co-v. sin. $(90-\theta)$=v. sin. $\theta$.

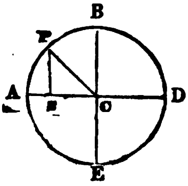
In order to show that the ratio of CS to the radius of the circle in the last figure, is the cosine of the angle PCA; that is, the sine of its complement,

Or that $$\frac{CS}{CA}=\text{cos. }\theta,$$

Draw a circle A′B′D′E′ equal to the circle ABDE, and from C′ the centre, draw C′P′, making with C′A′ the angle P′C′A′ equal to the angle PCB; that is, to the compliment of PCA, or to $(90°-\theta)$.

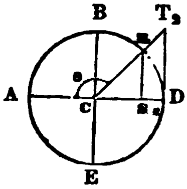


Then, since CP is equal to C′P′, and the angles at S and S′ are right angles, the angle CPS equal to the angle P′C′S′ the two triangles PCS, P′C′S′ are equal in every respect; PS=C′S′, CS=P′S′.



Therefore, $$\frac{CS}{CA}=\frac{P'S'}{CA}$$
=sin. P′C′A′ by Def.
=sin. $(90°-\theta)$ by construct.
=cos. $\theta$ by Def.

We have hitherto considered an angle PCA less than a right angle, but the same definitions are applied, whatever may be the magnitude of the angle.



Thus, for example, let us take an angle $P_2CA$ situated in the second quadrant, *that is*, an angle greater than one right angle, and less than two.

From $P_2$ let fall $P_2S_2$ perpendicular on AD, from D draw $DT_2$ a tangent to the circle at D, meeting $CP_2$ produced in $T_2$; then, as before,

$$\frac{P_2S_2}{CA}=\text{sin. } P_2CA$$

$$\frac{CS_2}{CA}=\text{cos. } P_2CA$$

$$\frac{DT_2}{CA}=\text{tan. } P_2CA$$

$$\frac{CT_2}{CA}=\text{sec. } P_2CA$$

$$\frac{AS_2}{CA}=\text{v. sin. } P_2CA.$$

Again, let the angle in question be situated in the third quadrant, that is, let it be an angle greater than two, and less than three right angles.

Making a construction analogous to that in the two former cases, we shall have

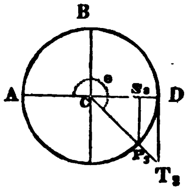
$$\frac{P_3S_3}{CA}=\text{sin. } P_3CA$$

$$\frac{CS_3}{CA}=\text{cos. } P_3CA$$

$$\frac{DT_3}{CA}=\text{tan. } P_3CA$$

$$\frac{CT_3}{CA}=\text{sec. } P_3AC$$

$$\frac{AS_3}{CA}=\text{v. sin. } P_3CA.$$

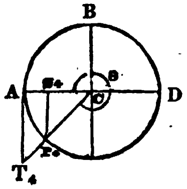


Lastly, let the angle be situated in the fourth quadrant; that is, let it be an angle greater than three, and less than four right angles, then as before,

$$\frac{P_4S_4}{CA}=\text{sin. } P_4CA$$

$$\frac{CS_4}{CA}=\text{cos. } P_4CA$$

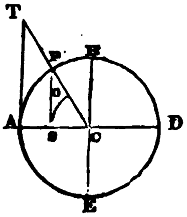
$$\frac{AT_4}{CA}=\text{tan. } P_4CA$$



$$\frac{CT_4}{CA}=\text{sec. } P_4CA$$

$$\frac{AS_4}{CA}=\text{v. sin. } P_4CA.$$

We shall now proceed to establish some important general relations, between the trigonometrical quantities which are immediately deducible from the above definitions, and from the principles of Geometry.



Resuming the figure of Def. (3):

Since CSP is a right-angled triangle, and CP the hypothenuse,

$$PS^2+CS^2=CP^2$$

Dividing by $CP^2$,

$$\frac{PS^2}{CP^2}+\frac{CS^2}{CP^2}=1$$

that is, $\sin.^2\theta+\cos.^2\theta=1$ - - - - - - - - - - - - - (1)

The triangles PSC, TAC, are equiangular and similar; hence,

$$\frac{PS}{CS}=\frac{AT}{CA}$$

Therefore $$\frac{\frac{PS}{CA}}{\frac{CS}{CA}}=\frac{AT}{CA}$$

that is, $$\frac{\sin.\ \theta}{\cos.\ \theta}=\tan.\ \theta \quad \text{- - - - - - - - - -} \quad (2)$$

In last case, for $\theta$ substitute $(90°-\theta)$; then

$$\frac{\sin.\ (90°-\theta)}{\cos.\ (90°-\theta)}=\tan.\ (90°-\theta)$$

Or, $$\frac{\cos.\ \theta}{\sin.\ \theta}=\cot.\ \theta \quad \text{- - - - - - - - - - -} \quad (3)$$

From (2) and (3) we have

$$\frac{\sin.\ \theta}{\cos.\ \theta}=\tan.\ \theta, \text{ and } \frac{\cos.\ \theta}{\sin.\ \theta}=\cot.\ \theta.$$

Hence, $\tan.\ \theta=\dfrac{1}{\cot.\ \theta},$ or, $\tan.\ \theta \cot.\ \theta=1$ - - - - - - - (4)

By similar triangles CTA, CPS.

$$\frac{CT}{CA}=\frac{CP}{CS}$$

$$=\frac{1}{\frac{CS}{CP}}$$

Or, $$\text{sec. }\theta=\frac{1}{\text{cos. }\theta}, \text{ or, sec. }\theta\text{ cos. }\theta=1 \quad - \quad - \quad (5)$$

By Definition,

$$\begin{aligned}\text{cosec. }\theta&=\text{sec. }(90^\circ-\theta)\\&=\frac{1}{\text{cos. }(90^\circ-\theta)}\text{ by the last case,}\\&=\frac{1}{\text{sin. }\theta}, \text{ or, cosec. }\theta\text{ sin. }\theta=1 \quad - \quad - \quad (6)\end{aligned}$$

Since CAT is a right angled triangle, and CT the hypothenuse

$$CA^2+AT^2=CT^2$$

Dividing by $CA^2$,

$$1+\frac{AT^2}{CA^2}=\frac{CT^2}{CA^2}$$

that is, $$1+\text{tan.}^2\,\theta=\text{sec.}^2\,\theta \quad - - - - - - - - - - \quad (7)$$

By (3) we have $$\text{cot. }\theta=\frac{\text{cos. }\theta}{\text{sin. }\theta}$$

Therefore, $$\text{cot.}^2\,\theta=\frac{\text{cos.}^2\,\theta}{\text{sin.}^2\,\theta}$$

Adding 1 to each side of the equation,

$$\begin{aligned}1+\text{cot.}^2\,\theta&=1+\frac{\text{cos.}^2\,\theta}{\text{sin.}^2\,\theta}\\&=\frac{\text{sin.}^2\,\theta+\text{cos.}^2\,\theta}{\text{sin.}^2\,\theta}\\&=\frac{1}{\text{sin.}^2\,\theta}\text{ by (1)}\\&=\text{cosec.}^2\,\theta\text{ by (6)} \quad - - - - \quad (8)\end{aligned}$$

By Definition,

$$\begin{aligned}\text{versin. }\theta&=\frac{SA}{CA}\\&=\frac{CA-CS}{CA}\\&=1-\frac{CS}{CA}\\&=1-\text{cos. }\theta \quad - - - - - - - - - \quad (9)\end{aligned}$$

By Definition,

$$\begin{aligned}\text{coversin. }\theta&=\text{versin. }(90^\circ-\theta)\\&=1-\text{cos.}(90^\circ-\theta,)\text{ by the last case.}\\&=1-\text{sin. }\theta \quad - - - - - - - \quad (10)\end{aligned}$$

The above results, which are of the highest importance in all trigonometrical investigations, are collected and arranged in the following table, which ought to be committed to memory:—

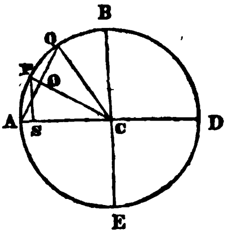
### TABLE I.

1. $\text{Sin.}^2\,\theta+\cos.^2\,\theta=1$
2. $\dfrac{\sin.\,\theta}{\cos.\,\theta}=\tan.\,\theta$
3. $\dfrac{\cos.\,\theta}{\sin.\,\theta}=\cot.\,\theta$
4. $\tan.\,\theta\ \cot.\,\theta=1$
5. $\sec.\,\theta\ \cos.\,\theta=1$
6. $\text{cosec.}\,\theta\ \sin.\,\theta=1$
7. $1+\tan.^2\,\theta=\sec.^2\,\theta$
8. $1+\cot.^2\,\theta=\text{cosec.}^2\,\theta$
9. $\text{v. sin.}\,\theta=1-\cos.\,\theta$
10. $\text{coversin.}\,\theta=1-\sin.\,\theta.$

12. The *chord* of an arc is the ratio of the straight line joining the two extremities of the arc to the radius of the circle.

#### PROPOSITION.

*The chord of any arc is equal to twice the sine of half the arc.*



Take any arc AQ, subtending at the centre of the circle the angle ACQ=$\theta$.

Draw the straight line CP bisecting the angle ACQ.

Join A, Q; from P let fall PS perpendicular on CA.

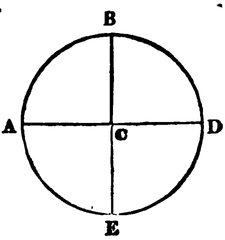
Since CP bisects ACQ, the vertical angle of the isosceles triangle ACQ, it bisects the base AQ at right angles.

Therefore, AO=OQ, and the angles at O are right angles.

Again, since the triangles AOC, PSC, have the angles CSP, COA, right angles, and the angle PCS common to the two triangles, and also the side CP of the one equal the side CA of the other, these triangles are in every respect equal.

$$\therefore\ PS=AO=OQ$$
$$\therefore\ AQ=2PS$$
$$\therefore\ \frac{AQ}{CA}=2\frac{PS}{CA}$$
$$\text{or, chord}\ \theta=2\sin.\ PCA$$
$$=2\sin.\frac{\theta}{2}$$

We shall now proceed to explain the principle by which the *signs* of the trigonometrical quantities are regulated—



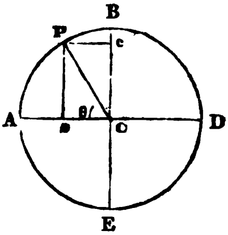
All lines measured from the point C along CA, that is, *to the left*, are considered positive, or have the sine +.

All lines measured from the point C along CD, that is, in the opposite direction *to the right* are considered negative, or have the sign —.

All lines measured from the point C along CB, that is, *upwards*, are considered positive or have sign +.

All lines measured from the point C along CE, that is, in the opposite direction *downwards*, are considered negative, or have the sign —.

Let us determine according to this principle, the signs of the sines and cosines of angles in the different quadrants.



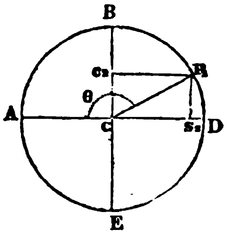
*In the first quadrant*, $\sin.\theta = \frac{PS}{CA}$

$$\cos.\theta = \frac{CS.}{CA}$$

Here PS=C*c* is reckoned from C along CB upwards, and is therefore positive.

CS is reckoned from C along CA, to the left and is therefore positive.

*In the first quadrant*, therefore the *sine and cosin are both positive.*



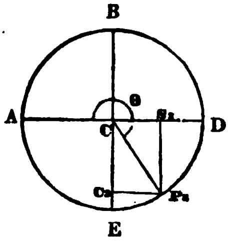
*In the second quadrant*, $\sin.\theta = \frac{P_2\,S_2}{C\ A}$

$$\cos.\theta = \frac{CS_2.}{CA}$$

Here $P_2S_2 = Cc_2$ is reckoned from C along CB upwards, and is therefore positive.

$CS_2$ is reckoned from C along CD to the right and is therefore negative.

*In the second quadrant*, therefore, *the sine is positive and the cosine negative.*



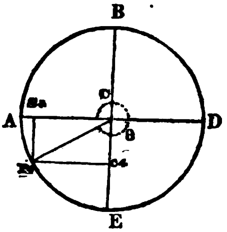
*In the third quadrant*, $\sin.\theta = \frac{P_3S_3}{CA}$

$$\cos.\theta = \frac{CS_3}{CA}$$

Here $P_3S_3 = Cc_3$ is reckoned from C along CE, downwards, and is therefore negative.

$CS_3$ is reckoned from C along CD, *to the right*, and is therefore negative.

*In the third quadrant,* therefore, *the sine and cosine are both negative.*



*In the fourth quadrant,* $\sin. \theta = \frac{P_4S_4}{CA}$

$$\cos. \theta = \frac{CS_4}{CA}$$

Here $P_4S_4 = Cc_4$ is reckoned from C along CE downwards and is therefore negative. $CS_4$ is reckoned from C along CA, to the left and is therefore positive.

*In the fourth quadrant,* therefore, *the sine is negative, and the cosine positive.*

Hence we conclude, that *the sine is positive in the first and second quadrants, and negative in the third and fourth; and the cosine is positive in the first and fourth, and negative in the second and third,* or in other words:

*The sine of an angle less than* 180° *is positive and the sine of an angle greater than* 180° *and less than* 360° *is negative.*

*The cosine of an angle less than* 90° *is positive, the cosine of an angle greater than* 90°, *and less than* 270°, *is negative, and the cosine of an angle greater than* 270°, *and less than* 360,° *is positive.*

The signs of the sine and the cosine being determined, the signs of all the other trigonometrical quantities may be at once established by referring to the relations in Table 1.

Thus, for the tangent,

$$\tan. \theta = \frac{\sin. \theta}{\cos. \theta}$$

Hence, it appears that when the sine and cosine have the same sign the tangent will be positive, and when they have different signs it will be negative.

Therefore, *the tangent is positive in the first and third quadrants and negative in the second and fourth.*

The same holds good for the cotangent; for

$$\cot. \theta = \frac{\cos. \theta}{\sin. \theta}$$

Again, since

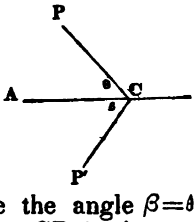
$$\sec. \theta = \frac{1}{\cos. \theta}$$

the *sign of the secant is always the same with that of the cosine;* and, since

$$\text{cosec.} \theta = \frac{1}{\sin. \theta}$$

in like manner, *the sign of the cosecant is always the same with that of the sine.*

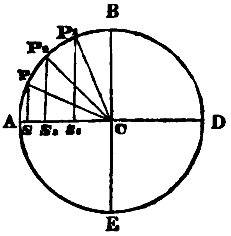
The versed sine is always positive, being reckoned from A always in the same direction.



It is sometimes convenient to give different signs to angles themselves. We have hitherto supposed angles of different magnitudes to be generated by the revolution of the moveable radius CP round C in a direction from left to right; and the angles so formed have been considered positive, or affected with the sign +. If we now suppose the angle $\beta=\theta$ to be generated by the revolution of the radius CP' in the opposite direction, we may, upon a principle analogous to the former, consider the angle $\beta$ as negative, and affect it with the sign —.

We shall now determine the variations in the magnitude of the sine and cosine for angles of different magnitudes.

*In the first quadrant:*



Let CP, $CP_2$, $CP_3$, ........ be different positions of the revolving radius in the first quadrant; and from P, $P_2$, $P_3$, ........ draw PS, $P_2S_2$, $P_3S_3$, perpendiculars on CA.

It is manifest, that as the angle increases the sine increases; for

$$\frac{P_2S_2}{CA}>\frac{PS}{CA} \text{ and } \frac{P_3S_3}{CA}>\frac{P_2S_2}{CA}$$

When the angle becomes very small, PS becomes very small also; and when the revolving radius coincides with CA, that is, when the angle becomes 0, then PS disappears altogether, and is =0.

Hence since, generally, sin. $\theta=\frac{PS}{CA}$ and since, when $\theta=0$, PS=0;

$$\therefore \text{ sin. } 0=\frac{0}{CA}$$
$$=0$$

On the other hand, when the angle becomes equal to 90°, PS coincides with CB, and is equal to it.

Hence since, generally, sin. $\theta=\frac{PS}{CA}$, and since, when $\theta=90°$, PS=CB;

$$\therefore \text{ sin. } 90°=\frac{CB}{CA}$$
$$=1; \therefore CB=CA.$$

Again, it is manifest, that as the angle increases the cosine diminishes; for

$$\frac{CS}{CA}>\frac{CS_2}{CA} \text{ and } \frac{CS_2}{CA}>\frac{CS_3}{CA}$$

When the angle is very small, CS is very nearly equal to CA; and when the revolving radius coincides with CA, that is, when the angle is 0, then CS coincides with CA and is equal to it.

Hence since, generally, $\cos.\ \theta=\frac{CS}{CA}$, and since, when $\theta=0$, $CS=CA$;

$$\therefore\ \cos.\ 0=\frac{CA}{CA}=1$$

On the other hand, as the angle increases, CS diminishes, and when the angle becomes equal to 90°, CS disappears altogether, and is $=0$.

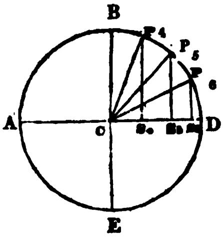
Hence since, generally, $\cos.\ \theta=\frac{CS}{CA}$; and since, when $\theta=90°$, $CS=0$;

$$\therefore\ \cos.\ 90°=\frac{0}{CA}=0$$

Let us now take different positions of the revolving radius *in the second quadrant.*

It is manifest, that as the angle increases the sine diminishes; for

$$\frac{P_4S_4}{CA}>\frac{P_5S_5}{CA}, \text{ and } \frac{P_5S_5}{CA}>\frac{P_6S_6}{CA}$$



As the angle goes on increasing, PS goes on diminishing; and when CP coincides with CD, that is, when the angle becomes equal to 180°, PS disappears altogether, and is equal 0.

Hence since, generally, $\sin.\ \theta=\frac{PS}{CA}$, and since, when $\theta=180°$, $PS=0$;

$$\therefore\ \sin.\ 180°=0.$$

On the other hand, as the angle increases the cosine increases; for

$$\frac{CS_4}{CA}<\frac{CS_5}{CA}, \text{ and } \frac{CS_5}{CA}<\frac{CS_6}{CA}$$

and when the revolving radius coincides with CD and the angle becomes 180°, CP coincides with CD and is equal to it.

Hence since, generally, $\cos.\ \theta=\frac{CS}{CA}$ and since, when $\theta=180°$, $CS=CD$;

$$\therefore\ \cos.\ 180°=\frac{CD}{CA}=-1;$$

$$\therefore\ CD=CA.$$

The negative sign here is employed, because the cosine is reckoned *to the right* along CD.

Reasoning in the same manner for the third and fourth quadrants, we shall find,

$$\begin{aligned}\sin.\ 270^\circ &= -1\\ \cos.\ 270^\circ &= 0\\ \sin.\ 360^\circ &= 0\\ \cos.\ 360^\circ &= 1.\end{aligned}$$

Thus, it appears,

That as the angle increases in the first quadrant, from 0 up to 90°,
  The sine, being positive, increases from 0 up to 1,
  The cosine being positive decreases from 1 down to 0.
That, as the angle increases in the second quadrant, from 90° up to 180°,
  The sine, being positive, decreases from 1 down to 0,
  The cosine, being negative, passes from 0 to —1.
That, as the angle increases in the third quadrant, from 180° up to 270°,
  The sine, being negative, passes from 0 to —1,
  The cosine, being negative, passes from —1 to 0.
That, as the angle increases in the fourth quadrant from 270° up to 360°,
  The sine, being negative, passes from —1 to 0,
  The cosine, being positive, increases from 0 up to 1.

The variations in the magnitude of the sines and cosines, being those of the other trigonometrical quantities may be determined by the means of the relations in Table I.

Thus, since, $\tan.\ \theta = \frac{\sin.\ \theta}{\cos.\ \theta}$

$$\tan.\ 0 = \frac{\sin.\ 0}{\cos.\ 0} = \frac{0}{1} = 0$$

$$\tan.\ 90^\circ = \frac{\sin.\ 90^\circ}{\cos.\ 90^\circ} = \frac{1}{0} = \textit{ad infinitum}, \text{ or } \infty.$$

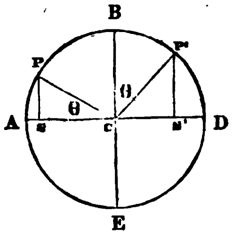
The truth of this last relation may be readily illustrated, by referring to the geometrical construction; when it will be seen that for the angle of 90° AT becomes parallel to CP; and therefore, the point T, in which the two lines meet, is at an infinite distance.

So, also, $\cot.\ 0 = \infty$

$$\cot.\ 90^\circ = 0$$

and so for all the rest.

We shall next proceed to point out some important general relations, which exist between the trigonometrical functions of angles less than 90° and those of angles greater than 90°



Draw CP, making with CA any angle PCA which we may call $\theta$; let fall PS perpendicular from P on CA. Draw CP', making with BC the angle $BCP' = PCA = \theta$; and from P' let fall P'S' perpendicular on CD.

Then the angle $P'CA = 90° + \theta$.

The two triangles PCS, P'CS', have the side PC of the one equal to the side P'C of the other, also the angles at S and S' right angles, and the angle CPS of the one equal to the angle P'CS' of the other; therefore the two triangles are in every respect equal; and

$$PS = CS', \quad CS = P'S'.$$

Therefore, $$\frac{P'S'}{CA} = \frac{CS}{CA}$$

Or, $$\sin. P'CA = \cos. PCA,$$

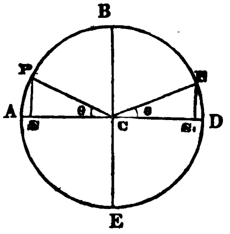
that is, $$\sin. (90° + \theta) = \cos. \theta.$$

Again,

$$\frac{CS'}{CA} = \frac{PS}{CA}$$

Or, $$-\cos. P'CA = \sin. PCA,$$

that is $$\cos. (90° + \theta) = -\sin. \theta.$$



As before, draw CP, making any angle $\theta$ with CA, and draw CP', making with CD the angle P'CD, equal to $\theta$.

Then the angle $P'CA = 180° - \theta$.

The two triangles PCS, P'CS' are manifestly in all respects equal; and

$$PS = P'S' \quad CS = CS'$$

Therefore, $$\frac{PS}{CA} = \frac{P'S'}{CA}$$

that is, $\sin. \theta = \sin. (180° - \theta)$

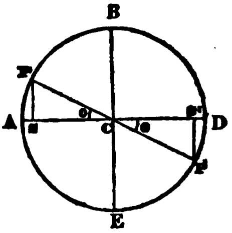
an important proposition which enunciated in words, is, *the sine of an angle is equal to the sine of its supplement.*

Again,

$$\frac{CS}{CA} = \frac{CS'}{CA}$$

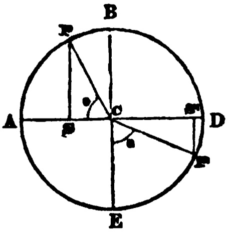
$$\cos. \theta = -\cos. (180° - \theta),$$

that is, *the cosine of an angle, and the cosine of its supplement are equal in absolute magnitude, but have opposite signs.*



If, as in the annexed figure, we draw CP', making with CD an angle DCP' equal to the angle $\theta$, we shall find in like manner,

$$\sin.\,(180^\circ+\theta)=-\sin.\,\theta$$
$$\cos.\,(180^\circ+\theta)=\cos.\,(180^\circ-\theta)$$
$$=-\cos.\,\theta.$$



If we draw CP', making with CE an angle ECP'$=\theta$, then

$$\sin.\,(270^\circ-\theta)=-\cos.\,\theta$$
$$\cos.\,(270^\circ-\theta)=-\sin.\,\theta$$

as is evident from Def. 8, and the rule for signs; and, in like manner, we may proceed for angles in the fourth quadrant. These relations being established between the sines and cosines, the corresponding relations between other trigonometrical functions may be deduced immediately from Table 1.

Thus,

$$\tan.\,(90^\circ+\theta)=\frac{\sin.\,(90^\circ+\theta)}{\cos.\,(90^\circ+\theta)}$$
$$=\frac{\cos.\,\theta}{-\sin.\,\theta}$$
$$=-\cot.\,\theta$$
$$\tan.\,(180^\circ-\theta)=\frac{\sin.\,(180^\circ-\theta)}{\cos.\,(180^\circ-\theta)}$$
$$=\frac{\sin.\,\theta}{-\cos.\,\theta}$$
$$=-\tan.\,\theta$$

and so for all the rest

The student may exercise himself by verifying such of the results in the following table as have not been formally demonstrated.

## TABLE II.

*$\sin. 0 = 0$
*$\cos. 0 = 1$
*$\tan. 0 = 0$
*$\cot. 0 = \infty$
$\sec. 0 = 1$
$\mathrm{cosec.}\ 0 = \infty$
*$\sin. (90^\circ - \theta) = \cos. \theta$
*$\cos. (90^\circ - \theta) = \sin. \theta$
*$\tan. (90^\circ - \theta) = \cot. \theta$
*$\cot. (90^\circ - \theta) = \tan. \theta$
$\sec. (90^\circ - \theta) = \mathrm{cosec.}\ \theta$
$\mathrm{cosec.}\ (90 - \theta) = \sec. \theta$
*$\sin. 90^\circ = 1$
*$\cos. 90^\circ = 0$
*$\tan. 90^\circ = \infty$
*$\cot. 90^\circ = 0$
$\sec. 90^\circ = \infty$
$\mathrm{cosec.}\ 90^\circ = 1$
*$\sin. (90^\circ + \theta) = \cos. \theta$
*$\cos. (90^\circ + \theta) = -\sin. \theta$
*$\tan. (90^\circ + \theta) = -\cot. \theta$
*$\cot. (90^\circ + \theta) = -\tan. \theta$
$\sec. (90^\circ + \theta) = -\mathrm{cosec.}\ \theta$
$\mathrm{cosec.}\ (90^\circ + \theta) = \sec. \theta$
*$\sin. (180^\circ - \theta) = \sin. \theta$
*$\cos (180^\circ - \theta) = -\cos. \theta$
*$\tan. (180^\circ - \theta) = -\tan. \theta$
*$\cot. (180^\circ - \theta) = -\cot. \theta$
$\sec. (180^\circ - \theta) = -\sec. \theta$
$\mathrm{cosec.}\ (180^\circ - \theta) = \mathrm{cosec.}$
*$\sin. 180^\circ = 0$
*$\cos. 180^\circ = -1$
*$\tan. 180^\circ = 0$
*$\cot. 180^\circ = -\infty$
$\sec. 180^\circ = -1$
$\mathrm{cosec.}\ 180^\circ = 0$

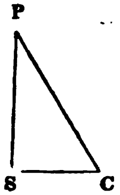
*$\sin. (180^\circ + \theta) = -\sin. \theta$
*$\cos. (180^\circ + \theta) = -\cos. \theta$
$\tan. (180^\circ + \theta) = \tan. \theta$
$\cot. (180^\circ + \theta) = \cot. \theta$
$\sec. (180^\circ + \theta) = -\sec. \theta$
$\mathrm{cosec.}\ (180^\circ + \theta) = -\mathrm{cosec.}\ \theta$
$\sin. (270^\circ - \theta) = -\cos. \theta$
$\cos. (270^\circ - \theta) = -\sin. \theta$
$\tan. (270^\circ - \theta) = \cot. \theta$
$\cot. (270^\circ - \theta) = \tan. \theta$
$\sec. (270^\circ - \theta) = -\mathrm{cosec.}\ \theta$
$\mathrm{cosec.}\ (270^\circ - \theta) = -\sec. \theta$
$\sin. 270^\circ = -1$
$\cos. 270^\circ = 0$
$\tan. 270^\circ = \infty$
$\cot. 270^\circ = 0$
$\sec. 270^\circ = \infty$
$\mathrm{cosec.}\ 270^\circ = -1$
$\sin. (270^\circ + \theta) = -\cos. \theta$
$\cos. (270^\circ + \theta) = \sin. \theta$
$\tan. (270^\circ + \theta) = -\cot. \theta$
$\cot. (270^\circ + \theta) = -\tan. \theta$
$\sec. (270^\circ + \theta) = \mathrm{cosec.}\ \theta$
$\mathrm{cosec.}\ (270^\circ + \theta) = -\sec. \theta$
$\sin. (360^\circ - \theta) = -\sin. \theta$
$\cos. (360^\circ - \theta) = \cos. \theta.$
$\tan. (360^\circ - \theta) = -\tan. \theta$
$\cot. (360^\circ - \theta) = -\cot. \theta$
$\sec. (360^\circ - \theta) = \sec. \theta$
$\mathrm{cosec.}\ (360 - \theta) = -\mathrm{cosec.}\ \theta$
$\sin. 360^\circ = 0$
$\cos. 360^\circ = 1$
$\tan. 360^\circ = 0$
$\cot. 360^\circ = \infty$
$\sec. 360^\circ = 1$
$\mathrm{cosec.}\ 360^\circ = \infty$

The results in the above table which are most frequently used, are marked with an asterisk, and ought to be committed to memory.

We have in the preceding pages confined ourselves to the consideration of angles not greater than 360°, but the student can find no difficulty in applying the above principles to angles of any magnitude whatsoever.

We shall conclude this introductory chapter, by demonstrating two propositions which are of the highest importance in our subsequent investigations. The first is,

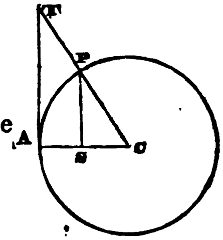
*In any right-angled triangle, the ratio which the side opposite to one of the acute angles bears to the hypothenuse, is the sine of that angle; the ratio which the side adjacent to one of the acute angles bears to the hypothenuse, is the cosine of that angle; and the ratio which the side opposite to one of the acute angles bears to the side adjacent to that angle, is the tangent of that angle.*



Let CSP be a plane triangle right-angled at S.

Then, $\frac{PS}{CP}=\sin. C, \frac{CS}{CP}=\cos. C, \frac{PS}{CS}=\tan. C,$

or $\frac{CS}{CP}=\sin. P. \frac{SP}{CP}=\cos. P. \frac{SC}{SP}=\tan. P.$



From C as a centre with the radius CP, describe a circle.

Produce CS, to meet the circumference in A.

From A draw AT a tangent to the circle at A.

Produce CP to meet AT in T.

Then, from Definitions (2) (3) (4),

$\frac{PS}{CP}=\sin. C, \frac{CS}{CP}=\cos. C, \frac{AT}{CP}=\tan. C,$

for CP=AC.

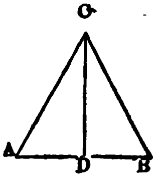
But the triangles TAC, PSC, are similar;

Therefore, $\frac{AT}{CP}=\frac{PS}{CS}=\tan. C,$

*Cor.* PS=CP sin. C=CP cos. P
CS=CP cos. C=CP sin. P
PS=CS tan. C=CS cot. P

The second proposition is,

*In any plane triangle, the ratio of any two of the sides is equal to the ratio of the sines of the angles opposite to them.*



Let ABC be a plane triangle; it is required to prove that

$\frac{CB}{CA}=\frac{\sin. A}{\sin. B}, \frac{CB}{BA}=\frac{\sin. A}{\sin. C}, \frac{CA}{BA}=\frac{\sin. B}{\sin. C},$

From C let fall the perpendicular CD on AB.

Then, since CDB is a plane triangle right-angled at D, by last proposition

$$CD=CB\ \sin.\ B \qquad (1)$$

Again, since CDA is a plane triangle right-angle at A,

$$CD=CA\ \sin.\ A \qquad (2)$$

Equating these two values of CD,

$$CB\ \sin B=CA\ \sin.\ A\,;$$

Therefore, $\frac{CB}{CA}=\frac{\sin.\ A.}{\sin.\ B.}$

In like manner, by dropping perpendiculars from B and A upon the side AC, CB we can prove,

$$\frac{CB}{BA}=\frac{\sin.\ A,}{\sin.\ C,}\quad \frac{CA}{BA}=\frac{\sin.\ B,}{\sin.\ C.}$$

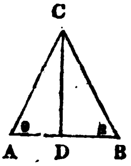
In treating of plane triangles, it is convenient to designate the three angles by the capital letters A, B, C, and the sides opposite to these angles by the corresponding small letters $a$, $b$, $c$. According to this notion, the last proposition will be

$$\frac{a}{b}=\frac{\sin.\ A}{\sin.\ B},\quad \frac{a}{c}=\frac{\sin.\ A}{\sin.\ C},\quad \frac{b}{c}=\frac{\sin.\ B}{\sin,\ c},$$

## CHAPTER II.

### GENERAL FORMULÆ.

*Given the sines and cosines of two angles, to find the sine of their sum.*



Let ABC be a plane triangle; from C let fall CD perpendicular on AB,

Let angle CAB=$\theta$,

and angle CBA=$\beta$,

$$\begin{aligned}\text{Then, } AB &= BD+DA\\ &= BC\cos.\ \beta + AC\cos.\ \theta,\end{aligned}$$

because BDC and ADC are right-angled triangles,

Dividing each member of the equation by AB,

$$\begin{aligned}1 &= \frac{BC}{AB}\cos.\ \beta+\frac{AC}{AB}\cos.\ \theta\\ &= \frac{\sin.\ \theta}{\sin.C}\cos\ \beta+\frac{\sin.\beta}{\sin.C}\cos.\ \theta, \text{ by last Prop. in Chap. I.}\end{aligned}$$

$$\therefore \sin.\ C = \sin.\ \theta\cos.\ \beta+\sin.\ \beta\cos.\ \theta.$$

But, since ABC is a plane triangle, $\theta+\beta+C = 180°$

$$\begin{aligned}\therefore C &= 180°-(\theta+\beta)\\ \sin.\ C &= \sin.\ \{180°-(\theta+\beta)\}\\ &= \sin(\theta+\beta,)\end{aligned}$$

because $180°-(\theta+\beta)$ is the supplement of $(\theta+\beta.)$

Hence, $\sin.\ (\beta+\theta) = \sin.\ \theta \cos \beta + \sin.\ \beta \cos.\ \theta$ - - - - ($a$)
That is, *the sine of the sum of two arcs or angles is equal the sine of the first multiplied by the cosine of the second, plus the sine of the second multiplied by the cosine of the first.*

This expression, from its great importance, is called the *fundamental formula* of Plane Trigonometry, and nearly the whole science may be derived from it.

### *Given the sines and cosines of two angles, to find the sine of their difference.*

By formula ($a$).

$$\sin.\ (\theta+\beta) = \sin.\ \theta \cos.\ \beta + \sin.\ \beta \cos.\ \theta.$$

For $\theta$ substitute $180°—\theta$, the above will become
$\sin.\ \{180° — (\theta — \beta)\} = \sin.\ (180° — \theta) \cos.\ \beta$
$+ \sin.\ \beta \cos. (180° — \theta)$
But $\sin. \{180° — (\theta — \beta)\} = \sin.\ (\theta — \beta)$ $\because$ $180° — (\theta — \beta)$ is the supplement of $(\theta — \beta.)$
And, $\sin. (180° — \theta) = \sin.\ \theta,$
And, $\cos. (180° — \theta) = — \cos.\ \theta$

Substitute, therefore, these values in the above expression, it becomes

$$\sin.\ (\theta — \beta) = \sin.\ \theta \cos.\ \beta — \sin.\ \beta \cos.\ \theta \text{ - - - - } (b)$$

That is, *the sine of the difference of two arcs or angles, is equal the sine of the first × cosine of the second, — the sine of the second × cosine of the first.*

### *Given the sines and cosines of two angles, to find the cosine of their sum.*

By formula ($a$)

$$\sin.\ (\theta+\beta) = \sin.\ \theta \cos.\ \beta + \sin.\ \beta \cos.\ \theta.$$

For $\theta$ substitute $90°+\theta$, the above will become
$\sin.\ \{90°+(\theta+\beta)\} = \sin.\ (90°+\theta) \cos.\ \beta$
$+ \sin.\ \beta \cos.\ (90°+\theta)$
But, $\sin.\ \{90°+(\theta+\beta)\} = \cos.\ (\theta+\beta)$ by Table II.
And, $\sin.\ (90°+\theta) = \cos.\ \theta.$
And, $\cos.\ (90°+\theta) = — \sin.\ \theta.$

Substituting, therefore, these values in the above expression, it becomes, $\cos.\ (\theta+\beta) = \cos.\ \theta \cos.\ \beta — \sin.\ \theta \sin. \beta$ - - ($c$)

That is, *the cosine of the sum of two arcs or angles, is equal to the cosine of the first multiplied by the cosine of the second, minus the sine of the first multiplied by the sine of the second.*

*Given the sines and cosines of two angles, to find the cosine of their difference.*

By formula ($a$:)

$$\sin.(\theta+\beta) = \sin.\theta\cos.\beta + \sin.\beta\cos.\theta,$$

For $\theta$ substitute $90°-\theta$, the above will become

$$\sin.\{90°-(\theta-\beta)\} = \sin.(90°-\theta)\cos.\beta + \sin.\beta\cos.(90°-\theta),$$

But, $\sin.\{90°-(\theta-\beta)\} = \cos.(\theta-\beta)$, By Table II.

$\sin.(90°-\theta) = \cos.\theta$ ………

$\cos.(90°-\theta) = \sin.\theta$ ………

Substituting, therefore, these values in the above expression, it becomes

$$\cos.(\theta-\beta) = \cos.\theta\cos.\beta + \sin.\theta\sin.\beta \quad - \; - \quad (d)$$

That is, *the cosine of the difference of two arcs or angles, is equal to the cosine of the first multiplied by the cosine of the second, plus sine of the first into the sine of the second.*

*Given the tangents of two angles, to find the tangent of their sum.*

By Table I.:

$$\tan.(\theta+\beta) = \frac{\sin.(\theta+\beta)}{\cos.(\theta+\beta)}$$

$$= \frac{\sin.\theta\cos.\beta + \sin.\beta\cos.\theta}{\cos.\theta\cos.\beta - \sin.\theta\sin.\beta} \text{ by } (a) \text{ and } (c)$$

Dividing both numerator and denominator of fraction by $\cos.\theta\cos.\beta$;

$$= \frac{\dfrac{\sin.\theta\cos.\beta}{\cos.\theta\cos.\beta} + \dfrac{\sin.\beta\cos.\theta}{\cos.\beta\cos.\theta}}{1 - \dfrac{\sin.\theta\sin.\beta}{\cos.\theta\cos.\beta}}$$

Simplifying, $$= \frac{\tan.\theta + \tan.\beta}{1 - \tan.\theta\tan.\beta} \quad - \; - \; - \; - \; - \; - \; - \; - \quad (e)$$

That is, *the tangent of the sum of two arcs or angles, is equal to the sum of the tangents of the two arcs, divided by* 1 *minus the product of the two tangents.*

*Given the tangents of two angles, to find the tangent of their difference.*

By Table I.:

$$\tan.(\theta-\beta) = \frac{\sin.(\theta-\beta)}{\cos.(\theta-\beta)}$$

$$= \frac{\sin.\theta\cos.\beta - \sin.\beta\cos.\theta}{\cos.\theta\cos.\beta + \sin.\theta\sin.\beta.} \text{ by } (b) \text{ and } (d)$$

Dividing both numerator and denominator by cos. $\theta$ cos. $\beta$:

$$=\frac{\frac{\sin.\theta\cos.\beta}{\cos.\theta\cos.\beta}-\frac{\sin.\beta\cos.\theta}{\cos.\theta\cos.\beta}}{1+\frac{\sin.\theta\sin.\beta}{\cos.\theta\cos.\beta}}$$

Simplifying, $$=\frac{\tan.\theta-\tan.\beta}{1+\tan.\theta\tan.\beta}\quad\text{- - - - - - - -}\quad(f)$$

Hence, *the tangent of the difference of two arcs or angles, is equal to the difference of the tangents of the two arcs, divided by 1 plus the product of the two tangents.*

The student will have no difficulty in deducing the following:

$$\cot.(\theta+\beta)=\frac{\cot.\theta\cot.\beta-1}{\cot.\beta+\cot.\theta}$$

$$\cot.(\theta-\beta)=\frac{\cot.\theta\cot.\beta+1}{\cot.\beta-\cot.\theta}$$

$$\sec.(\theta+\beta)=\frac{\sec.\theta\sec.\beta\operatorname{cosec.}\theta\operatorname{cosec.}\beta}{\operatorname{cosec.}\theta\operatorname{cosec.}\beta-\sec.\theta\sec.\beta}$$

$$\sec.(\theta-\beta)=\frac{\sec.\theta\sec.\beta\operatorname{cosec.}\theta\operatorname{cosec.}\beta}{\operatorname{cosec.}\theta\operatorname{cosec.}\beta+\sec.\theta\sec.\beta}$$

$$\operatorname{cosec.}(\theta+\beta)=\frac{\sec.\theta\sec.\beta\operatorname{cosec.}\theta\operatorname{cosec.}\beta}{\sec.\theta\operatorname{cosec.}\beta+\sec.\beta\operatorname{cosec.}\theta}$$

$$\operatorname{cosec.}(\theta-\beta)=\frac{\sec.\theta\sec.\beta\operatorname{cosec.}\theta\operatorname{cosec.}\beta}{\sec.\theta\operatorname{cosec.}\beta-\sec.\beta\operatorname{cosec.}\theta}$$

*To determine the sine of twice a given angle.*

By formula $(a)$:

$$\sin.(\theta+\beta)=\sin.\theta\cos.\beta+\sin.\beta\cos.\theta.$$

Let $\theta=\beta$, then the above becomes

$$\sin.2\theta=\sin.\theta\cos.\theta+\sin.\theta\cos.\theta$$
$$=2\sin.\theta\cos.\theta\quad\text{- - - - - - -}\quad(g1)$$

That is, *the sine of twice a given angle, is equal to twice the sine of the given angle multiplied by its cosine.*

In the last formula, for $\theta$ substitute $\frac{\theta}{2}$; then,

$$\sin.2\times\frac{\theta}{2}=2\sin.\frac{\theta}{2}\cos.\frac{\theta}{2}$$

$$\text{Or, }\sin.\theta=2\sin.\frac{\theta}{2}\cos.\frac{\theta}{2}\quad\text{- - - - - - -}\quad(g2)$$

### *To determine the cosine of twice a given angle.*

By formula (*c*):

$$\cos.\ (\theta+\beta)=\cos.\ \theta\cos.\ \beta-\sin.\ \theta\sin.\ \beta$$

Let $\theta=\beta$ then the above becomes

$$\cos.\ 2\theta=\cos.^2\theta-\sin.^2\theta \quad \text{- - - - - - - -} \quad (h1)$$

By table I. $\sin.^2\theta=1-\cos.^2\theta$; substituting this for $\sin.^2\theta$:

$$\cos.\ 2\theta=2\cos.^2\theta-1 \quad \text{- - - - - - - - -} \quad (h2)$$

Again, since $\cos.^2\theta=1-\sin.^2\theta$, substitute this for $\cos.^2\theta$:

$$\cos.\ 2\theta=1-2\sin.^2\theta \quad \text{- - - - - - - - -} \quad (h3)$$

Hence, *the cosine of twice a given arc or angle, is equal to* 1 *minus twice the square of the sine of the given angle.*

### *To determine the tangent of twice a given angle.*

By formula (*e*):

$$\tan.\ (\theta+\beta)=\frac{\tan.\ \theta+\tan.\ \beta}{1-\tan.\ \theta\tan.\ \beta}$$

Let $\theta=\beta$, the above becomes

$$\tan.\ 2\theta=\frac{2\tan.\ \theta}{1-\tan.^2\theta} \quad \text{- - - - - - - - - - - -} \quad (i)$$

*The tangent of twice a given arc, is equal to twice the tangent of the given arc, divided by* 1 *minus the square of the tangent.*

The student will easily deduce the following:

$$\cot.\ 2\theta=\frac{\cot.^2\theta-1}{2\cot.\ \theta}=\frac{\cot.\ \theta-\tan.\ \theta}{2}$$

$$\sec.\ 2\theta=\frac{\sec.^2\theta\ \mathrm{cosec.}^2\theta}{\mathrm{cosec.}^2\theta-\sec^2\theta}$$

$$\mathrm{cosec.}\ 2\theta=\frac{\sec.^2\theta\ \mathrm{cosec.}^2\theta}{2\sec.\ \theta\ \mathrm{cosec.}\ \theta}=\frac{\sec.\ \theta\ \mathrm{cosec.}\ \theta}{2}$$

### *To determine the sine of half a given angle.*

By formula (*h3*):

$$\cos.\ 2\theta=1-2\sin.^2\theta$$

For $\theta$ substitute $\frac{\theta}{2}$; the above becomes,

$$\cos.\ 2\frac{\theta}{2}=1-2\sin.^2\frac{\theta}{2}$$

$$\text{Or,}\quad \cos.\ \theta=1-2\sin.^2\frac{\theta}{2}$$

$$\therefore\ 2\sin.^2\frac{\theta}{2}=1-\cos.\ \theta$$

$$\sin.\ \frac{\theta}{2}=\sqrt{\frac{1-\cos.\ \theta}{2}} \quad \text{- - - - - - - -} \quad (j)$$

That is, *the sine of half a given angle or arc is equal to the square root of* 1 *minus the cosine of the arc divided by two.*

*To determine the cosine of half a given angle.*

By formal ($h2$):

$$\cos. 2\theta = 2\cos.^2\theta - 1$$

For $\theta$ substitute $\frac{\theta}{2}$; the above becomes,

$$\cos. 2\frac{\theta}{2} = 2\cos.^2\frac{\theta}{2} - 1$$

Or, $$\cos. \theta = 2\cos.^2\frac{\theta}{2} - 1$$

$$\therefore \quad 2\cos.^2\frac{\theta}{2} = 1 + \cos. \theta$$

$$\cos. \frac{\theta}{2} = \sqrt{\frac{1+\cos. \theta}{2}} \quad \text{- - - -} \quad (k)$$

*To determine the tangent of half a given angle.*

Divide formula ($j$) by ($k$):

$$\frac{\sin. \frac{\theta}{2}}{\cos. \frac{\theta}{2}} = \sqrt{\frac{1-\cos. \theta}{1+\cos. \theta}}$$

Or, $$\tan. \frac{\theta}{2} = \sqrt{\frac{1-\cos. \theta}{1+\cos. \theta}} \quad \text{- - - -} \quad (l1)$$

Multiply both numerator and denominator by $\sqrt{1-\cos. \theta}$; the above becomes,

$$\tan. \frac{\theta}{2} = \frac{1-\cos. \theta}{\sin. \theta} \quad \text{- - - - -} \quad (l2)$$

Multiply both numerator and denominator of ($l$ 1) by $\sqrt{1+\cos. \theta}$; we have,

$$\tan. \frac{\theta}{2} = \frac{\sin. \theta}{1+\cos. \theta} \quad \text{- - - - - -} \quad (l3)$$

The student will easily deduce the following:

$$\cot. \frac{\theta}{2} = \sqrt{\frac{1+\cos. \theta}{1-\cos. \theta}}$$

$$= \frac{1+\cos. \theta}{\sin. \theta}$$

$$= \frac{\sin. \theta.}{1-\cos. \theta}$$

$$\sec.\frac{\theta}{2} = \sqrt{\frac{2\sec.\theta}{\sec.\theta+1}}$$

$$\text{cosec.}\frac{\theta}{2} = \sqrt{\frac{2\sec.\theta}{\sec.\theta-1}}$$

*To determine the sine of $(n+1)\,\theta$, in terms of $n\,\theta$, $(n-1)\,\theta$ and $\theta$.*

By formula (*a*) and (*b*):

$$\sin.(\beta+\theta) = \sin.\beta\cos.\theta + \sin.\theta\cos.\beta$$
$$\sin.(\beta-\theta) = \sin.\beta\cos.\theta - \sin.\theta\cos.\beta$$

Add these two equations,

$$\sin.(\beta+\theta) + \sin.(\beta-\theta) = 2\sin.\beta\cos.\theta$$

Subtract $\sin.(\beta-\theta)$ from each member,

$$\sin.(\beta+\theta) = 2\sin.\beta\cos.\theta - \sin.(\beta-\theta)$$

Let $\beta = n\,\theta$, the above becomes

$$\sin.(n+1)\theta = 2\sin.n\,\theta\cos.\theta - \sin.(n-1)\,\theta \,..\, (m)$$

In the above formula, let $n=1$; $\therefore n+1=2$, $n-1=0$

$$\therefore \sin.2\,\theta = 2\sin.\theta\cos.\theta - \sin.0$$
$$= 2\sin.\theta\cos.\theta, \text{ the same result as in } (g).$$

Let $n=2$; $\therefore n+1=3$, $n-1=1$;

$$\therefore \sin.3\,\theta = 2\sin.2\,\theta\cos.\theta - \sin.\theta$$
$$= 2\times2\sin.\theta\cos.\theta\times\cos.\theta - \sin.\theta$$
$$= 4\sin.\theta\cos.^2\theta - \sin.\theta$$
$$= 4\sin.\theta(1-\sin.^2\theta) - \sin.\theta$$
$$= 3\sin.\theta - 4\sin.^3\theta \;\text{- - - - - -}\; (n)$$

Let $n=3$; $\therefore n\times1=4$, $n-1=2$;

$\therefore$ By formula (*m*):

$$\sin.4\,\theta = 2\times\sin.3\,\theta\times\cos.\theta - \sin.2\,\theta$$
$$= 2(3\sin.\theta - 4\sin.^3\theta)\cos.\theta - 2\sin.\theta\cos.\theta$$
$$= (8\cos.^3\theta - 4\cos.\theta)\sin.\theta$$

It is manifest that, by continuing the same process, we may find in succession, $\sin.5\,\theta$, $\sin.6\,\theta$, - - - &c.

*To determine the cosine of $(n+1)\,\theta$, in terms of $n\,\theta$, $(n-1)\,\theta$, and $\theta$.*

By formula (*c*) and (*d*):

$$\cos.(\beta+\theta) = \cos.\beta\cos.\theta - \sin.\beta\sin.\theta$$
$$\cos.(\beta-\theta) = \cos.\beta\cos.\theta + \sin.\beta\sin.\theta$$

Add these two equations,

$$\cos.(\beta+\theta) + \cos.(\beta-\theta) = 2\cos.\beta\cos.\theta$$

Subtract $\cos.(\beta-\theta)$ from each member,

$$\cos.\ (\beta+\theta)=2\cos.\ \beta\cos.\ \theta-\cos.\ (\beta-\theta);$$

Let $\beta=n\,\theta$, the above becomes

$$\cos.\ (n+1)\,\theta=2\cos.\ n\,\theta\cos.\ \theta-\cos.\ (n-1)\,\theta \quad \text{- -} \quad (o)$$

In the above formula, let $n=1$; $\therefore\ n+1=2,\ n-1=0$;

Then, $\cos.\ 2\,\theta=2\cos.\ \theta\cos.\ \theta-\cos 0$

$=2\cos.^2\theta-1$, the same result as in ($h2$).

Let $n=2$, $\therefore\ n+1=3,\ n-1=1$;

$$\therefore\quad \cos.\ 3\,\theta=2\cos.\ 2\,\theta\cos.\ \theta-\cos.\ \theta$$
$$=2\,(2\cos.^2\theta-1)\cos.\ \theta-\cos.\ \theta$$
$$=4\cos.^3\theta-3\cos.\ \theta \quad \text{- - - - -} \quad (p)$$

Let $n=3$; $\therefore\ n+1=4,\ n-1=2$;

$$\therefore\quad \cos.\ 4\,\theta=2\cos.\ 3\,\theta\cos.\ \theta-\cos.\ 2\,\theta$$
$$=2\,(4\cos.^3\theta-3\cos.\ \theta)\cos.\ \theta-(2\cos.^2\theta-1)$$
$$=8\cos.^4\theta-8\cos.^2\theta+1$$

It is manifest that, by continuing the same process, we may find, in succession, $\cos.\ 5\,\theta$, $\cos.\ 6\,\theta$, - - - &c.

By adding and subtracting ($a$) and ($b$), and by adding and subtracting ($c$) and ($d$), we obtain the following formulæ, which are of considerable utility.

$$\left.\begin{array}{l} \sin.\ (\theta+\beta)+\sin.\ (\theta-\beta)=\ \ 2\sin.\ \theta\cos.\ \beta \\ \sin.\ (\theta+\beta)-\sin.\ (\theta-\beta)=\ \ 2\sin.\ \beta\cos.\ \theta \\ \cos.\ (\theta+\beta)+\cos.\ (\theta-\beta)=\ \ 2\cos.\ \theta\cos.\ \beta \\ \cos.\ (\theta+\beta)-\cos.\ (\theta-\beta)=-2\sin.\ \theta\sin.\ \beta \end{array}\right\} \text{- - - -} \quad (q)$$

Any angle $\theta$ may, by a simple artifice, be put under the form,

$$\theta=\frac{\theta+\beta}{2}+\frac{\theta-\beta}{2}$$

And, in like manner,

$$\beta=\frac{\theta+\beta}{2}-\frac{\theta-\beta}{2}$$

$$\therefore\quad \sin.\ \theta=\sin.\left\{\frac{\theta+\beta}{2}+\frac{\theta-\beta}{2}\right\}$$

$$=\sin.\ \frac{\theta+\beta}{2}\cos.\ \frac{\theta-\beta}{2}+\sin.\ \frac{\theta-\beta}{2}\cos.\ \frac{\theta+\beta}{2} \quad \text{- - -} \quad (1)$$

$$\sin.\ \beta=\sin.\left\{\frac{\theta+\beta}{2}-\frac{\theta-\beta}{2}\right\}$$

$$=\sin.\ \frac{\theta+\beta}{2}\cos.\ \frac{\theta-\beta}{2}-\sin.\ \frac{\theta-\beta}{2}\cos.\ \frac{\theta+\beta}{2} \quad \text{- - -} \quad (2)$$

$$\cos.\ \theta=\cos.\left\{\frac{\theta+\beta}{2}+\frac{\theta-\beta}{2}\right\}$$

$$=\cos.\ \frac{\theta+\beta}{2}\cos.\ \frac{\theta-\beta}{2}-\sin.\ \frac{\theta+\beta}{2}\sin.\ \frac{\theta-\beta}{2} \quad \text{- - -} \quad (3)$$

$$\cos.\,\beta=\cos.\left\{\frac{\theta+\beta}{2}-\frac{\theta-\beta}{2}\right\}$$

$$=\cos.\frac{\theta+\beta}{2}\cos.\frac{\theta-\beta}{2}+\sin.\frac{\theta+\beta}{2}\sin.\frac{\theta-\beta}{2}\quad\text{. . . (4)}$$

Add together (1) and (2):

$$\sin.\,\theta+\sin.\,\beta=2\sin.\frac{\theta+\beta}{2}\cos.\frac{\theta-\beta}{2}\quad\text{. . . . . . . . (}r\text{)}$$

Subtract (2) from (1),

$$\sin.\,\theta-\sin.\,\beta=2\sin.\frac{\theta-\beta}{2}\cos.\frac{\theta+\beta}{2}\quad\text{. . . . . . . (}s\text{)}$$

Add together (3) and (4),

$$\cos.\,\theta+\cos.\,\beta=2\cos.\frac{\theta+\beta}{2}\cos.\frac{\theta-\beta}{2}\quad\text{. . . . . . . (}t\text{)}$$

Subtract (4) from (3),

$$\cos.\,\theta-\cos.\,\beta=-2\sin.\frac{\theta+\beta}{2}\sin.\frac{\theta-\beta}{2}\quad\text{. . . . . . (}v\text{)}$$

These formulæ, which are of the greatest importance, might have been immediately deduced from the group ($q$), by changing $\theta+\beta$ into $\theta$, $\theta-\beta$ into $\beta$, $\theta$ into $\frac{\theta+\beta}{2}$, $\beta$ into $\frac{\theta-\beta}{2}$.

Divide ($r$) by ($s$):

$$\frac{\sin.\,\theta+\sin.\,\beta}{\sin.\,\theta-\sin.\,\beta}=\frac{2\sin.\frac{\theta+\beta}{2}\cos.\frac{\theta-\beta}{2}}{2\sin.\frac{\theta-\beta}{2}\cos.\frac{\theta+\beta}{2}}$$

$$=\frac{\tan.\frac{\theta+\beta}{2}}{\tan.\frac{\theta-\beta}{2}}=\frac{\tan.\,\frac{1}{2}(\theta+\beta)}{\tan.\,\frac{1}{2}(\theta-\beta)}\quad\text{. . . . (}w\text{)}$$

Multiply ($a$) by ($b$); then,

$$\sin.\,(\theta+\beta)\sin.\,(\theta-\beta)=\sin.^2\theta\cos.^2\beta-\sin.^2\beta\cos.^2\theta$$

$$=\sin.^2\theta-\sin.^2\beta\quad\text{. . . . . . (}x\text{)}$$

Multiply ($c$) by ($d$); then,

$$\cos.\,(\theta+\beta)\cos.\,(\theta-\beta)=\cos.^2\theta\cos.^2\beta-\sin.^2\theta\sin.^2\beta$$

$$=\cos.^2\theta-\sin.^2\beta\quad\text{. . . . . (}y\text{)}$$

We will now investigate a few properties where *more* than two arcs or angles are concerned, and which may be of use in the subsequent part of this work.

Let $\theta$, $\beta$, $\gamma$, be any three arcs or angles.

Then,

$$\sin.\ (\beta+\gamma)=\frac{\sin.\ \theta\ \sin.\ \gamma+\sin.\ \beta\ \sin.\ (\theta+\beta+\gamma)}{\sin.\ (\theta+\beta)}$$

For by formula (*a*)

$\sin.\ (\theta+\beta+\gamma)=\sin.\ \theta\ \cos.\ (\beta+\gamma)+\cos.\ \theta\ \sin.\ (\beta+\gamma)$ which [putting $\cos.\ \beta\ \cos.\ \gamma-\sin.\ \beta\ \sin.\ \gamma$, for $\cos.\ (\beta+\gamma)$], is $=\sin.\ \theta\ \cos.\ \beta\ \cos.\ \gamma-\sin.\ \theta\ \sin.\ \beta\ \sin.\ \gamma+\cos.\ \theta\ \sin.\ (\beta+\gamma)$; and, multiplying by $\sin.\ \beta$, and adding $\sin.\ \theta\ \sin\ \gamma$, there results $\sin.\ \theta\ \sin.\ \gamma+\sin.\ \beta\ \sin.\ (\theta+\beta+\gamma)=\sin.\ \theta\ \cos.\ \beta\ \cos.\ \gamma\ \sin.\ \beta+\sin.\ \theta\ \sin.\ \gamma\ \cos.^2\ \beta+\cos.\ \theta\ \sin.\ \beta\ \sin.\ (\beta+\gamma)=\sin.\ \theta\ \cos.\ \beta\ (\sin.\ \beta\ \cos.\ \gamma+\cos.\ \beta\ \sin.\ \gamma)+\cos.\ \theta\ \sin.\ \beta\ \sin.\ (\beta+\gamma)=(\sin.\ \theta\ \cos.\ \beta+\cos.\ \theta\ \sin.\ \beta)\ \sin.\ (\beta+\gamma)=\sin.\ (\theta+\beta)\ \sin.\ (\beta+\gamma)$.

Hence, dividing by $\sin.\ (\theta+\beta)$, we have,

$$\sin.\ (\beta+\gamma)=\frac{\sin.\ \theta\ \sin.\ \gamma+\sin.\ \beta\ \sin.(\theta+\beta+\gamma)}{\sin.\ (\theta+\beta)}$$

In a similar manner it may be shown, that

$$\sin.\ (\beta-\gamma)=\frac{\sin.\ \theta\ \sin.\ \gamma-\sin.\ \beta\ \sin.\ (\theta-\beta+\gamma)}{\sin.\ (\theta-\beta)}$$

If $\theta$, $\beta$, $\gamma$, $\delta$, represents any four arcs or angles, then writing $\gamma+\delta$ for $\gamma$ in the preceding investigation, there will result

$$\sin.\ (\beta+\gamma+\delta)=\frac{\sin.\ \theta\ \sin.\ (\gamma+\delta)+\sin.\ \beta\ \sin.\ (\theta+\beta+\gamma+\delta)}{\sin.\ (\theta+\beta)}$$

A like process for five arcs or angles will give

$$\sin.\ (\beta+\gamma+\delta+\zeta)=\frac{\sin.\theta\ \sin.(\gamma+\delta+\zeta)+\sin.\beta\sin.(\theta+\beta+\gamma+\delta+\zeta)}{\sin.\ (\theta+\beta)}$$

And for any number $\theta$, $\beta$, $\gamma$, &c. . . . $\lambda$

$$\sin.(\beta+\gamma+..\lambda)=\frac{\sin.\theta\ \sin.(\gamma+\delta+\ldots\lambda)+\sin.\beta\ \sin.(\theta+\beta+\gamma+\ldots\lambda)}{\sin.\ (\theta+\beta)}$$

Taking again the three $\theta$, $\beta$, $\gamma$, we have

$$\sin.\ (\beta-\gamma)=\sin.\ \beta\ \cos.\ \gamma-\sin.\ \gamma\ \cos.\ \beta$$
$$\sin.\ (\gamma-\theta)=\sin.\ \gamma\ \cos.\ \theta-\sin.\ \theta\ \cos.\ \gamma$$
$$\sin.\ (\theta-\beta)=\sin.\ \theta\ \cos.\ \beta-\sin.\ \beta\ \cos.\ \theta$$

Multiplying the first of these equations by $\sin.\ \theta$, second by $\sin.\ \beta$, third by $\sin.\ \gamma$; then adding together the equations thus transformed; there will result,

$$\sin.\ \theta\ \sin\ (\beta-\gamma)+\sin.\ \beta\ \sin.\ (\gamma-\theta)+\sin.\ \gamma\ \sin.\ (\theta-\beta)=0$$
$$\sin.\ \theta\ \sin.\ (\beta-\gamma)+\cos.\ \beta\ \sin.\ (\gamma-\theta)+\cos.\ \gamma\ \sin.\ (\theta-\beta)=0$$

These two equations resulting from any three angles whatever may evidently be applied to the three angles of any triangles.

Let the series of arcs or angles $\theta, \beta, \gamma, \delta$ - - - - - $\lambda$, be contemplated, then we have formula ($x$)

$$\begin{aligned}
&\sin.\,(\theta+\beta)\,\sin.\,(\theta-\beta)=\sin.^2\theta-\sin.^2\beta\\
&\sin.\,(\beta+\gamma)\,\sin.\,(\beta-\gamma)=\sin.^2\beta-\sin.^2\gamma.\\
&\sin.\,(\gamma+\delta)\,\sin.\,(\gamma-\delta)=\sin.^2\gamma-\sin.^2\delta\\
&\qquad\&c.\ldots\ldots=\&c.\ldots\ldots\\
&\sin.\,(\lambda+\theta)\,\sin.\,(\lambda-\theta)=\sin.^2\lambda-\sin.^2\theta
\end{aligned}$$

Adding these equations together, we have $\sin.\,(\theta+\beta)\,\sin.\,(\theta-\beta)+\sin.\,(\beta+\gamma)\,\sin.\,(\beta-\gamma)+\sin.\,(\gamma+\delta)\,\sin.\,(\gamma-\delta)+\ldots\ldots\ldots\sin.\,(\lambda+\theta)\,\sin.\,(\lambda-\theta)=0$

Proceeding in a similar manner with the $\sin.\,(\theta-\beta)$, $\cos.\,(\theta+\beta)$, $\sin.\,(\beta-\gamma)$, $\cos.\,(\beta+\gamma)$, &c., there will at length be obtained $\cos.(\theta+\beta)\,\sin.(\theta-\beta)+\cos.(\beta+\gamma)\,\sin.(\beta-\gamma)+\ldots\cos.\,(\lambda+\theta)\,\sin.\,(\lambda-\theta)=0$

If the arcs $\theta, \beta, \gamma$ - - - - $\lambda$ form an arithmetical progression of which the first term is 0 the ratio $\rho$ and the last term $\lambda$, any number $n$ of circumferences, then will $\beta-\theta=\rho$, $\gamma-\beta=\rho$, &c., $\theta+\beta=\rho$, $\beta+\gamma=3\rho$, &c.; dividing the whole by the $\sin.\,\rho$, the preceding equations will become

$$\left.\begin{aligned}
&\sin.\,\rho+\sin.\,3\rho+\sin.\,5\rho+\&c.=0\\
&\cos.\,\rho+\cos.\,3\rho+\cos.\,5\rho+\&c.=0
\end{aligned}\right\}\text{- - - - - }(z)$$

If $\xi$ were equal $2\rho$, these equations would become

$$\begin{aligned}
&\sin.\,\rho+\sin.\,(\rho+\xi)+\sin.\,(\rho+2\xi)+\sin.\,(\rho+3\xi)+\&c.=0\\
&\cos.\,\rho+\cos.\,(\rho+\xi)+\cos.\,(\rho+2\xi)+\cos.\,(\rho+3\xi)+\&c.=0
\end{aligned}$$

The last equations, however, only show the sums of the sines and cosines of arcs or angles in arithmetical progression when the common difference is to the first term in the ratio of 2 to 1. To find a general expression for an infinite series of this kind, let

$$S+\sin.\,\theta+\sin.\,(\theta+\beta)+\sin.\,(\theta+2\beta)+\sin.\,(\theta+3\beta)+\text{ - - - - }\&c.$$

Then since this series is a recurring series whose scale of relations is $2\cos.\,\beta-1$, it will arise from the development of a fraction whose denominator $1-2\chi\cos.\,\beta+\chi^2$ making $\chi=1$.

Now this fraction will be,

$$=\frac{\sin.\,\theta+\chi(\sin.(\theta+\beta)-2\sin.\,\theta\,\cos.\,\beta}{1-2\chi\cos.\,\beta+\chi^2}$$

Therefore, when $\chi=1$, we have,

$$S=\frac{\sin.\,\theta+\sin.\,(\theta+\beta)-2\sin.\,\theta\cos.\,\beta}{2-2\cos.\,\beta};$$

And this because, $2\sin.\,\theta\,\cos.\,\beta=\sin.\,(\theta+\beta)+\sin.\,(\theta-\beta)$

$$=\frac{\sin.\,\theta-\sin.\,(\theta-\beta)}{2\,(1-\cos.\,\beta)}\text{ by formula }(q).$$

Now putting $\theta'$ for $(\theta+\beta)$ and $\beta'$ for $(\theta-\beta)$ we have from formula ($s$):

$$\sin.\,\theta' - \sin.\,\beta' = 2\cos.\tfrac{1}{2}(\theta'+\beta')\sin.\tfrac{1}{2}(\theta'-\beta')$$

Hence, it follows that,

$$\sin.\,\theta - \sin.\,(\theta-\beta) = 2\cos.\,(\theta-\tfrac{1}{2}\beta)\sin.\tfrac{1}{2}\beta$$

Besides which we have,

$$1-\cos.\,\beta = 2\sin.^2\tfrac{1}{2}\beta$$

Consequently the preceding expression becomes,

$$S = \sin.\,\theta + \sin.\,(\theta+\beta) + \sin.\,(\theta+2\beta) + \sin.\,(\theta+3\beta)\ \&c.\ \textit{ad infinitum}, = \frac{\cos.\,(\theta-\frac{1}{2}\beta)}{2\sin.\frac{1}{2}\beta}\ \text{- - - - - - - -}\ (z2)$$

To find the sum of $n+1$ terms of this series, we have simply to consider that the sum of the terms past the $(n+1)$th, that is the sum of

$\sin.\,(\theta+(n+1)\beta) + \sin.\,(\theta+(n+2)\beta) + \sin.(\theta+(n+3)\beta) +$ &c. *ad infinitum*, is by the preceeding theorem,

$$= \frac{\cos.\,(\theta+(n+\frac{1}{2})\beta)}{2\sin.\frac{1}{2}\beta}$$

Deducting this from the former expression, there will remain

$$\sin.\,\theta + \sin.\,(\theta+\beta) + \sin.\,(\theta+2\beta) + \sin.\,(\theta+3\beta) + \text{- - - -}$$
$$\text{- - - -}\ \sin.\,(\theta+n\beta) = \frac{\cos.\,(\theta-\frac{1}{2}\beta) - \cos.\,(\theta+(n+\frac{1}{2})\beta)}{2\sin.\frac{1}{2}\beta}$$
$$= \frac{\sin.\,(\theta+\frac{1}{2}n\beta)\sin.\frac{1}{2}(n+1)\beta}{\sin.\frac{1}{2}\beta}\ \text{-}\ (z3)$$

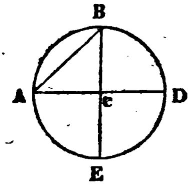
By like means it may be found, that the sum of the cosines of arcs or angles in arithmetical progression, is $\cos.\,\theta + \cos.\,(\theta+\beta) + \cos.\,(\theta+2\beta) + \cos.\,(\theta+3\beta) +$ &c. *ad infinitum*,

$$= -\frac{\sin.(\theta+\frac{1}{2}\beta)}{2\sin.\frac{1}{2}\beta}\ \text{- - - -}\ (z4)$$

Also,

$$\cos.\,\theta + \cos.\,(\theta+\beta) + \cos.\,(\theta+2\beta) + \cos.\,(\theta+3\beta) + \text{- - - -}$$
$$\text{- - -}\ (\cos.\,\theta+n\beta) = \frac{\cos.\,(\theta+\frac{1}{2}\beta(\sin.\frac{1}{2}(n+1)\beta}{\sin.\frac{1}{2}\beta}\ \text{- -}\ (z5)$$

*To find the numerical value of the sine, cosine, &c., of 45°.*



In the circle ABD, draw CA, CB, radii at right angles; join AB.

Then by Definition (12)

$$\text{Chord ABC }(90°) = \frac{AB}{AC}$$

$$\text{Chord}^2\ 90° = \frac{AB^2}{AC^2}$$

$$= \frac{AC^2+BC^2}{AC^2}$$

$$=\frac{2\,AC^2}{AC^2} \quad \because BC=AC$$

$$= 2 \quad - - - - - - - - - - - (1)$$

Now, the chord of an arc is equal to twice the sine of half the arc; therefore,

$$2 \sin. 45^\circ = \text{chord } 90^\circ$$

$$4 \sin.^2 45^\circ = \text{chord}^2\, 90^\circ$$

$$= 2, \text{ by Equation (1);}$$

$$\therefore \quad \sin. 45^\circ = \frac{1}{\sqrt{2}}$$

Again, by table I.:

$$\sin.^2 \theta + \cos.^2 \theta = 1$$

$$\therefore \quad \cos.^2 45^\circ = 1 - \sin.^2 45^\circ$$

$$= 1 - \frac{1}{2}$$

$$= \frac{1}{2}$$

$$\therefore \quad \cos. 45^\circ = \frac{1}{\sqrt{2}} = \sin. 45^\circ.$$

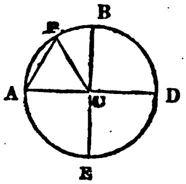
Also,

$$\tan. 45^\circ = \frac{\sin. 45^\circ}{\cos. 45^\circ}$$

$$= 1 = \cot. 45^\circ.$$

*To find the numerical value of the sine, cosine, &c., of* 30°.

In the circle ABD, draw CP, making with CA the angle ACP=60°; join A, P.



Now,

$$2 \sin. 30^\circ = \text{chord } 60^\circ$$

$$= \frac{AP}{AC}$$

$$= \frac{AC}{AC} \quad \because AP = AC, \because \text{the triangle APC is equiangular, and therefore equilateral.}$$

$$= 1$$

$$\therefore \quad \sin. 30^\circ = \frac{1}{2}$$

Again,

$$\cos. 30^\circ = \sqrt{1 - \sin.^2 30^\circ}$$

$$= \sqrt{1 - \tfrac{1}{4}} = \frac{\sqrt{3}}{2} = \frac{1}{2}\sqrt{3}$$

Also,

$$\tan. 30^\circ = \frac{\sin. 30^\circ}{\cos. 30^\circ}$$
$$= \frac{1}{\sqrt{3}}$$
$$\cot. 30^\circ = \frac{1}{\tan. 30^\circ} = \sqrt{3}$$

*To find the numerical value of the sine, cosine, &c., of* 60°.

$$\sin. 60^\circ = \cos. (90^\circ - 60^\circ)$$
$$= \cos. 30^\circ$$
$$= \frac{\sqrt{3}}{2}, \text{ by last art.}$$

Again,

$$\cos. 60^\circ = \sin. (90^\circ - 60^\circ)$$
$$= \sin. 30^\circ$$
$$= \frac{1}{2}$$

Also,

$$\tan. 60^\circ = \sqrt{3}$$
$$\cot. 60^\circ = \frac{1}{\sqrt{3}}$$

It is required to find the sum of all the natural sines to every minute in the quadrant, radius $=1$. In this problem, the actual addition of all the terms would be a very tiresome labor, but the solution by means of formula (z3), is rendered very easy.

Applying that formula we have $\sin. (\theta + \frac{1}{2} n\beta) = \sin. 45^\circ$, $\sin. \frac{1}{2}(n+1)\beta = \sin. 45^\circ, 0', 30''$ and $\sin. \frac{1}{2}\beta = \sin. 30'$,
$\frac{\sin. 45^\circ \sin. 45^\circ\ 0'\ 30''}{\sin. 39''} = 3438.2467465$, the same sum required.

Let it be required to find the sum of the sines to every minute of the arc of 60°.

Here the numerical expression in the equation would become
$$\frac{\sin. 30^\circ \times \sin. 30^\circ\ 0'\ 30''}{\sin. 30''} = .5 \times .500126 \div .00014545953$$
$= 1719.123373.25$ equal the sum of all the natural sines to every minute of the arc of 60°.

It may be useful to exhibit the most useful results in this chapter, in the following table.

## TABLE III.

(1.) $\sin.\ (\theta\pm\beta) = \sin.\ \theta\cos.\ \beta\pm\sin.\ \beta\cos.\ \theta$

(2.) $\cos.\ (\theta\pm\beta) = \cos.\ \theta\cos.\ \beta\mp\sin.\ \theta\sin.\ \beta$

(3.) $\tan.\ (\theta\pm\beta) \quad \dfrac{\tan.\ \theta\pm\tan.\ \beta}{1\mp\tan.\ \theta\tan.\ \beta}$

(4.) $\sin.\ 2\ \theta = 2\sin.\ \theta\cos.\ \theta$

(5.) $\cos.\ 2 = \cos.^2\theta-\sin.^2\theta = 2\cos.^2\theta-1 = 1-2\sin.^2$

(6.) $\tan.\ 2\ \theta = \dfrac{2\tan.\ \theta}{1-\tan.^2\theta}$

(7.) $\sin.\ \dfrac{\theta}{2} = \sqrt{\dfrac{1-\cos.\ \theta}{2}}$

(8.) $\cos.\ \dfrac{\theta}{2} = \sqrt{\dfrac{1+\cos.\ \theta}{2}}$

(9.) $\tan.\ \dfrac{\theta}{2} = \sqrt{\dfrac{1-\cos.\ \theta}{1+\cos.\ \theta}} = \dfrac{1-\cos.\ \theta}{\sin.\ \theta} = \dfrac{\sin.\ \theta}{1+\cos.\ \theta}$

(10.) $\sin.\ \theta = 2\sin.\ \dfrac{\theta}{2}\cos.\ \dfrac{\theta}{2}$

(11.) $\sin.\ 3\ \theta = 3\sin.\ \theta-4\sin.^3\theta$

(12.) $\cos.\ 3\ \theta = 4\cos.^3\theta-3\cos.\ \theta$

(13.) $\sin.\ (n+1)\ \theta = 2\sin.\ n\theta\cos.\ \theta-\sin.\ (n-1)\ \theta$

(14.) $\cos.\ (n+1)\ \theta = 2\cos.\ n\theta\cos.\ \theta-\cos.\ (n-1)\ \theta$

(15.) $\sin.\ \theta+\sin.\ \beta = 2\sin.\ \dfrac{\theta+\beta}{2}\cos.\ \dfrac{\theta-\beta}{2}$

(16.) $\sin.\ \theta-\sin.\ \beta = 2\sin.\ \dfrac{\theta-\beta}{2}\cos.\ \dfrac{\theta+\beta}{2}$

(17.) $\cos.\ \theta+\cos.\ \beta = 2\cos.\ \dfrac{\theta+\beta}{2}\cos.\ \dfrac{\theta-\beta}{2}$

(18) $\cos.\ \theta-\cos.\ \beta = -2\sin.\ \dfrac{\theta+\beta}{2}\sin.\ \dfrac{\theta-\beta}{2}$

(19.) $\dfrac{\sin.\ \theta+\sin.\ \beta}{\sin.\ \theta-\sin.\ \beta} = \dfrac{\tan.\ \dfrac{\theta+\beta}{2}}{\tan.\ \dfrac{\theta-\beta}{2}}$

(20.) $\sin.\ (\theta+\beta)+\sin.\ (\theta-\beta) = 2\sin.\ \theta\cos.\ \beta$

(21.) $\sin.\ (\theta+\beta)-\sin.\ (\theta-\beta) = 2\sin.\ \beta\cos.\ \theta$

(22.) $\cos.\ (\theta+\beta)+\cos.\ (\theta-\beta) = 2\cos.\ \theta\cos.\ \beta$

(23.) $\cos.\,(\theta+\beta)-\cos.\,(\theta-\beta)=-2\sin.\,\theta\sin.\,\beta$

(24.) $\sin.\,(\theta+\beta)\cos.\,(\theta-\beta)=\sin.^2\theta-\sin.^2\beta=\cos.^2\beta-\cos.^2\theta$

(25.) $\cos.\,(\theta+\beta)\cos.\,(\theta-\beta)=\cos.^2\theta-\sin.^2\beta=\cos.^2\theta+\cos.^2\beta-1$

(26.) $\sin.\,\theta+\sin.\,(\theta+\beta)+\sin.\,(\theta+2\beta)+\sin.\,(\theta+3\beta)+\cdots\cdots\sin.\,(\theta+n\beta)=\dfrac{\sin.\,(\theta+\frac{1}{2}n\beta)\sin.\,\frac{1}{2}(n+1)\beta}{\sin.\,\frac{1}{2}\beta}$

(27.) $\cos.\,\theta+\cos.\,(\theta+\beta)+\cos.\,(\theta+2\beta)+\cos.\,(\theta+3\beta)+\cdots\cdots\cos.\,(\theta+n\beta)=\dfrac{\cos.\,(\theta+\frac{1}{2}\beta)\sin.\,\frac{1}{2}(n+1)\beta}{\sin.\,\frac{1}{2}\beta}$

(28.) $\sin.\,45^\circ=\cos.\,45^\circ=\dfrac{1}{\sqrt{2}}$

(29.) $\tan.\,45^\circ=\cot.\,45^\circ=1$

(30.) $\sin.\,30^\circ=\cos.\,60^\circ=\dfrac{1}{2}$

(31.) $\cos.\,30^\circ=\cos.\,60^\circ=\dfrac{\sqrt{3}}{2}$

(32.) $\tan.\,30^\circ=\cot.\,60^\circ=\dfrac{1}{\sqrt{3}}$

(33.) $\cot.\,30^\circ=\tan.\,60^\circ=\sqrt{3}$

The formulæ of Trigonometry may be multiplied to almost any extent, and the same quantity may be expressed in a vast number of different ways. An intimate acquaintance with those given in the above table is *essential* to the progress of the student.

The following, although of less frequent occurrence, may occasionally be found useful, and can be readily deduced from the above.

(34.) $\left.\begin{matrix}\sin.\,(45^\circ\pm\theta)\\ \cos.\,(45^\circ\mp\theta)\end{matrix}\right\}=\dfrac{\cos.\,\theta\pm\sin.\,\theta}{\sqrt{2}}$

(35.) $\tan.\,(45^\circ\pm\theta)=\dfrac{1\pm\tan.\,\theta}{1\mp\tan.\,\theta}$

(36.) $\tan.^2\left(45^\circ\pm\dfrac{\theta}{2}\right)=\dfrac{1\pm\sin.\,\theta}{1\mp\sin.\,\theta}$

(37.) $\tan.\left(45^\circ\pm\dfrac{\theta}{2}\right)=\dfrac{1\pm\sin.\,\theta}{\cos.\,\theta}=\dfrac{\cos.}{1+\sin.\,\theta}$

(38.) $\dfrac{\sin.\,(\theta+\beta)}{\sin.\,(\theta-\beta)}=\dfrac{\tan.\,\theta+\tan.\,\beta}{\tan.\,\theta-\tan.\,\beta}=\dfrac{\cot.\,\beta\mp\cot.\,\theta}{\cot.\,\beta-\cot.\,\theta}$

$$(39.)\ \frac{\cos.\ (\theta+\beta)}{\cos.\ (\theta-\beta)} = \frac{\cot.\ \beta-\tan.\ \theta}{\cot.\ \beta+\tan.\ \theta} = \frac{\cot.\ \theta-\tan.\ \beta}{\cot.\ \theta+\tan.\ \beta}$$

$$(40.)\ \frac{\sin.\ \theta+\sin.\ \beta}{\cos.\ \theta+\cos.\ \beta} = \tan.\ \frac{\theta+\beta}{2}$$

$$(41.)\ \frac{\sin.\ \theta+\sin.\ \beta}{\cot.\ \theta-\cos.\ \beta} = -\cot.\ \frac{\theta-\beta}{2}$$

$$(42.)\ \frac{\sin.\ \theta-\sin.\ \beta}{\cos.\ \theta+\cos.\ \beta} = \tan.\ \frac{\theta-\beta}{2}$$

$$(43.)\ \frac{\sin.\ \theta-\sin.\ \beta}{\cos.\ \theta-\cos.\ \beta} = -\cot.\ \frac{\theta+\beta}{2}$$

$$(44.)\ \frac{\cos.\ \theta+\cos.\ \beta}{\cos.\ \theta-\cos.\ \beta} = -\cot.\ \frac{\theta+\beta}{2}\cot.\ \frac{\theta-\beta}{2}$$

$$(45.)\ \tan.\ \theta+\tan.\ \beta = \frac{\sin.\ (\theta+\beta)}{\cos.\ \theta\cos.\ \beta}$$

$$(46.)\ \cot.\ \theta+\cot.\ \beta = \frac{\sin.\ (\theta+\beta)}{\sin.\ \theta\sin.\ \beta}$$

$$(47.)\ \tan.\ \theta-\tan.\ \beta = \frac{\sin.\ (\theta-\beta)}{\cos.\ \theta\cos.\ \beta}$$

$$(48.)\ \cot.\ \theta-\cot.\ \beta = -\frac{\sin.\ (\theta-\beta)}{\sin.\ \theta\sin.\ \beta}$$

$$(49.)\ \tan.^2\ \theta-\tan.^2\ \beta = \frac{\sin.\ (\theta+\beta)\sin.\ (\theta-\beta)}{\cos.^2\ \theta\cos.^2\ \beta}$$

$$(50.)\ \cot.^2\ \theta-\cot.^2\ \beta = -\frac{\sin.\ (\theta+\beta)\sin.\ (\theta-\beta)}{\sin.^2\ \theta\sin.^2\ \beta}$$

In order to become familiar with the various combinations, and dexterous in the application of these expressions, the student will do well to exercise himself by verifying the following values of Sin. $\theta$, Cos $\theta$, Tan. $\theta$, which are extracted from the large work of Cagnoli.

## TABLE OF THE MOST USEFUL ANALYTICAL VALUES OF SIN. $\theta$, COS. $\theta$, TAN. $\theta$.

| VALUES OF SIN. $\theta$. | VALUES OF COS. $\theta$. |
|---|---|
| 1. $\cos.\theta\,\tan.\theta$ | 16. $\frac{\sin.\theta}{\tan.\theta}$ |
| 2. $\frac{\cos.\theta}{\cot.\theta}$ | 17. $\sin.\theta\,\cot.\theta$ |
| 3. $\sqrt{1-\cos.^2\theta}$ | 18. $\sqrt{1-\sin.^2\theta}$ |
| 4. $\frac{1}{\sqrt{1+\cot.^2\theta}}$ | 19. $\frac{1}{\sqrt{1+\tan.^2\theta}}$ |
| 5. $\frac{\tan.\theta}{\sqrt{1+\tan.^2\theta}}$ | 20. $\frac{\cot.\theta}{\sqrt{1+\cot.^2\theta}}$ |
| 6. $2\sin.\frac{\theta}{2}\cos.\frac{\theta}{2}$ | 21. $\cos.^2\frac{\theta}{2}-\sin.^2\frac{\theta}{2}$ |
| 7. $\frac{\sqrt{1-\cos.2\theta}}{2}$ | 22. $1-2\sin.^2\frac{\theta}{2}$ |
| 8. $\frac{2\tan.\frac{\theta}{2}}{1+\tan.^2\frac{\theta}{2}}$ | 23. $2\cos.^2\frac{\theta}{2}-1$ |
| 9. $\frac{2}{\cot.\frac{\theta}{2}+\tan.\frac{\theta}{2}}$ | 24. $\sqrt{\frac{1+\cos.2\theta}{2}}$ |
| 10. $\frac{\sin.(30^\circ+\theta)-(\sin.30^\circ-\theta)}{\sqrt{3}}$ | 25. $\frac{1-\tan.^2\frac{\theta}{2}}{1+\tan.^2\frac{\theta}{2}}$ |
| 11. $2\sin.^2(45^\circ+\frac{\theta}{2})-1$ | 26. $\frac{\cot.\frac{\theta}{2}-\tan.\frac{\theta}{2}}{\cot.\frac{\theta}{2}+\tan.\frac{\theta}{2}}$ |
| 12. $1-2\sin.^2(45^\circ-\frac{\theta}{2})$ | 27. $\frac{1}{1+\tan.\theta\,\tan.\frac{\theta}{2}}$ |
| 13. $\frac{1-\tan.^2(45^\circ-\frac{\theta}{2})}{1+\tan.^2(45^\circ-\frac{\theta}{2})}$ | 28. $\frac{2}{\tan.(45^\circ+\frac{\theta}{2})+\cot.(45^\circ+\frac{\theta}{2})}$ |
| 14. $\frac{\tan.(45^\circ+\frac{\theta}{2})-\tan.(45^\circ-\frac{\theta}{2})}{\tan.(45^\circ+\frac{\theta}{2})+\tan.(45^\circ-\frac{\theta}{2})}$ | 29. $2\cos.(45^\circ+\frac{\theta}{2})\cos.(45^\circ-\frac{\theta}{2})$ |
| 15. $\sin.(60^\circ+\theta)-\sin.(60^\circ-\theta)$ | 30. $\cos.(60^\circ+\theta)+\cos.(60^\circ-\theta)$ |

VALUES OF TAN. $\theta$.

31. $\dfrac{\sin.\theta}{\cos.\theta}$

32. $\dfrac{1}{\cot.\theta}$

33. $\sqrt{\dfrac{1}{\cos.^2\theta}-1}$

34. $\dfrac{\sin.\theta}{\sqrt{1-\sin.^2\theta}}$

35. $\dfrac{\sqrt{1-\cos.^2\theta}}{\cos.\theta}$

36. $\dfrac{2\tan.\frac{\theta}{2}}{1-\tan.^2\frac{\theta}{2}}$

37. $\dfrac{2\cot.\frac{\theta}{2}}{\cot.^2\frac{\theta}{2}-1}$

38. $\dfrac{2}{\cot.\frac{\theta}{2}-\tan.\frac{\theta}{2}}$

39. $\cot.\theta-2\cot.2$

40. $\dfrac{1-\cos.2\theta}{\sin.2\theta}$

41. $\dfrac{\sin.2\theta}{1+\cos.2\theta}$

42. $\sqrt{\dfrac{1-\cos.2\theta}{1+\cos.2\theta}}$

43. $\dfrac{\tan.(45^\circ+\frac{\theta}{2})-\tan.(45^\circ-\frac{\theta}{2})}{2}$

From certain properties of the circles to be discussed in another volume, other important trigonometrical formulæ, may be deduced, furnishing us with more expeditious means of determining, numerically, the values of some of the trigonometrical lines, and ratios, all of which will occur in their order.

*To develop sin.* x *and cos.* x *in a series ascending by the powers of* x.

The series for sin. $x$ must vanish when $x=0$, and therefore no term in the series can be independent of $x$, nor can the even powers of $x$ occur in the series; for if we suppose

$$\sin.x= a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+\dots.$$

$$\text{then }\sin.(-x)=-a_1x+a_2x^2-a_3x^3+a_4x^4-a_5x^5+\dots$$

$$\text{but }\sin.(-x)=-\sin.x$$

$$=-a_1x-a_2x^2-a_3x^3-a_4x^4-a_5x^5-\dots$$

$$\therefore\ a_2=-a_2,\ a_4=-a_4,\dots.;\ \text{hence } a_2=0,\ a_4=0\dots.$$

$$\therefore \sin.x= a_1x+a_3x^3+a_5x^5+a_7x^7+\dots\dots \quad (1)$$

Again, the series for cos. $x$ must $= 1$ when $x = 0$, and therefore the series must contain a term independent of $x$, and it must be 1; also the series can contain no odd powers of $x$, for if we suppose

$$\cos.\ x = 1 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots\ldots$$

$$\text{then } \cos.\ (-x) = 1 - a_1x + a_2x^2 - a_3x^3 + a_4x^4 - \ldots\ldots$$

$$\text{but } \cos.\ (-x) = \cos.\ x$$

$$= 1 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots\ldots$$

$$\therefore\ a_1 = -a_1,\ a_3 = -a_3,\ \ldots\quad \therefore a_1 = 0,\ a_3 = 0\ \ldots$$

$$\therefore\ \cos.\ x = 1 + a_2x^2 + a_4x^4 + a_6x^6 + \quad - - - - - \quad (2)$$

$$\text{Hence } \cos.\ x + \sin.\ x = 1 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 \quad - - \quad (3)$$

$$\cos.\ x - \sin.\ x = 1 - a_1x + a_2x^2 - a_3x^3 + a_4x^4 - a_5x^5 + \quad - - \quad (4)$$

Now in equation (3) write $x + h$ for $x$, and we have

$$\cos.(x+h) + \sin(x+h) = 1 + a_1(x+h) + a_2(x+h)^2 + a_3(x+h)^3 + \quad (5)$$

$$\text{but } \cos.\ (x+h) + \sin.\ (x+h) = \cos.\ x \cos.\ h - \sin.\ x \sin.\ h + \sin.\ x \cos.\ h + \cos.\ x \sin.\ h$$

$$= \cos.\ h\ (\cos.\ x + \sin.\ x) + \sin.\ h\ (\cos.\ x - \sin.\ x)$$

$$= (1 + a_2h^2 + a_4h^4 + \ldots).(1 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots)$$

$$+ \ (a_1h + a_3h^3 + a_5h^5 + \ldots)(1 - a_1x + a_2x^2 - a_3x^3 + \ldots)$$

$$\left.\begin{array}{r} = 1 + a_1x + a_2x^2 + a_3x^3 \quad + \ldots \\ + a_1h - a_1^2xh + a_1a_2x^2h + \ldots \\ + a_2h^2 + a_1a_2xh^2 \quad + \ldots \\ + a_3h^3 \quad + \ldots \\ + \ldots \end{array}\right\} \quad - - - \quad (6)$$

Comparing equations (5) and (6) we have

$$\left.\begin{array}{r} 1 + a_1x + a_2x^2 \ + a_3x^3 \ + \\ + a_1h + 2a_2xh + 3a_3x^2h + \\ + \ a_2h^2 + 3a_3xh^2 + \\ + \ a_3h^3 \ + \\ + \end{array}\right\} = \left\{\begin{array}{r} 1 + a_1x + a_2x^2 \quad + a_3x^3 \quad + \\ + a_1h - a_1a_1xh + a_1a_2x^2h - \\ + a_2h^2 \quad + a_1a_2xh^2 + \\ + \ a_3h^3 \quad - \\ + \end{array}\right.$$

and equating the coefficients of the terms involving the same powers of $x$ and $h$, we have

$$2a_2 = -a_1a_1; \text{ therefore } a_2 = -\frac{a_1a_1}{2} = -\frac{a_1^2}{1.2}$$

$$3a_3 = \ a_1a_2 \ .\ .\ .\ .\ a_3 = \frac{a_1a_2}{3} = -\frac{a_1^3}{1.2.3}$$

$$4a_4 = -a_1a_3 \ .\ .\ .\ .\ a_4 = -\frac{a_1a_3}{4} = +\frac{a_1^4}{1.2.3.4}$$

$$5a_5 = \ a_1a_4 \ .\ .\ .\ .\ a^5 = \frac{a_1a_4}{5} = +\frac{a_1^5}{1.2.3.4.5}$$

$$\text{hence sin. } x=a_1x-\frac{a_1^3}{1.2.3}x^3+\frac{a_1^5}{1.2.3.4.5}x^5-\frac{a_1^7}{1.2.3.4.5.6.7}x^7+$$

$$\text{cos. } x=1\quad-\frac{a_1^2}{1.2}x^2+\frac{a_1^4}{1.2.3.4}x^4-\frac{a_1^6}{1.2.3.4.5.6}x^6+\ldots$$

and we have only to determine the value of $a_1$. To effect this, we have

$$\text{sin. } x=a_1x-\frac{a_1^3}{1.2.3}x^3+\frac{a_1^5}{1.2.3.4.5}x^5-\ldots$$

$$=a_1x\left(1-\frac{a_1^2}{1.2.3}x^2+\frac{a_1^4}{1.2.3.4.5}x^4-\ldots\right.$$

Now the value of $x$ may be assumed so small that the series in the parenthesis, and sin. $x$, shall differ from 1 and $x$ respectively, by less than any assignable quantities; hence ultimately

$x=a_1x$, and therefore $a_1=1$; whence

$$\text{sin. } x=x-\frac{x^3}{1.2.3}+\frac{x^5}{1.2.3.4.5}-\frac{x^7}{1.2.3.4.5.6.7}+\ldots$$

$$\text{cos. } x=1-\frac{x^2}{1.2}+\frac{x^4}{1.2.3.4}-\frac{x^6}{1.2.3.4.5.6}+\ldots.$$

*To develop tan.* x *and cot.* x *in a series ascending by the powers of* x.

The development may be obtained from those of sin. $x$ and cos. $x$, already found.

$$\text{tan. } x=\frac{\text{sin. } x}{\text{cos. } x}=\frac{x-\dfrac{x^3}{1.2.3}+\dfrac{x^5}{1.2.3.4.5.}-\&c.}{1-\dfrac{x^2}{1.2}+\dfrac{x^4}{1.2.3.4}-\&c.}$$

and the series will therefore be of the form

$$x+a_3x^3+a_5x^5+a_7x^7+\ldots.$$

$$\text{Hence, let } x+a_3x^3+a_5x^5+\ldots=\frac{x-\dfrac{x^3}{1.2.3}+\dfrac{x^5}{1.2.3.4.5}-\ldots}{1-\dfrac{x^2}{1.2}+\dfrac{x^4}{1.2.3.4}-\ldots}$$

$$\therefore\ x-\frac{x^3}{1.2.3}+\frac{x^5}{1.2.3.4.5}-\ldots.$$

$$\ldots=\left(1-\frac{x^2}{1.2}+\frac{x^4}{1.2.3.4}-\ldots\right)\left(x+a_3x^3+a_5x^5+\ldots\right)$$

$$\begin{array}{rl|rl|l} =x+ & a_3 & x^3+ & a_5 & x^5+ \\ & -\frac{1}{1.2} & & -\frac{a_3}{1.2} & - \\ & & & +\frac{1}{1.2.3.4} & + \\ & & & & - \end{array}$$

Hence, equating the coefficients of the like terms, we have

$$a_3 - \frac{1}{1.2} = - \frac{1}{1.2.3} \qquad \therefore a_3 = \frac{2}{1.2.3}$$

$$a_5 - \frac{a_3}{1.2} + \frac{1}{1.2.3.4} = \frac{1}{1.2.3.4.5} \qquad \therefore a_5 = \frac{2^4}{1.2.3.4.5} \text{ \&c.}$$

$$\therefore \tan. x = x + \frac{2x^3}{1.2.3} + \frac{2^4x^5}{1.2.3.4.5} + \dots\dots$$

$$\text{Sim. } \cot. x = \frac{1}{x} - \frac{2x}{1.2.3} - \frac{2^4x^3}{1.2.3.4.5} - \dots\dots$$

## CHAPTER III.

### FORMULÆ FOR THE SOLUTION OF TRIANGLES.

We shall here repeat the enunciations of the two propositions established in Chapter I.

### PROPOSITION I.

*In any right-angled plane triangle,*

1°. *The ratio which the side opposite to one of the acute angles has to the hypothenuse, is the sine of that angle.*

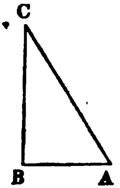
2°. *The ratio which the side adjacent to one of the acute angles has to the hypothenuse, is the cosine of that angle.*

3°. *The ratio which the side opposite to one of the acute angles has to the side adjacent to that angle, is the tangent of that angle.*

Thus, in any right-angled triangle ABC,

$$\frac{CB}{CA} = \text{sin. } A, \quad \frac{BA}{AC} = \text{cos. } A, \quad \frac{CB}{BA} = \text{tan. } A$$

$$= \text{cos. } C, \qquad = \text{sin. } C, \qquad = \text{cot. } C$$



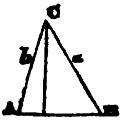
$$\text{Or,} \quad \left.\begin{array}{l} CB = AC \text{ sin. } A \\ \quad\; = AC \text{ cos. } C \\[1ex] BA = AC \text{ cos. } A \\ \quad\; = AC \text{ sin. } C \\[1ex] CB = BA \text{ tan. } A \\ \quad\; = BA \text{ cot. } C \end{array}\right\} \cdots\cdots\cdots\cdots (\alpha)$$

PROPOSITION II.

*In any plane triangle, the sides are to each other as the sines of the angles opposite to them.*

We shall, frequently in treating of triangles, make use of the following notation; denoting the angles of the triangle by the large letters at the angular points, and the sides of the triangle opposite to these angles, by the corresponding small letters.

Thus, in the triangle ABC, we shall denote the angles, BAC, CBA, BCA, by the letters, A, B, C, respectively, and the sides BC, AC, AB, by the letters *a*, *b*, *c*, respectively.



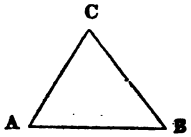
According to this, we shall have, by the proposition,

$$\left.\begin{array}{l} \dfrac{a}{b} = \dfrac{\text{sin. } A}{\text{sin. } B} \\[2ex] \dfrac{a}{c} = \dfrac{\text{sin. } A}{\text{sin. } C} \\[2ex] \dfrac{b}{c} = \dfrac{\text{sin. } B}{\text{sin, } C} \end{array}\right\} \cdots\cdots\cdots\cdots (\beta)$$

### PROPOSITION III.

*In any plane triangle, the sum of any two sides, is to their difference, as the tangent of half the sum of the angles opposite to them, is to the tangent of half their difference.*

Let ABC be any plane triangle, then, by Proposition II



$$\frac{a}{b}=\frac{\text{sin. A}}{\text{sin. B}}$$

$$\frac{a+b}{a-b}=\frac{\text{sin. A}+\text{sin. B}}{\text{sin. A}-\text{sin. B}}$$

But, by Trigonometry, Chap. II. (*r*)

$$\text{sin. A}+\text{sin. B} = 2\ \text{sin.}\frac{A+B}{2}\ \text{cos.}\frac{A-B}{2}$$

$$\text{sin. A}-\text{sin. B} = 2\ \text{cos.}\frac{A+B}{2}\ \text{sin.}\frac{A-B}{2}$$

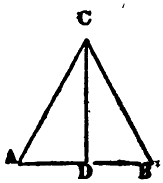
$$\therefore\ \frac{a+b}{a+b}=\frac{2\ \text{sin.}\frac{A+B}{2}\ \text{cos.}\frac{A-B}{2}}{\text{cos.}\frac{A+B}{2}\ \text{sin.}\frac{A-B}{2}}$$

$$= \text{tan.}\frac{A+B}{2}\ \text{cot.}\frac{A-B}{2}$$

$$\left.\begin{array}{l} = \dfrac{\text{tan.}\frac{A+B}{2}}{\text{tan.}\frac{A-B}{2}} \\ \text{And in like manner,} \\ \dfrac{a+c}{a-c}=\dfrac{\text{tan.}\frac{A+C}{2}}{\text{tan.}\frac{A-C}{2}} \\ \dfrac{b+c}{b-c}=\dfrac{\text{tan.}\frac{B+C}{2}}{\text{tan.}\frac{B-C}{2}} \end{array}\right\}\ \cdots\cdots\ (\gamma)$$

### PROPOSITION IV.

*To express the cosine of an angle of a plane triangle in terms of the sides of the triangle.*



Let ABC be a triangle; A, B, C, the three angles; $a, b, c$, the corresponding sides.

1. Let the proposed (A) be acute.

From C draw CD perpendicular to AB, the base of the triangle.

Then,

$$BC^2 = AC^2 + AB^2 - 2AB.\ AD$$ (Prop. XXVI. B. IV. *El. Geom.*)

Or,

$$a^2 = b^2 + c^2 - 2c\ .\ AD$$

But, since CDA is a right-angled triangle,

$$AD = AC \text{ cos. } CAD = b \text{ cos. } A$$

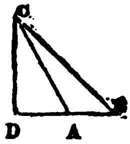
$$\therefore\ a^2 = b^2 + c^2 - 2bc \text{ cos. } A$$

$$\therefore \text{ cos. } A = \frac{b^2 + c^2 - a^2}{2bc}$$

which is the expression required.

2 Let the proposed angle (A) be abtuse.

From C, draw CD perpendicular to AB produced.



Then,

$$BC^2 = AC^2 + AB^2 + 2AB\,.\,AD$$

Or,

$$a^2 = b^2 + c^2 + 2c\ .\ AD$$

But, since CDA is a right angled triangle,

$$AD = AC \text{ cos. } CAD$$

$$= AC \times - \text{cos. } CAB$$

$\because$ CAB is the supplement of CAD.

$$= - b \text{ cos } A$$

$$\therefore\ a^2 = b^2 + c^2 - 2bc \text{ cos. } A$$

$$\therefore \text{ cos. } A = \frac{b^2 + c^2 - a^2}{2bc}$$

It will be seen that this result is identical with that which we deduced in the last case, so that, whether A be acute or obtuse, we shall have,

$$\left.\begin{array}{l} \cos. A = \dfrac{b^2+c^2-a^2}{2bc} \\ \text{Proceeding in the same manner} \\ \text{for the other angles, we shall} \\ \text{find,} \\ \cos. B = \dfrac{a^2+c^2-b^2}{2ac} \\ \cos. C = \dfrac{a^2+b^2-c^2}{2ab} \end{array}\right\} \cdots\cdots\cdots (\delta)$$

### PROPOSITION V.

*To express the sine of an angle of a plane triangle in terms of the sides of the triangle.*

Let A be the proposed angle ; then by last prop.,

$$\cos. A = \frac{b^2+c^2-a^2}{2\,b\,c}$$

Adding unity to each member of the equation,

$$\begin{aligned} 1+\cos. A &= 1+\frac{b^2+c^2-a^2}{2\,b\,c} \\ &= \frac{b^2+2bc+c^2-a^2}{2\,b\,c} \\ &= \frac{(b+c)^2-a^2}{2\,b\,c} \\ &= \frac{(b+c+a)\,(b+c-a)}{2\,b\,c} \quad \cdots\cdots\cdots \quad (1) \end{aligned}$$

$$\text{Again, } \cos. A = \frac{b^2+c^2-a^2}{2\,b\,c}$$

Substracting each member of the equation from unity,

$$\begin{aligned} 1-\cos. A &= 1-\frac{b^2+c^2-a^2}{2bc} \\ &= \frac{2bc-b^2-c^2+a^2}{2bc} \\ &= \frac{a^2-(b^2-2bc+c^2)}{2bc} \\ &= \frac{a^2-(b-c)^2}{2bc} \\ &= \frac{(a+b-c)\,(a+c-b)}{2bc} \quad \cdots\cdots\cdots \quad (2) \end{aligned}$$

Multiplying together equations (1) and (2,)

$$(1+\cos A).(1-\cos A)=\frac{(a+b+c)\ (b+c-a)\ (a+c-b)\ (a+b-c)}{4b^2\,c^2}$$

But $(1+\cos.\ A)\ (1-\cos.\ A)=\ 1-\cos.^2 A$

$$= \sin.^2 A \quad \text{(Table I.)}$$

$$\therefore\ \sin.^2 A\ =\frac{(a+b+c)\ (b+c-a)\ (a+c-b)\ (a+b-c)}{4b^2c^2}$$

Extracting the root on both sides,

$$\sin.\ A = \frac{1}{2bc}\sqrt{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}..(3$$

The above expresison, for the sine of an angle of a triangle in terms of the sides, is sometimes exhibited under a form some what different.

Let $s$ denote the semiperimeter, that is to say, half the sum of the sides of the triangle ; then

$$s = \frac{a+b+c}{2}, \text{ and, } \quad 2s = a+b+c$$

$$s-a = \frac{b+c-a}{2} \quad \ldots\ 2\ (s-a) = b+c-a$$

$$s-b = \frac{a+c-b}{2} \quad \ldots\ 2\ (s-b) = a+c-b$$

$$s-c = \frac{a+b-c}{2} \quad \ldots\ 2\ (s-c) = a+b-c$$

Substituting $2\ s, 2\ (s-a), \ldots\ldots$ for $a+b+c, b+c-a, \ldots\ldots$ in the expression for $\sin.^2$ A, it becomes

$$\sin.^2 A = \frac{16\ s\ (s-a)\ (s-b)\ (s-c)}{4\ b^2c^2}$$

And extracting the root on both sides,

$$\sin.\ A = \frac{2}{bc}.\sqrt{s\ (s-a)\ (s-b)\ (s-c)}$$

Proceeding in the same manner for the other angles, we shall find . . . (4)

$$\sin.\ B = \frac{2}{ac}.\sqrt{s\ (s-a)\ (s-b)\ (s-c}$$

$$\sin.\ C = \frac{2}{ab}.\sqrt{s\ (s-a)\ (s-b)\ (s-c)}$$

By equation (1) we have

$$1+\cos. A = \frac{(a+b+c)\ (b+c-a)}{2\,b\,c}$$

$$= \frac{4\,s\,(s-a)}{2\,b\,c}$$

But, by Chap. II,

$$1+\cos. A = 2\cos.^2\frac{A}{2}$$

$$\therefore\quad 2\cos.^2\frac{A}{2} = \frac{4\,s\,(s-a)}{2\,bc}$$

Extracting the root on both sides,

$$\left.\begin{aligned} &\cos.\frac{A}{2} = \sqrt{\frac{s\,(s-a)}{bc}} \\ &\text{And in like manner,} \\ &\cos.\frac{B}{2} = \sqrt{\frac{s\,(s-b)}{ac}} \\ &\cos.\frac{C}{2} = \sqrt{\frac{s\,(s-s)}{a\,b}} \end{aligned}\right\} \cdots\cdots\cdots\cdots (\varepsilon)$$

By equation (2) we have

$$1-\cos. A = \frac{(a+c-b)\ (a+b-c)}{2bc}$$

$$= \frac{4\,(s-b)\ (s-c)}{2bc}$$

But, by Chap. II,

$$1-\cos. A = 2\sin.^2\frac{A}{2}$$

$$\therefore\quad 2\sin.^2\frac{A}{2} = \frac{4\,(s-b)\,(s-c)}{2bc}$$

Extracting the root on both sides,

$$\left.\begin{aligned} &\sin.\frac{A}{2} = \sqrt{\frac{(s-b)\ (s-c)}{bc}} \\ &\text{And in like manner,} \\ &\sin.\frac{B}{2} = \sqrt{\frac{(s-a)\ (s-c)}{ac}} \\ &\sin.\frac{C}{2} = \sqrt{\frac{(s-a)\ (s-b)}{ab}} \end{aligned}\right\} \cdots\cdots\cdots (\zeta)$$

Dividing the formulæ marked (ζ) by those marked (σ) we have

$$\left.\begin{array}{l}
\tan.\dfrac{A}{2}=\sqrt{\dfrac{(s-b)\,(s-c)}{s\,(s-a)}}\\
\tan.\dfrac{B}{2}=\sqrt{\dfrac{(s-a)\,(s-c)}{s\,(s-b)}}\\
\tan.\dfrac{C}{2}=\sqrt{\dfrac{(s-a)\,(s-b)}{s\,(s-c)}}
\end{array}\right\}\cdots\cdots\quad(\eta)$$

## CHAPTER IV.

### ON THE CONSTRUCTION OF TRIGONOMETRICAL TABLES.

Before proceeding to apply the formulæ deduced in the last chapter to the solution of triangles, we shall make a few remarks upon the construction of those tables, by means of which we are enabled to reduce our trigonometrical calculations to numerical results.

It is manifest, from definitions 1°, 2°, 3°, &c. that the various trigonometrical quantities, the sine, the cosine, the tangent, &c. are abstract numbers representing the comparative length of certain lines. We have already obtained the numerical value of these quantities in a few particular cases, and we shall now show how the numbers, corresponding to angles of every degree of magnitude, may be obtained by the application of the most simple principles.

The numbers corresponding to the sine, cosine, &c. of all angles from 1″ up to 90°, when arranged in a table, form what is called the *Trigonometrical canon.*

The first operation to be performed is

*To compute the numerical value of the sine and cosine of* 1′.

We have seen, Chap. II. formula ($j$) that

$$\sin.\frac{\theta}{2}=\sqrt{\frac{1-\cos.\theta}{2}}$$

$$=\sqrt{\tfrac{1}{2}-\tfrac{1}{2}\sqrt{1-\sin.^2\theta}}$$

By which formula the sine of any angle is given in terms of the sine of twice that angle.

Now substitute $\frac{\theta}{2}$ for $\theta$ and it becomes

$$\text{Sin. } \frac{\theta}{4} \text{ or sin. } \frac{\theta}{2^2} = \sqrt{\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - \text{sin.}^2 \frac{\theta}{2}}}$$

$$\text{In like manner, sin.} \frac{\theta}{2^3} = \sqrt{\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - \text{sin.}^2 \frac{\theta}{2^2}}}$$

$$\text{sin. } \frac{\theta}{2^4} = \sqrt{\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - \text{sin.}^2 \frac{\theta}{2^3}}}$$

&c. = &c.

$$\text{And generally, sin. } \frac{\theta}{2^n} = \sqrt{\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - \text{sin.}^2 \frac{\theta}{2^{n-1}}}}$$

$$\text{Now let } \theta = 30^\circ \quad \therefore \frac{\theta}{2} = 15^\circ$$

and applying the above formula, we have

$$\text{sin. } 15^\circ = \sqrt{\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - \text{sin.}^2 30^\circ}}$$

But by Chap. II. sin. $30^\circ = \frac{1}{2}$ $\therefore$ sin.$^2$ $30^\circ = \frac{1}{4}$

$$\begin{aligned} \therefore \quad \text{sin. } 15^\circ &= \sqrt{\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - \tfrac{1}{4}}} \\ &= \tfrac{1}{2}\sqrt{2 - \sqrt{3}} \\ &= .2588190 \text{ - - - -} \end{aligned}$$

$$\begin{aligned} \text{Similarly, } \quad \text{sin. } 7^\circ 30' &= \sqrt{\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - \text{sin.}^2 15^\circ}} \\ &= \sqrt{\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - (.2588190)^2}} \\ &= .1305268 \text{ - - - - -} \end{aligned}$$

&c. = &c.

It is manifest, that, by continuing the process, we shall obtain in succession the sines of 3°45′, of 1°52′30″, &c.

In this way we find

$$\text{Sin. } \frac{30^\circ}{2^{11}} \text{ or sin. } 1'\ 45''\ 28'''\ 7^{\text{iv}}\ 30^{\text{v}} = .0005113269, \text{ \&c.}$$

$$\text{Sin. } \frac{30^\circ}{2^{12}} \text{ or sin. } 52''\ 44'''\ 3^{\text{iv}}\ 45^{\text{v}} = .0002556634, \text{ \&c.}$$

From which it appears, that, when the operation above mentioned has been repeated so many times, the sine of the arc is halved at the same time that the arc itself is bisected: that is, *The sines of very small arcs are nearly proportional to the arcs themselves.*

Hence we shall have

$$\begin{array}{llll} \text{Sin. } 52'' \, 44''' \, 3^{iv} \, 45^{v} & : \text{sin. } 1' & :: 52'' \, 44''' \, 3^{iv} \, 45^{v} & : 1' \\ & & :: \dfrac{60}{2^{11}} & : \dfrac{60}{60 \times 60} \\ & & :: 3600 & : 4096 \end{array}$$

$$\therefore \text{ sin. } 1' = \frac{\text{sin. } 52'' \, 44''' \, 3^{iv} \, 45^{v} \times 4096}{3600}$$

$$= \frac{.0002556634 \times 4096}{3600}$$

$$= .000290888204 \ldots\ldots = \text{cos. } 89^{\circ} \, 59'$$

$$\because \text{ sin. } \theta = \text{cos. } (90^{\circ} - \theta)$$

$$\text{Again, } \because \text{ cos. } \theta = \sqrt{1 - \text{sin. } \theta)}$$

$$\text{cos. } 1' = \sqrt{1 - (.000290888204 \ldots)^{2}}$$

$$= .999999915384$$

The sine and cosine of 1′ being thus determined, we shall proceed to show in what manner we shall now be enabled to compute the sines and cosines of all superior angles.

By formula ($m$) Chap. II.

$$\text{Sin. } (n+1) \, \theta = 2 \text{ cos. } \theta \text{ sin. } n \, \theta - \text{sin. } (n-1) \, \theta$$

If we suppose $\theta = 1'$ and $n$ to be taken = to the numbers 1, 2, 3, ....... in succession, we find

$\text{Sin. } 2' = 2 \text{ cos. } 1' \text{ sin. } 1' - \text{sin. } 0 = .0005817764 \ldots = \text{cos. } 89^{\circ} \, 58'$

$\text{Sin. } 3' = 2 \text{ cos. } 1' \text{ sin. } 2' - \text{sin. } 1' = .0008726645 \ldots = \text{cos. } 89^{\circ} \, 57'$

$\text{Sin. } 4' = 2 \text{ cos. } 1' \text{ sin. } 3' - \text{sin. } 2' = .0011635526 \ldots = \text{cos. } 89^{\circ} \, 56'$

&c. = &c.

Again, by employing formula ($o$), Chap. II.

$$\text{Cos. } (n+1) \, \theta = 2 \text{ cos. } \theta \text{ cos. } n \, \theta - \text{cos. } (n-1) \, \theta$$

If, as before, we suppose $\theta = 1'$ and $n = 1, 2, 3, \ldots..$ in succession,

$\text{Cos. } 2' = 2 \text{ cos.}^{2} \, 1' - \text{cos. } 0 = .999999830 \ldots = \text{sin. } 89^{\circ} \, 58'$

$\text{Cos. } 3' = 2 \text{ cos. } 1' \text{ cos. } 2' - \text{cos. } 1' = .999999619 \ldots = \text{sin. } 89^{\circ} \, 57'$

$\text{Cos. } 4' = 2 \text{ cos. } 1' \text{ cos. } 3' - \text{cos. } 2' = .999999323 \ldots = \text{sin. } 89^{\circ} \, 56'$

&c. = &c.

It is manifest, that, by continuing the above processes, we shall obtain the numerical values of the sines and cosines of all angles from 1′ up to 90°. These being determined, the tangents, cotangents, &c. may be calculated by means of the relations established in table I.

The above operations are exceedingly laborious, but require a knowledge of the fundamental rules of arithmetic alone. It is manifest that, in employing this method, an error committed in the sine or cosine of an inferior arc, will entail errors on the sines or cosines of all succeeding arcs. Hence is created the necessity of some check on the computist, and of some independent mode of examining the accuracy of the computation. For this purpose, formulæ, derived immediately from established properties, are employed; if the numerical results from these formulæ agree with the results obtained by a regular process of computation, then it is almost a certain conclusion that the latter process has been rightly conducted.

Formulæ employed for this purpose are called *formulæ of verification*, and of these any number may be obtained; it will be sufficient for our present purpose to give one.

$$\text{Sin.}^2\,\theta+\cos.^2\,\theta=1 \quad - \; - \; - \; - \; - \; - \; - \; - \; - \; - \; - \quad \text{tab. I.}$$

And $2\sin.\theta\cos.\theta=\sin.2\theta$

Hence $\sin.\theta=\frac{1}{2}\sqrt{1+\sin.2\theta}\pm\frac{1}{2}\sqrt{1-\sin.2\theta}$

$$\cos.\theta=\frac{1}{2}\sqrt{1+\sin.2\theta}\mp\frac{1}{2}\sqrt{1-\sin.2\theta}$$

Now if we suppose $\theta=12^\circ\,30'$

$$\sin.12^\circ\,30'=\frac{1}{2}\sqrt{1+\sin.25^\circ}\pm\frac{1}{2}\sqrt{1-\sin.25^\circ}$$

$$\cos.12^\circ\,30'=\frac{1}{2}\sqrt{1+\sin.25^\circ}\mp\frac{1}{2}\sqrt{1-\sin.25^\circ}$$

Hence, if the values of the sine and cosine of 12° 30′, and of the sine of 25° obtained by the method already explained, when substituted in these equations, render the two members identical, we conclude that our operations are correct.

The values of the sine and cosine of 30°, 45°, 60°, &c. which were obtained in Chap. II., may be employed as formulæ of verification.

We can obtain finite expressions, although under an incommensurable form, for the sines of arcs of 3°, and all the multiples of 3°, *i. e.* for

3°, 6°, 9°, 12°, 15°,18°, 21°, 24°, 27°, 30°, 33°, 36°, 39°, 42°, 45°, 48°, 51°, 54°, 57°, 60°, 63°, 66°, 69°, 72°, 75°, 78°, 81°, 84°, 87°, 90°.

We first obtain the values of the sines 30°, 45°, 60°, 18°, and from these we obtain all the others, by means of the formulæ, for

$$\text{Sin.}\,(\theta+\beta),\ \sin.\,(\theta-\beta),\ \&c.$$

The numerical value of the trigonometrical functions have been calculated by some to ten places of figures, by others as far as twelve. We must have tables calculated to ten places to have the seconds and tenths of a second with precision, when we make use of the sines of angles which differ but little from 90°, or of the cosines of angles of a few seconds only. Tables in general, however, are calculated as far as seven places only, and these give results sufficiently accurate for all ordinary purposes.

Such is the formation of the trigonometrical canon. The labor of the application of this canon, may be much facilitated by the application of a system of artificial numbers called logarithms, a description of which, forms the subject of the next chapter.

## CHAPTER V.

### LOGARITHMS.

#### DEFINITIONS AND ILLUSTRATIONS.

1. Logarithms are certain functions of natural numbers, by the use of which the tedious operations of multiplication and division are performed by the addition and subtraction of those functions; which consist of artificial numbers having such relations to certain natural numbers, that the sum of any two of those artificial numbers will be a similar function of the product of the natural numbers to which they have such relation; and the difference of any two will be a similar function of the quotient arising from the division of such natural numbers.

Or more definitely, logarithms are the numerical exponents of ratios, being a series of numbers in arithmetical progression corresponding to another series in geometrical progression.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Thus, | 0, | 1, | 2, | 3, | 4, | 5, | 6, | Indices or logarithms, |
| | 1, | 2, | 4, | 8, | 16, | 32, | 64, | Geometrical progression. |
| Or, | 0, | 1, | 2, | 3, | 4, | 5, | 6, | Logarithms, |
| | 1, | 3, | 9, | 27, | 81, | 243, | 729, | Geometric progression. |
| Or, | 0, | 1, | 2, | 3, | 4, | 5 | | Logarithms, |
| | 1, | 10, | 100, | 1000, | 10000, | 100000 | | Geom. progress. |

Where it is evident that the same indices answer for any geometric series, and therefore there may be an endless variety of systems of logarithms to the same natural numbers by changing the second term 2, 3 or 10, &c., of geometrical series of whole numbers; and by interpolation the whole system of numbers may be made to enter the geometric series and receive their proportional logarithms, whether intregers or decimals.

It also appears from the construction of these series, that if any two indices be added together, their sum will be the index of that number which is equal to the product of the two terms in the geometric series to which those indices belong. Thus, the indices 2 and 3 being added together, make 5, and the product of 4 and 8, being the terms corresponding to those indices is 32, which is the number corresponding to the index 5.

In like manner, if any index be subtracted from another, the difference will be the index of that number which is equal to the quotient of the two terms to which those indices belong. Thus the index 6 — the index 4 is=2; and the terms corresponding to those indices are 64 and 16, whose quotient is=4, which is the number answering to the index 2.

For the same reason, if the logarithm of any number be multiplied by the index of its power, the product will be equal to the logarithm of that power. Thus, the index or logarithm of 4, in the above series is 2; and if this number be multiplied by 3, the product will be=6, which is the logarithm of 64, or the third power of 4.

And, if the logarithms of any number be divided by the index of its root, the quotient will be equal to the logarithm of that root. Thus, the index or logarithm of 64 is 6, and if this number be divided by 2, the quotient will be 3, which is the logarithm of 8, or the equal root of 64.

The logarithms most convenient for practice, are such as are adapted to a series increasing in a tenfold ratio, as in the last of the above forms, and are those which are usually found in most of the common tables on the subject.

2. *In a system of logarithms all numbers are considered as the powers of some one number, arbitrarily chosen, which is called the base of the system, and the exponent of that power of the base which is equal to any given number, is called the logarithm of that number.*

Thus, if $a$ be the base of a system of logarithms, N any number, and $x$ such that

$$N = a^x$$

then $x$ is called the logarithm of N in the system whose base is $a$.

The base of the common system of logarithms, (called from their inventor "Briggs's Logarithms"), is the number 10. Hence since

$(10)^0 = 1$ , 0 is the logarithm of 1 in this system.
$(10)^1 = 10$ , 1 ———————— 10 ————————
$(10)^2 = 100$ , 2 ———————— 100 ————————
$(10)^3 = 1000$ , 3 ———————— 1000 ————————
$(10)^4 = 10000$ , 4 ———————— 10000 ————————
&c. = &c. &c. . . . . . . . . . .

From this it appears, that in the common system the logarithms of every number between 1 and 10, is some number between 0 and 1, *i. e.* is 1 plus a fraction. The logarithm of every number between 10 and 100, is some number between 1 and 2, *i. e.* is 1 plus a fraction. The logarithm of every number between 100 and 1000, is some number between 2 and 3, *i. e.* is 2 plus a fraction, and so on.

In the common tables the fractional part alone of the logarithm is registered, and from what has been said above, the rule usually given for finding the *characteristic*, or, *index*, *i. e.* the integral part of the logarithm will be readily understood, viz. *The index of the logarithm of any number greater than unity is equal to one less than the number of integral figures in the given number.* Thus, in searching for the logarithm of such a number as 2970, we find in the tables opposite to 2970 the number 4727564; but since 2970 is a number between 1000 and 10000, its logarithm must be some number between 3 and 4, *i. e.* must be 3 plus a fraction; the fractional part is the number 4727564, which we have found in the tables, affixing to this the index 3, and interposing a decimal point, we have 3,4727564, the logarithm of 2970.

We must not, however, suppose that the number 3.4727564 is the exact logarithm of 2970, or that

$$2970=(10)^{3.4727564}$$

accurately. The above is only an approximate value of the logarithm of 2970; we can obtain the exact logarithm of very few numbers, but taking a sufficient number of decimals, we can approach as nearly as we please to the true logarithm, as will be seen when we come to treat of the construction of tables.

It has been shown that in Briggs' system the logarithm of 1 is 0, consequently, if we wish to extend the application of logarithms of fractions, we must establish a convention by which the logarithms of numbers less than 1 may be represented by numbers less than zero, *i. e.* by *negative numbers.*

*Extending*, therefore, the above principles to negative exponents, since

$\frac{1}{10}$ or $(10)^{-1}=0.1$, $-1$ is the logarithm of .1 in this system

$\frac{1}{100}$ or $(10)^{-2}=0.01$, $-2$ ——— .01 ———

$\frac{1}{1000}$ or $(10)^{-3}=0.001$. $-3$ ——— .001 ———

$\frac{1}{10000}$ or $(10)^{-4}=0.0001$, $-4$ ——— .0001. ———

&c. &c.

It appears, then, from the *convention*, that the logarithm of every number between 1 and .1, is some number between 0 and —1; the logarithm of every number between .1 and .01, is some number between —1 and —2; the logarithm of every number between .01 and .001, is some number between —2 and —3; and so on.

From this will be understood the rule given in books, of tables, for finding the *characteristic* or *index* of the logarithm of a decimal fraction, viz. *The index of any decimal fraction is a negative number, equal to unity, added to the number of zeros immediately following the decimal point.* Thus, in searching for a logarithm of the number such as .00462, we find in the tables opposite to 462 the number 6646420; but since .00462 is a number between .001 and .0001. its logarithm must be some number between —3 and —4, *i. e.* must be —3 plus a fraction, the fractional part is the number 6646420, which we have found in the tables, affixing to this the index —3, and interposing a decimal point, we have —3.6646420, the logarithm of .00462.

### *General Properties of Logarithms.*

Let N and N′ be any two numbers, $x$ and $x'$ their respective logarithms, $a$ the base of the system. Then, by def. (2),

$$N = a^x \quad \cdots\cdots \quad (1)$$
$$N' = a^{x'} \quad \cdots\cdots \quad (2)$$

I. Multiply equations (1) and (2) together,

$$N\,N' = a^x a^{x'}$$
$$= a^{x+x'}$$

∴ by def. 2, $x+x'$ is the logarithm of N N′, that is to say,

*The logarithm of the product of the two or more factors is equal to the sum of the logarithms of these factors.*

II. Divide equation (1) by (2),

$$\frac{N}{N'} = \frac{a^x}{a^{x'}}$$
$$= a^{x-x'}$$

∴ def. (2), $x - x'$ is the logarithm of $\frac{N}{N'}$, that is to say, *The logarithm of a fraction, or of the quotient of two numbers, is equal to the logarithm of the numerator minus the logarithm of the denominator.*

III. Raise both members of equation (1) to the power of $n$.

$$N^{n} = a^{nx}$$

∴ by def. (2), $n\,x$ is the logarithm of $N^{n}$, that is to say,

*The logarithm of any power of a given number is equal to the logarithm of the number multiplied by the exponent of the power.*

IV. Extract the $n^{th}$ root of both members of equation (1).

$$N^{\frac{1}{n}} = a^{\frac{x}{n}}$$

∴ by def. (2), $\frac{x}{n}$ is the logarithm of $N^{\frac{1}{n}}$ that is to say,

*The logarithm of any root of a given number is equal to the logarithm of the number divided by the index of the root.*

Combining the two last cases, we shall find,

$$N^{\frac{m}{n}} = a^{\frac{mx}{n}}$$

whence, $\frac{mx}{n}$ is the logarithm of $N^{\frac{m}{n}}$.

It is of the highest importance to the student to make himself familiar with the application of the above principles to algebraic calculations. The following examples will afford a useful exercise:

Ex. 1. $\log.\ (a.\,b.\,c.\,d.\dots) = \log.\ a + \log.\ b + \log.\ c + \log.\ d + \dots$

Ex. 2. $\log.\ \left(\frac{a\,b\,c}{d\,e}\right) = \log.\ a + \log.\ b + \log.\ c - \log.\ d - \log\ e.$

Ex. 3. $\log.\ (a^{m} b^{n} c^{p} \dots) = m \log.\ a + n \log.\ b + p \log.\ c \dots$

Ex. 4. $\log.\ \left(\frac{a^{m} b^{n}}{c^{p}}\right) = m \log.\ a + n \log.\ b - p \log.\ c \dots$

Ex. 5. $\log.(a^{2} - x^{2}) = \log.(a+x).(a-x) = \log.(a+x) + \log.(a-x)$

Ex. 6. $\log.\ \sqrt{a^{2} - x^{2}} = \frac{1}{2} \log.\ (a+x) + \frac{1}{2}\ (a-x)$

Ex. 7. $\log.(a^3\sqrt[4]{a^3})=\log.\,a^3+\frac{1}{4}\log.\,a^3=3\log.\,a+\frac{3}{4}\log.\,a$

$$=\frac{15}{4}\log.\,a$$

Ex. 8. $\log.\,\sqrt[n]{(a^3-x^3)^m}=\frac{m}{n}\log.\,(a-x)+\frac{m}{n}\log.(a^2+ax+x^2)$

$$=\frac{m}{n}\{\log.\,(a-x)+\log.\,(a+x+z)+\log.\,(a+x-z)\}$$

where $z^2=ax$.

Ex. 9. $\log.\,\sqrt{a^2+x^2}=\frac{1}{2}\{\log.\,(a+x+z)+\log.\,(a+x-z)\}$,

where $z^2=2ax$

Ex. 10. $\log.\,\frac{\sqrt{a^2-x^2}}{(a+x)^2}=\frac{1}{2}\{\log.\,(a-x)-3\log.\,(a+x)\}$

Let us resume the equation,

$$N=a^x$$

1°. If $a>1$, making $x=0$, we have $N=1$; the hypothesis $x=1$ gives $N=a$. As $x$ passes from 0 up to 1, and from 1 up infinity, N will increase from 1 up to $a$, and from $a$ up to infinity; so that $x$ being supposed to pass through all intermediate values, according to the law of continuity, N increases also, but with much greater rapidity. If we attribute negative values to $x$. we have $N=a^{-x}$, or $N=\frac{1}{a^x}$ Here, as $x$ increases, N diminishes, so that $x$ being supposed to increase negatively, N will decrease from 1 towards 0, the hypothesis $x=\infty$ gives $N=0$.

2°. If $a<1$, put $a=\frac{1}{b}$, where $b>1$, and we shall then have $N=\frac{1}{b^x}$ or $N=b^x$, according as we attribute positive or negative values to $x$. We here arrive at the same conclusion as in the former case, with this difference, that when $x$ is positive $N<1$, and when $x$ is negative $N>1$.

3°. If $a=1$, then $N=1$. whatever may be the value $x$.

From this it appears, that,

1. *In every system of logarithms the logarithm of* 1 *is* 0, *and the logarithm of the base* is 1.

II. *If the base be* $>1$, *the logarithms of numbers* $>1$ *are positive, and the logarithms of numbers* $<1$ *are negative. The contrary takes place if the base be* $<1$.

III. *The base being fixed, any number has only one real logarithm; but the same number has manifestly a different logarithm for each value of the base, so that every number has an infinite number of real logarithms.* Thus, since $9^2=81$, and $3^4=81$, 2 and 4 are the logarithms of the same number 81, according as the base is 9 or 3.

IV. *Negative numbers have no real logarithms, for attributing to* $x$ *all values from* $-\infty$ *up to* $+\infty$, *we find that the corresponding values of* N *are positive numbers only, from* 0 *up to* $+\infty$.

The formation of a table of logarithms consists in determining and registering the values of $x$ which correspond to $N=1, 2, 3, \ldots$ in the equation,

$$N=a^x$$

If we suppose $m=a^\alpha$, making

$x=0, \alpha, 2\alpha, 3\alpha, 4\alpha, 5\alpha,$ - - - - - *logarithms.*

$y=1, m, m^2; m^3, m^4, m^5,$ - - - - - *numbers.*

the logarithms increase in arithmetical progression, while the numbers increase in geometrical progression; 0 and 1 being the first terms of the corresponding series, and the arbitrary numbers $\alpha$ and $m$ the common difference and the common ratio.

We may, therefore, consider the systems of values of $x$ and $y$, which satisfy the equation $N=a^x$, as ranged in these two progressions.

In order to solve the equation

$$c=a^x$$

where $c$ and $a$ are given, and where $x$ is unknown, we equate the logarithms of the two members, which gives us

$$\log. c=x \log. a$$

Whence,

$$x=\frac{\log. c}{\log. a}$$

To determine the value of $x$ in the equation

$$Aa^x+Ba^{x-b}+Ca^{x-c}+\ldots\ldots=P$$

we have

$$a^x\left(A+\frac{B}{a^b}+\frac{C}{a^c}\right.\quad+\ldots\ldots=P$$

Or,

$$Qa^{x} \qquad = P$$

$$\therefore \quad x = \frac{\text{log. P} - \text{log. Q}}{\text{log. } a}$$

If we have an equation $a^{z} = b$, where $z$ depends upon an unknown quantity $x$, and we have

$$z = Ax^{n} + Bx^{n-1} + \dots .$$

Since $z = \frac{\text{log. } b}{\text{log. } a} = K$ some known number, the problem depends upon the solution of the equation of the $n^{\text{th}}$ degree.

$$K = Ax^{n} + Bx^{n-1} + \dots .$$

For example, let

$$4\left(\frac{2}{3}\right)^{x^2 - 5x + 4} = 9$$

Hence,

$$(x^2 - 5x + 4) \text{ log. } \left(\frac{2}{3}\right) = \text{log. } \frac{9}{4}$$

$$\therefore \quad x^2 - 5x + 4 \qquad = -2$$

an equation of the second degree, from which we find $x = 2$, $x = 3$.

To find the value of $x$ from the equation

$$b^{n - \frac{a}{x}} = c^{mx} f^{x-p}$$

Taking the logarithms of each member,

$$(n - \frac{a}{x}) \text{ log. } b = m\, x \text{ log. } c + (x - p) \text{ log. } f$$

Or,

$$(m \text{ log. } c + \text{log. } f)\, x^2 - (n \text{ log. } b + p \text{ log. } f)\, x + a \text{ log. } b = 0$$

a quadratic equation, from which the value of $x$ may be determined.

In like manner, from the equation

$$c^{mx} = ab^{nx-1}$$

we find

$$x = \frac{\text{log. } a - \text{log. } b}{m \text{ log. } c - n \text{ log. } b}$$

Equations of this nature are called *Exponential Equations.*

Let N and N + 1 be two consecutive numbers, the difference of their logarithms, taken in any system, will be

$$\text{log. } (N+1) - \text{log. } N = \text{log. } \left(\frac{N+1}{N}\right) = \text{log. } (1 + \frac{1}{N})$$

a quantity which approaches to the logarithm of 1, or zero, in proportion as $\frac{1}{N}$ decreases, that is, as N increases. Hence it appears, that

*The difference of the logarithms of two consecutive numbers is less in proportion as the numbers themselves are greater.*

When we have calculated a table of logarithms for any base $a$, we can easily change the system, and calculate another table for a new base $b$.

Let $c=b^x$, $x$ is the log of $c$ in the system whose base is $b$; Taking the logs. in the known system, whose base is $a$, we have

$$x=\frac{\log. c}{\log. b}=\log. c\left(\frac{1}{\log. b}\right) \ldots\ldots (A) \text{ hence}$$

*The log. of* c *in the system whose base is* b, *is the quotient arising from dividing the log. of* c *by the log. of the new base* b, *both these last logs. being taken in the system whose base is* a.

In order ∴ to have $x$ the log. of $c$ in the new system, we must multiply log. $c$ by $\frac{1}{\log. b}$; this last factor $\frac{1}{\log. b}$ is constant for all numbers, and is called the *Modulus*; that is to say, if we divide the logs. of the same number $c$ taken in two systems, the quotient will be invariable for these systems, whatever may be the value of $c$, and will be the modulus, the constant multiplier which reduces the first system of logs. to the second.

If we find it inconvenient to make use of a log. calculated to the base 10, we can in this manner, by aid of a set of tables calculated to the base 10, discover the logarithm of the given number in any required system.

For example, let it be required, by aid of Briggs' tables, to find the log. of $\frac{2}{3}$ in a system whose base is $\frac{5}{7}$

Let $x$ be the log. sought, then by (A)

$$x=\frac{\log. \frac{2}{3}}{\log. \frac{5}{7}}$$

$$=\frac{\log. 2-\log. 3}{\log. 5-\log. 7}$$

Taking these logs. in Briggs' system, and reducing, we find.

$$= \frac{-0.17609125}{-0.14612804}$$

$$= 1.2050476 = \log. \frac{2}{3} \text{ to base } \frac{5}{7}.$$

Similarly, the log. of $\frac{2}{3}$, in the system whose base is $\frac{3}{2}$, is

$$x = \frac{\log. 2 - \log. 3}{\log. 3 - \log. 2}$$
$$= -1$$

which is manifestly the true result; for in this case the general equation $N = a^x$ becomes $\frac{2}{3} = \left(\frac{3}{2}\right)^x = \left(\frac{2}{3}\right)^{-x}$, and $x$ is evidently $= -1$.

In a system whose base is $a$, we have

$$n = a^{\log. n}$$

for, by the definition of a log. in the equation $n = a^x$, $x$ is the log. $n$.

In like manner,

$$n^h = a^{\log. (n^h)} = n^{h \log. n}.$$

## Examples for Exercise.

Ex. 1. Given $2^{2x} + 2^x = 12$ to find the value of $x$.
Ans. $x = 1.584962$, or $x = \log. (-4) \div \log. 2$.
Ex. 2. Given $x + y = a$, and $m^{(x-y)} = n$ to find $x$ and $y$.
Ans. $x = \frac{1}{2}\{a + \log. n \div \log. m\}$ and $y = \frac{1}{2}\{a - \log. n \div \log. m\}$.
Ex. 3. Given $m^x n^x = a$, and $hx = ky$ to find $x$ and $y$.

$$\text{Ans.} \left\{ \begin{array}{l} x = \log. a \div (\log. m + \log. n) \\ \quad \text{and } y = \frac{h}{k} \log. a \div (\log. m + \log. n). \end{array} \right.$$

*To find the logarithm of any given number.*

Let N be any given number whose logarithm is $x$, in a system whose base is $a$; then

$$a^x = N \text{ and } a^{xz} = N^z;$$

hence, by the exponential theorem, we have from the last equation

$$1 + Axz + A^2 \frac{x^2 z^2}{1.2} + \ldots = 1 + A_1 z + A_1^2 \frac{z^2}{1.2} + \ldots$$

and equating the coefficients of $z$, we get $Ax=A_1$; hence

$$x=\frac{A_1}{A}=\frac{(N-1)-\frac{1}{2}(N-1)^2+\frac{1}{3}(N-1)^3-\ldots}{(a-1)-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\ldots}$$

because $A=(a-1)-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\ .\ .\ .\ .$ in the expansion of $a^{xz}$.

and $A_1=(N-1)-\frac{1}{2}(N-1)^2+\frac{1}{3}(N-)^3-\ .\ .\ .$ in the expansion of $N^z$.

### *To find the logarithm of u number in a converging series.*

We have seen that if $a^x=N^1$, then

$$x=\frac{(N^1-1)-\frac{1}{2}(N^1-1)^2+\frac{1}{3}(N^1-1)^3-\frac{1}{4}(N^1-1)^4+\ldots}{(a-1)-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\frac{1}{4}(a-1)^4+\ldots}$$

Now the reciprocal of the denominator is the *modulus* of the system; and, representing the modulus by M, we have
$x=\log.N^1=M\{(N^1-1)-\frac{1}{2}(N^1-1)^2+\frac{1}{3}(N^1-1)^3-\frac{1}{4}(N^1-1)^4+\}$

Put $N^1=1+n$; then $N^1-1=n$, and we have

$$\log.(1+n)=M(+n-\tfrac{1}{2}n^2+\tfrac{1}{3}n^3-\tfrac{1}{4}n^4+\tfrac{1}{5}n^5-\ldots\}$$

Similarly $\log.(1-n)=M(-n-\frac{1}{2}n^2-\frac{1}{3}n^3-\frac{1}{4}n^4-\frac{1}{5}n^5-\ldots)$

$\therefore\ \log.(1+n)-\log.(1-n)=2M(n+\frac{1}{3}n^3+\frac{1}{5}n^5+\frac{1}{7}n^7+\ldots)$

$$\text{or }\log.\frac{1+n}{1-n}=2M(n+\tfrac{1}{3}n^3+\tfrac{1}{5}n^5+\tfrac{1}{7}n^7+\ldots)$$

$$\text{Put }n=\frac{1}{2P+1};\text{ then }1+n=\frac{2P+2}{2P+1},\ 1-n=\frac{2P}{2P+1},\text{ and }\frac{1+n}{1-n}=\frac{P+1}{P};$$

consequently

$$\log.(P+1)-\log.P=2M\left\{\frac{1}{2P+1}+\frac{1}{3(2P+1)^3}+\frac{1}{5(2P+1)^5}+\right\}$$

$$\therefore\log(P+1)=\log P+2M\left\{\frac{1}{2P+1}+\frac{1}{3(2P+1)^3}+\frac{1}{5(2P+1)^5}+\right\}$$

Hence, if log. P be known, the log of the next greater number can be found by this rapidly converging series.

### *To find the Napierian logarithms of numbers.*

In the preceding series, which we have deduced for log. (P+1), we find a number M, called the *modulus* of the system; and we must assign some value to this number before we can compute the value of the series. Now, as the value of M is arbitrary, we may follow the steps of the celebrated Lord Napier, the inventor of logarithms, and assign to M the simplest possible value. This value will therefore be unity; and we have

$$\text{log. }(P+1)=\text{log. }P+2\left\{\frac{1}{2P+1}+\frac{1}{3(2P+1)^3}+\frac{1}{5(2P+1)^5}+\cdots\right\}$$

Expounding P successively by 1,2,3,4, &c., we find

$$2=\text{log. }1+2\left(\frac{1}{3}+\frac{1}{3\cdot3^3}+\frac{1}{5\cdot3^5}+\frac{1}{7\cdot3^7}+\cdots\right)=\ .6931472$$

$$\text{log. }3=\text{log. }2+2\left(\frac{1}{5}+\frac{1}{3\cdot5^3}+\frac{1}{5\cdot5^5}+\frac{1}{7\cdot5^7}+\cdots\right)=1.0986123$$

$$\text{log. }4=2\text{ log. }2\ \ -\ \ -\ \ -\ \ -\ \ -\ \ =1.3862944$$

$$\text{log. }5=\text{log. }4+2\left(\frac{1}{9}+\frac{1}{3\cdot9^3}+\frac{1}{5.9^5}+\frac{1}{7\cdot9^7}+\cdots\right)=1.6094379$$

$$\text{log. }6=\text{log. }2+\text{log. }3\ \ -\ \ -\ \ -\ \ -\ \ =1.7917595$$

$$\text{log. }7=\text{log. }6+2\left(\frac{1}{3}+\frac{1}{3.13^3}+\frac{1}{5.13^5}+\cdots\cdots\right)=1.9459101$$

$$\text{log. }8=\text{log. }2+\text{log. }4,\text{ or }3\text{ log. }2\ \ -\ \ -\ \ =2.0794415$$

$$\text{log. }9=2\text{ log. }3\ \ -\ \ -\ \ -\ \ -\ \ -\ \ =2.1972246$$

$$\text{log. }10=\text{log. }2+\text{log. }5\ \ -\ \ -\ \ -\ \ -\ \ =2.3025851$$

In this manner the Napierian logarithms of all numbers may be computed.

### *To find the common logarithms of numbers.*

Let $a^x = N$ and $b^y = N$; then we have

$x$ = log. N to the base $a$, or $x$=log. ${}_aN$
$y$ = log. N to the base $b$, or $y$=log. ${}_bN$
hence, log. ${}_aN$=log. ${}_ab_y$ =log. ${}_ab$ (Gen. properties logarithms.)

$$\therefore\ x=y\text{ log. }{}_ab$$

$$\text{and }y=\frac{1}{\text{log. }{}_ab}\cdot x$$

and by means of this equation we can pass from one system of logs. to another, by multiplying $x$, the log. of any number in the system whose base is $a$, by the reciprocal of log. $b$ in the same system; and thus we shall obtain the log. of the same number in the system whose base is $b$.

Let the two systems be the Napierian and the common, in which the base of the former is $\epsilon$=2.718281828 . . . and the base of the latter is $b$=10, the base of the common system of arithmetic; then we have $b$=10, and $a$=$\epsilon$=2.718281828 . . .

and consequently if N denote any number, we shall have

$$\log._{10}N=\frac{1}{\log._{\epsilon}10}\cdot\log_{\epsilon}N \text{ ; that is,}$$

$$\text{com. log } N=\frac{\text{nap. log.}N}{\text{nap.log.}10}=\frac{\text{nap. log.}N}{2.3025851}=;43429448\times\text{nap. log.}N;$$

and the modulus of the common system is, therefore,

$$M=\frac{1}{2.3025851}=.43429448 \therefore 2M=.86858896$$

Hence, to construct a table of common logarithms, we have

$$\log(P+1)=\log P+.86858896\left\{\frac{1}{2P+1}+\frac{1}{3(2P+1)^3}+\frac{1}{5(2P+1)^5}\ldots\right.$$

Expounding P successively by 1,2,3, &c., we get

$$\log.\ 2=.86858896\left(\frac{1}{3}+\frac{1}{3^4}+\frac{1}{5.3^5}+\ldots\right)$$

$$=.86858896\times.6931472 \quad - \quad - \quad - \quad =.3010300$$

$$\log.\ 3=\log.2+.86858896\left(\frac{1}{5}+\frac{1}{3.5^3}+\frac{1}{5^5}+\ldots\right)=.4771213$$

$$\log.\ 4=2\log.2 \quad - \quad - \quad - \quad - \quad - \quad - \quad =.6020600$$

$$\log.\ 5=\log.\tfrac{10}{2}=\log.\ 10-\log 2=1-\log.2 \quad - \quad =.6989700$$

$$\log.\ 6=\log.\ 2+\log.\ 3 \quad - \quad - \quad - \quad - \quad - \quad - \quad =.7781513$$

$$\log.\ 7=\log.6+.86858896\left(\frac{1}{13}+\frac{1}{3.13^3}+\frac{1}{5.13^5}+\ldots\right)=.8450980$$

$$\log.\ 8=\log.2^3=3\log.2 \quad - \quad - \quad - \quad - \quad =.9030900$$

$$\log.\ 9=\log.3^2=2\log.3 \quad - \quad - \quad - \quad - \quad =.9542426$$

$$\log.10= \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad =1.0000000$$

&c. &c.

$$\text{Since } \log.\frac{1+n}{1-n}=2M\left(n+\tfrac{1}{3}n^3+\tfrac{1}{5}n^5+\tfrac{1}{7}n^7+\ldots\right)$$

$$\text{let } \frac{1+n}{1-n}=P\text{ ; then } 1+n=P(1-n) \text{ or } n=\frac{P-1}{P+1}$$

$$\therefore \log.\ P=2M\left\{\frac{P-1}{P+1}+\frac{1}{3}\cdot\left(\frac{P-1}{P+1}\right)^3+\frac{1}{5}\cdot\left(\frac{P-1}{P+1}\right)^5+\ldots\right\}$$

and thus we have a series for computing the logs. of all numbers, without knowing the log. of the previous number.

### EXAMPLES IN LOGARITHMS.

(1.) Given the log. of $2=0.3010300$, to find the logs. of 25 and .0125.

Here $25=\frac{100}{4}=\frac{10^2}{2^2}$;

therefore log. $25=2$ log. $10—2$ log. $2=1.3979406$

Again $.0125=\frac{125}{10000}=\frac{1}{80}=\frac{1}{10\times 2^3}$

$\therefore$ log. $.0125=$log. $1—$log. $10—3$ log. $2=—1—3$ log. $2$
$=\bar{2}.0969100$

(2.) Calculate the common logarithm of 17.
Ans. 1.2304489.

(3.) Given the logs. of 2 and 3 to find the logarithm of 12.5.
Ans. $1+2$ log. $3—2$ log. 2.

(4.) Having given the logs. of 3 and .21, to find the logarithm of 83349.
Ans. $6+2$ log. $3+3$ log. .21.

### ON EXPONENTIAL EQUATIONS.

An *exponential equation* is an equation in which the unknown quantity appears in the form of an exponent or index; thus, the following are exponential equations:

$$a^x = b,\ x^x = a,\ a^{b^x} = c,\ x^{x^x} = a,\ \&c.$$

When the equation is of the form $a^x = b$, or $a^{b^x} = c$, the value of $x$ is readily obtained by logarithms, as we have already seen above. But if the equation be of the form $x^x = a$, the value of $x$ may be obtained by a rule of *approximation*, as in the following example:

*Ex.* Given $x^x = 100$, to find an approximate value of $x$.

The value of $x$ is evidently between 3 and 4, since $3^3 = 27$ and $4^4 = 256$; hence, taking the logs. of both sides of the equation, we have

$$x \log. x = \log. 100 = 2.$$

| | |
|---|---|
| First, let $x_1 = 3.5$; then | Second, let $x_2 = 3.6$; then |
| $3.5 \log. 3.5 = 1.9042380$ | $3.6 \log. 3.6 = 2.0026890$ |
| true no. $= 2.0000000$ | true no. $= 2.0000000$ |
| error $= -.0957620$ | error $= +.0026890$ |

Then, as the difference of the results is to the difference of the assumed numbers, so is the least error to a correction of the assumed number corresponding to the least error; that is,

$$.098451 : .1 :: .002689 : .00273;$$

hence $x = 3.6 - .00273 = 3.59727$, nearly

Again, by forming the value of $x^x$ for $x = 3{\cdot}5972$, we find the error to be $-.0000841$, and for $x = 3.5973$, the error is $+ .0000149$;

$$\text{hence, as } .000099 : .0001 :: .0001 : .0000151;$$

therefore $x = 3.5973 - .0000151 = 3.5972849$, the value nearly.

### Examples for Practice.

(1.) Find $x$ from the equation $x^x = 5$ Ans. 2.129372.

(2.) Solve the equation $x^x = 123456789$ Ans. 8.6400268.

(3.) Find $x$ from the equation $x^x = 2000$. Ans. 4.8278226.

Since the properties of logarithms afford great facilities in performing complicated arithmetical operations upon large numbers, it becomes desirable to have the logarithms of sines, cosines, tangents, &c. computed and arranged in tables; but most of these numbers being less than unity, their logarithms would, of course, be negative. To avoid this inconvenience, all the trigonometrical functions calculated in the manner explained in Chap. IV, are multiplied by a large number, and, the operation being performed upon all, their relative value is not altered. This number may, of course, be any whatever, provided it be so large, that, when the numerical values of trigonometrical quantities are multiplied by it, their logarithms may be positive numbers.

The number employed for this purpose in the common tables is 10000000000 or $10^{10}$, which is usually represented by the symbol R.

The sine of 1′, as computed above, is

$$\text{Sin. } 1' = .0002908882 \ldots.$$

a number much smaller than unity, and whose logarithm would consequently be negative.

When multiplied by $10^{10}$ it becomes

$$= 2908882\ldots.$$

a number whose logarithm is 6.4637261, and consequently we find in our tables, log. sin. $1' = 6.4637261$.

A table constituted upon this principle is called a *Table of Logarithmic Sines, Cosines, Tangents,* &c. and by this nearly all the practical operations of trigonometry are usually performed.

It is manifest, from these remarks, that before we can apply formulæ deduced in the preceding chapters to practical purposes, we must transform them in such a manner as to render the several trigonometrical quantities identical with those registered in our tables. The sines, cosines, &c. we have hitherto employed, are called *Trigonometrical qnantities calculated to a radius unity;* those registered in the tables, *Trigonometrical quantities calculated to radius* R.

The problem to be solved therefore is

*To transform an expression calculated to a radius unity, to another calculated to a radius* R

Let us represent sin. $\theta$ to radius unity by $m$.
.. .. .. R by $n$.

Then the relation between them is

$$n = \mathrm{R}\, m$$

$$m = \frac{n}{\mathrm{R}}$$

and so for all the other trigonometrical quantities.

Hence, *in order to transform an expression calculated to radius unity, to another calculated to radius* R, *we must divide each of the trigonometrical quantities by* R.

If any of the trigonometrical quantities enter in the square, cube, &c. these must of course be divided by $\mathrm{R}^2$ $\mathrm{R}^3$, &c. ......

As observed above, R may be any given number whatever, the number usually employed in the ordinary tables being $10^{10}$, and therefore

$$\log. \mathrm{R} = 10$$

Take as an example such an expression as

$$a \sin.\theta = b \tan.^2 \varphi$$

in order to reduce this to an expression which we can compute by our tables we must, according to the above rule, divide each of the trigonometrical quantities by the proper power of R: the expression then becomes

$$a\frac{\sin.\theta}{\mathrm{R}} = b\,\frac{\tan.^2\varphi}{\mathrm{R}^2}$$

Or, clearing of fractions,

$$a\,R \sin.\,\theta = b \tan.^2 \varphi.$$

Or,

$$\log.\,a + \log.\,R + \log.\sin.\,\theta = \log.\,b + 2 \log.\tan.\,\varphi$$

an expression which may be calculated by the tables.

If the expression calculated to radius unity be of the form

$$m = \frac{\sin.\,\theta}{\sin.\,\varphi}$$

it requires no modification, for if we divide both terms of the fraction $\frac{\sin.\,\theta}{\sin.\,\varphi}$ by R, we shall not alter its value.

We need not prosecute this subject farther, as numerous examples of these transformations will occur at every step in the succeeding chapters.

## CHAPTER VI.

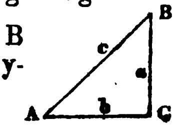
### ON THE SOLUTION OF RIGHT-ANGLED TRIANGLES.

Every plane triangle being considered to consist of 6 parts, the three sides, and the three angles, if any of these three parts be given, we can, in general, determine the remaining parts by trigonometry.

In right-angled triangles, the right angle is always known, and therefore any two other parts being given, we can, in general, determine the rest. We shall thus have five different cases.

1. When one of the acute angles and the hypothenuse is given.
2. When one of the acute angles and a side is given.
3. When the hypothenuse and one side is given.
4. When the two sides are given.
5. When the two acute angles are given.

Let ABC be a right-angled triangle, C the right angle.



Let the sides opposite to the angles A and B be denoted respectively by $a$, $b$, and let the hypothenuse be called $c$.

*Case* 1, Given A, $c$, required B, $a$. $b$.

Since C is a right angle

$$A+B=90^\circ$$

$$\therefore \qquad B=90^\circ-A \text{ whence B is known} \quad (1)$$

By Chap. III. prop. 1,

$$a=c \sin.\,A$$

Adapting this expression to computation by the tables.

$$\log.\ a = c\,\frac{\sin.\ A}{R}$$

$\therefore \log.\ a = \log.\ c + \log.\ \sin. A - \log. R$, whence $a$ is known - (2)

In like manner,

$$b = c\,\frac{\cos.\ A}{R}$$

$\therefore \log.\ b = \log.\ c + \log.\ \cos.\ A - \log.\ R$, whence $b$ is known (3)

If B, $c$ are given, and A, $a$, $b$ required, we shall have precisely in the same manner,

$A = 90° - B$ - - - - - - - - - - - - - - (4)

$\log.\ a = \log.\ c + \log.\ \cos.\ B - \log.\ R$ - - - - - - - (5)

$\log.\ b = \log.\ c + \log.\ \sin.\ B - \log.\ R$ - - - - - - - (6)

*Case* 2. Given A, $a$, required B, $b$, $c$.

$B = 90° - A$, whence B is known - - - - - - (7)

$b = a \cot.\ A$

Adapting the expression to computation by the tables.

$$b = a\,\frac{\cot.\ A}{R}$$

$\log.\ b = \log.\ a + \log.\ \cot.\ A - \log.\ R$, whence $b$ is known (8)

Again,

$a = c \sin.\ A$

$$\therefore\ c = \frac{a}{\sin.\ A}$$

Adapting it to computation,

$$c = R\ .\ \frac{a}{\sin.\ A}$$

$\therefore \log.\ c = \log.\ R + \log.\ a - \log.\ \sin.\ A$, whence $c$ is known (9)

If A, $b$, be given, B, $a$, $c$, we shall have in like manner,

$B = 90° - A$ - - - - - - - - - - - - - - (10)

$\log.\ a = \log.\ b + \log.\ \tan.\ A - \log.\ R$ - - - - - - (11)

$\log.\ c = \log.\ R + \log.\ b - \log.\ \cos.\ A$ - - - - - - (12)

If B, $b$ be given, and A, $a$, $c$ required,

$A = 90° - B$ - - - - - - - - - - - - - (13)

$\log.\ a = \log.\ b + \log.\ \cot.\ B - \log.\ R$ - - - - - - (14)

$\log.\ c = \log.\ R + \log.\ b - \log.\ \sin.\ B$ - - - - - - (15)

If B, $a$ be given, and A, $b$, $c$, required.

$A = 90° - B$ - - - - - - - - - - - - - - (16)

$\log.\ b = \log.\ a + \log.\ \tan.\ B - \log.\ R$ - - - - - - (17)

$\log.\ c = \log.\ R + \log.\ a - \log.\ \cos.\ B$ - - - - - - (18)

*Case* 3. **Let $a$, $c$ be given, required $b$, A, B.**

$$b^2=c^2-a^2$$
$$=(c+a)\ (c-a)$$

$\therefore$ 2 log. $b$=log. $(c+a)$+log. $(c-a)$ whence $b$ is known - - (19)

$$\text{sin. A}=\frac{a}{c}$$

Adapting the expression to computation.

$$\frac{\text{sin. A}}{\text{R}}=\frac{a}{c}$$

$\therefore$ log. sin. A=log. R+log. $a$−log. $c$, whence A is known, (20)

So also,

$$\frac{\text{cos. B}}{\text{R}}=\frac{a}{c}$$

$\therefore$ log. cos. B=log. R+log. $a$−log. $c$, whence B is known, (21)

If $b$, $c$ be given, and $a$, A, B required, we shall have

2 log. $a$=log. $(c+b)$+log. $(c-b)$ - - - - - - (22)
log. cos. A=log. R+log $b$−log. $c$ - - - - - - (23)
log. sin. B=log. R+log. $b$−log. $c$ - - - - - - (24)

*Case* 4. Let $a$, $b$ be given, required A, B, $c$.

$$\text{tan. A}=\frac{a}{b}$$

Adapting the expression to computation.

$$\frac{\text{tan. A}}{\text{R}}=\frac{a}{b}$$

$\therefore$ log. tan. A=log. R+log. $a$−log. $b$, whence A is known, (25)

So also,

$$\frac{\text{tan. B}}{\text{R}}=\frac{b}{a}$$

$\therefore$ log. tan. B=log. R+log. $b$−log. $a$, whence B is known, (26)

$c=\sqrt{a^2+b^2}$, whence $c$ is known, - - - - (27)

*Case* 5. Given A, B, required $a$, $b$, $c$.

It is manifest that this case does not admit of a solution, for *any number* of unequal similar triangles may be constructed, having their angles equal to the angles A, B, C.

We shall conclude this chapter by giving some numerical examples.

*Example* 1. Given A=26° 41′ 6″, $c$=6539.76 yards, required $a$.

Then by (2).

$$\log. a = \log. c + \log. \sin. A - \log. R$$

By the tables

$$\begin{array}{rl} \log. c = & 3.8155618 \\ \log. \sin. A = & 9.6523286 \\ \hline & 13.4678904 \\ \log. R = & 10. \\ \hline \therefore \quad \log. a = & 3.4678904 \end{array}$$

The number in the tables corresponding to the logarithm 3.4678904 is found to be 2936.91.

$$\therefore \qquad a = 2936.91 \text{ yards.}$$

In like manner, the side $b$ may be determined, if required.

*Example* 2. Given $c=6539.76$ yards, $a=2936.91$ yards, required $b$, A, B.

By (19).

$$2 \log. b = \log. (c+a) + \log. (c-a) \qquad \begin{array}{l} c+a=9476.67 \\ c-a=3602.85 \end{array}$$

By the tables,

$$\begin{array}{rl} \log. (c+a) = & 3.9766557 \\ \log. (c-a) = & 3.5566462 \\ \hline \therefore \quad 2 \log. b = & 7.5333019 \\ \log. b = & 3.7666509 \end{array}$$

The number in the tables corresponding to the logarithm 3.7666509 is 5843.2

$$\therefore \qquad b = 5843.2 \text{ yards.}$$

To determine A we have (20).

$$\log. \sin. A = \log. R + \log. a - \log. c$$

By the tables,

$$\begin{array}{rl} \log. a = & 3.4678904 \\ \log. R = & 10. \\ \hline & 13.4678904 \\ \log. c = & 3.8155618 \\ \hline \therefore \quad \log. \sin. A = & 9.6523286 \end{array}$$

On referring to our tables, we shall find that the angles whose logarithmic sine is 9.6523286 is 26° 41′ 6″, which is consequently the value of A.

A being known, B is determined at once by subtracting the value of A from 90°, or B may be determined independently of A by (21).

$$\log. \cos. B = \log. R + \log. a - \log. c.$$

*Example* 3. Let $a$, $b$, in the last number be given, and $c$ required.

Then by (27).

$$c=\sqrt{a^2+b^2} \text{ or } c^2=a^2+b^2$$

The calculation in this case is not so simple, for the quantity under the radical cannot be easily adapted to logarithmic calculation.

$$\begin{array}{llll}
\text{We have,} & \log.\ a^2=6.9357808 & \therefore & a^2=\phantom{0}8625400 \\
 & \log.\ b^2=7.5333019 & \therefore & b^2=34143000 \\
 & & & \overline{\phantom{b^2=42768400}} \\
 & & \therefore & c^2=42768400 \\
\therefore & \log.\ c^2=7.6311230 & & \\
 & \log.\ c\ =3.8155615 & & \\
 & \quad\ \ c\ =6539.76 & &
\end{array}$$

*Example* 4. Given $c=6512.4$ yards, $b=6510.6$, to find A. By (23).

$$\log.\ \cos.\ \mathrm{A}=\log.\ \mathrm{R}+\log.\ b-\log.\ c$$

$$\begin{array}{lrl}
\text{Now,} & \log.\ \mathrm{R}= & 10. \\
 & \log.\ b= & 3.8136210 \\
\hline
 & & 13.8136210 \\
 & \log.\ c= & 3.8137411 \\
\hline
\therefore & \log.\ \cos.\ \mathrm{A}= & 9.9998799 \\
\therefore & \mathrm{A}= & 1^\circ\ 20'\ 50''
\end{array}$$

Upon inspecting the tables that are calculated to seven places of decimals only, it will be seen that, when the angles becomes very small, the cosines differ very little from each other. The same remark applies, of course, to the sines of angles nearly 90°. In cases, therefore, where great accuracy is required, we may commit an important error by calculating a small angle from its cosine, or a large one from its sine. We must consequently endeavor to avoid this, by transforming our expression by help of the relations established in chapter first and second.

In the example before us, A is a small angle which has been calculated from its cosine; we must therefore, if possible, calculate this angle by means of its sine, or some other trigonometrical function.

Now, by formula ($j$), chap. II. we have generally

$$\sin.\ \frac{\mathrm{A}}{2}=\sqrt{\frac{1-\cos.\ \mathrm{A}}{2}}$$

In the present case, cos. $A=\frac{a}{c}$, substituting this in the above equation,

$$\sin.\frac{A}{2}=\sqrt{\frac{c-a}{2a}}$$

$$\therefore \qquad \log.\sin.\frac{A}{2}=\frac{1}{2}\log.(c-a)-\frac{1}{2}\log.2\,a+\log.R.$$

From which we find,

$$\frac{A}{2}=40'\ 24''$$

$$\text{And} \qquad \therefore \qquad A=1^\circ\ 20'\ 48''$$

Instead of 1° 20′ 50″, as obtained by the former process.

No angle which is nearly 90° ought to be calculated from its tangent, for the tangents of all large angles increase with so much rapidity, that the results derived from the column of proportional parts found in the tables cannot be depended on as accurate.

## CHAPTER VII.

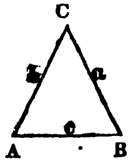
### ON THE SOLUTION OF OBLIQUE ANGLED TRIANGLES.

Six different cases present themselves.

1. When two angles and the side between them are given.
2. When two angles and the side opposite to one of them are given.
3. When two sides and the included angle are given.
4. When two sides and the angle opposite to one of them are given.
5. When the three sides are given.
6. When the three angles are given.

Let A, B, C be a plane triangle.

Let the angles be denoted by the large letters A, B, C, and the sides opposite to these angles by the corresponding small letters $a$, $b$, $c$.



*Case* 1. Given A, B, $c$, required C, $a$, $b$.

Since $A+B+C=180^\circ$

$$C=180^\circ-(A+B,) \text{ whence C is known.}$$

C being thus determined, we have, by chap. III. prop. 2,

$$\frac{a}{c}=\frac{\sin. A}{\sin. C}$$

$$a=c.\frac{\sin. A}{\sin. C}$$

An expression which is in a form adapted to computation by the tables.

$\therefore$ log. $a$=log. $c$+log. sin. A−log. sin. C, whence $a$ is known. Again,

$$\frac{b}{c}=\frac{\sin. B}{\sin. C}$$

$$b=c.\frac{\sin. B}{\sin. C}$$

$\therefore$ log. $b$=log. $c$+log. sin. B—log. sin. C, whence $b$ is known.

If any other two angles and the side between them be given, we may determine the remaining angle and sides in a manner precisely similar.

*Case* 2. Given A, B, $a$, required C, $b$, $c$.

Since A+B+C=180°

$\therefore$ C=180°—(A+B,) whence C is known.

Again, $$\frac{b}{a}=\frac{\sin. B}{\sin. A}$$

$$\therefore \quad b=a.\frac{\sin. B}{\sin. A}$$

$\therefore$ log. $b$=log. $a$+log. sin. B—log. sin. A, whence $b$ is known.

Also, C being known,

$$\frac{c}{a}=\frac{\sin. C}{\sin. A}$$

$$\therefore \quad c=a\frac{\sin. C}{\sin. A}$$

$\therefore$ log. $c$=log. $a$+log. sin. C—log. sin. A, whence $c$ is known.

If any two other angles and the side opposite to one of them are given, the remaining angle and sides may be determined in a manner precisely similar.

*Case* 3. Given $a$, $b$, C, required A, B, $c$.

By prop. 3, chap. III.

$$\frac{\tan. \frac{A+B}{2}}{\tan. \frac{A-B}{2}}=\frac{a+b}{a-b} \quad - - - - - - - - - - \quad (1)$$

Now, $A+B+C=180°$

$$\therefore \quad \frac{A+B}{2}=90°-\frac{C}{2}$$

$$\therefore \quad \tan.\frac{A+B}{2}=\tan.\left(90°-\frac{C}{2}\right)$$

$$=\cot.\frac{C}{2}$$

Substituting this value of $\tan.\frac{A+B}{2}$ in (1.)

$$\frac{\cot.\frac{C}{2}}{\tan.\frac{A-B}{2}}=\frac{a+b}{a-b}$$

$$\therefore \quad \tan.\frac{A-B}{2}=\frac{a-b}{a+b}\cot.\frac{C}{2}$$

$$\log.\tan.\frac{A-B}{2}=\log.(a-b)+\log.\cot.\frac{C}{2}-\log.(a+b)$$

And we can thus calculate the value of the angle $\frac{A-B}{2}$ from our tables; let the angle thus found be called $\varphi$ $\therefore$ $A-B=2\varphi$,

$$\begin{array}{ll} \text{Now} & A+B=180°-C \\ \text{And} & \underline{A-B=2\varphi} \end{array}$$

$$\therefore \text{ adding and subtracting } A=90°+\varphi-\frac{C}{2}$$

$$B=90°-\left(\varphi+\frac{C}{2}\right)$$

The angles A and B will thus become known, and, these being determined, we can find the side $c$ from the relation,

$$\frac{c}{a}=\frac{\sin.C}{\sin.A}$$

$$c=a.\frac{\sin.C}{\sin.A}$$

$$\therefore \quad \log.c=\log.a+\log.\sin.C-\log.\sin.A$$

If $a$, $c$, B, or $b$, $c$, A be given, the remaining angles and side may be determined in a similar manner by aid of the formula (j) in chap. III.

*Case* 4. Given $a$, $b$, A to determine B, C, $c$.

$$\frac{\sin. B}{\sin. A}=\frac{b}{a}$$

$$\therefore \quad \sin. B=\sin. A.\frac{b}{a}$$

$\therefore$ log. sin. B = log. sin. A + log. $b$ — log. $a$, whence $c$ is known.
B being known, C = 180° —(A+B,) whence C is known.

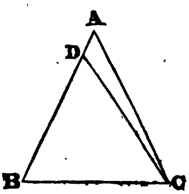
$$\text{C being known, } \frac{c}{a}=\frac{\sin. C}{\sin. A}$$

$$c=a.\frac{\sin. C}{\sin. A}$$

$\therefore$ log. $a$ = log. sin. C — log. sin. A, whence $b$ is known.

If any two other sides and the angle opposite to one of them be given, the remaining angles and side may be determined in a manner precisely similar.

It must be remarked, that, in the above case, we determine the angle B from the logarithm of its sine; but since the sine of any angle, and the sine of its supplement are equal to one another, and since it is not always possible for us to ascertain *a priori* whether the angle B is acute or obtuse, the solution will be sometimes ambiguous.



In fact, two different and unequal triangles may be constructed, having two sides and the angle opposite to one of those sides in one triangle, equal to the corresponding sides and angle of the other; one of these triangles will be obtuse-angled, and the other acute-angled, and the angles opposite the remaining given sides in each will be supplemental.

Thus let ABC, be a plane triangle.

With centre C and radius equal to CB describe a circle cutting AB in D.

Join CD.

Then it is manifest that the two unequal triangles CBA, CDA, have the two sides CB, CA of the one, equal to the two sides CD, CA of the other, and the angle A, opposite the equal sides CB, CD, in each, common.

It is manifest from this, that it is impossible to determine generally, from the data of this case, which of the two triangles is the solution of the problem. There are certain considerations, however, by which the ambiguity may sometimes be removed.

1. If the angle be obtuse, then both of the remaining angles must be acute, and the species of B will be determined.

2. If the given angle be acute, but the side opposite the given angle greater than the given side opposite the required angle, then the required angle is acute. For since in every triangle the greater side has the greater angle opposite to it, and since the side opposite to the given angle, which is acute, is greater than the side opposite to the required angle, it follows, *a fortiori*, that the required angle is acute.

But if the given angle be acute, and the side opposite to the given angle less than the side opposite to the required angle, then we have no means of ascertaining the species of the required angle, and the solution in this case is ambiguous.

*Case* 5. Given the three sides, *a*, *b*, *c*, required the three angles A, B, C.

By formula ($\varepsilon$) chap. III.

$$\text{sin. } A = \frac{2}{bc} \cdot \sqrt{s\,(s-a)\,(s-b)\,(s-c)}$$

$$\text{sin. } B = \frac{2}{ac} \cdot \sqrt{s\,(s-a)\,(s-b)\;s-c)}$$

$$\text{sin. } C = \frac{2}{ab} \cdot \sqrt{s\,(s-a)\,(s-b)\,(s-c)}$$

Adapting these expressions to computation by the tables, and taking the logs.

log. sin. A

$$=\text{log. } R+\text{log. } 2+\tfrac{1}{2}\,\{\text{log. } s+\text{log. } (s-a)+\text{log. } (s-b)+\text{log. } (s-c)\}-\text{log } b-\text{log. } c$$

log. sin. B

$$=\text{log. } R+\text{log. } 2+\tfrac{1}{2}\{\text{log. } s+\text{log. } (s-a)+\text{log. } (s-b)+\text{log. } (s-c)\}-\text{log. } a-\text{log. } c$$

log. sin. C

$$=\text{log. } R+\text{log. } 2+\tfrac{1}{2}\{\text{log. } s+\text{log. } (s-a)+\text{log. } (s-b)+\text{log. } (s-c)\}-\text{log. } a-\text{log. } b$$

Whence the three angles are known.

The three angles may also be obtained from any of the groups of formulæ ($\sigma$), ($\zeta$), ($\eta$), in chap. III.

It is manifest, from the remarks made at the conclusion of the last chapter, that, when one or more of the required angles is very small, the group ($\sigma$) may be used with the greatest advantage, and when one or more of the angles is nearly 90°, we ought to employ the group ($\zeta$.) The group ($\eta$) may be made use of in any case.

*Case* 6. Given the three angles A, B, C, required the three sides $a$, $b$, $c$.

It is manifest that this case does not admit of solution, for *any number* of unequal similar triangles may be constructed, having their angles equal to the angles A, B, C.

We shall conclude this chapter by giving some numerical examples.

*Example* 1. Given $A = 68° 2' 24''$, $B = 57° 53' 16''.8$, $a = 3754$ feet, required C, $b$, $c$.

Then by case 2.

$$\begin{aligned} C &= 180° - (A+B) \\ &= 180° - 125° 55' 40''.8 \\ &= 54° 4' 19''.2 \\ b &= a \cdot \frac{\text{sin. B}}{\text{sin. A}} \end{aligned}$$

$$\text{log. } b = \text{log. } a + \text{log. sin. B} - \text{log. sin. A}$$

$$\begin{aligned} \text{Now} \quad \text{log. } a &= 3.5744943 \\ \text{log. sin. B} &= \underline{9.9278888} \\ &\phantom{=} 13.5023831 \\ \text{log. sin. A} &= \underline{9.9672882} \\ \therefore \quad \text{log. } b &= 3.5350949 = \text{log. } 3428.43 \\ \therefore \quad b &= 3428.43 \end{aligned}$$

Similarly,

$$\begin{aligned} \text{log. } c &= \text{log. } a + \text{log. sin. C} - \text{log. sin. A} \\ \text{log. } a &= 3.5744943 \\ \text{log. sin. C} &= \underline{9.9083536} \\ &\phantom{=} 13.4828479 \\ \text{log. sin. A} &= \underline{9.9672882} \\ \therefore \quad \text{log. } c &= 3.5155597 = \text{log. } 3277.628 \\ \therefore \quad c &= 3277.628 \text{ feet.} \end{aligned}$$

*Example* 2. Given $a = 145$, $b = 178.3$, $A = 41° 10'$, required B, C

This example belongs to case 4, and since the given angle A is acute, and the side $b$ opposite to the required angle B greater than the side $a$, the solution will be ambiguous.

$$\begin{aligned} \text{We have} \quad \text{log. sin. B} &= \text{log. sin. A} + \text{log. } b - \text{log. } a \\ \text{log. sin. A} &= 9.8183919 \\ \text{log. } b &= \underline{2.2511513} \\ &\phantom{=} 12.0695432 \\ \text{log. } a &= \underline{2.1613680} \\ \therefore \quad \text{log. sin. B} &= 9.9081752 \end{aligned}$$

The angle in the tables corresponding to this logarithm, is $54° \, 2' \, 22''$, but we cannot determine *a priori* whether the angle sought be this angle, or its supplement $125° \, 57' \, 38''$.

$$\begin{aligned} \therefore \quad B &= 54° \, 2' \, 22'' \\ \text{Or} \quad B &= 125° \, 57' \, 38'' \end{aligned}$$

If we take the 1st value,

$$\left.\begin{array}{l} C=84° \, 47' \, 38'' \text{ and the triangle required is ABC} \\ \quad\text{If we take the second value,} \\ C=12° \, 52' \, 22'' \text{ and the triangle required is ADC} \end{array}\right\} \text{see last fig.}$$

*Example* 3. Given $a = 178.3$, $b = 145$, $A = 41° \, 10'$, required B.

This example also belongs to case 4, but since the given angle A is acute, and the side $b$ opposite the required angle B less than the side $a$, it follows that the angle B must be an acute angle, and the solution will not be ambiguous.

$$\begin{array}{lrl} \text{We have} & \log.\sin. B = & \log.\sin. A + \log. b - \log. a \\ \text{But} & \log.\sin. A = & 9.8183919 \\ & \log. b = & \underline{2.1613680} \\ & & 11.9797599 \\ & \log. a = & \underline{2.2511513} \\ \therefore & \log.\sin. B = & 9.7286086 \end{array}$$

The angle in the tables corresponding to this logarithm is $32° \, 21' \, 54''$, and since, in the present instance, the supplement of $32° \, 21' \, 54''$ cannot belong to the case proposed, the solution is not ambiguous.

*Example* 4. Given $a = 374$, $b = 3277.628$, and the included angle $57° \, 53' \, 16''.8$: required A, C, $b$.

By case 3 we have

$$\log.\tan. \frac{A-B}{2} = \log.(a-b) + \log.\cot. \frac{C}{2} - \log.(a+b)$$

$$\begin{array}{lrl} a-b = 476.372, \therefore & \log.(a-b) = & 2.6779444 \\ & \log.\cot. \frac{C}{2} = & \underline{10.2572497} \\ & & 12.9351941 \\ a+b = 7031.628, & \log.(a+b) = & \underline{3.8470543} \\ \therefore & \log.\tan. \frac{A-B}{2} = & 9.0881398 \end{array}$$

$$\begin{array}{lrl} \text{Whence} & \frac{A-B}{2} = & 6° \, 59' \, 2''.4 \\ \text{And since} & A+B = & 122° \, 6' \, 43''.2 \\ \text{And} & A-B = & 13° \, 58' \, 4''.8 \\ \therefore & 2A = & 136° \, 4' \, 48'' \\ & 2B = & 108° \, 8' \, 38''.4 \end{array}$$

$$\therefore \quad A = 68° \, 2' \, 24'', \; B = 54° \, 4' \, 19''.2$$

The angles A and B being determined, the side $c$ may be readily found from the equation.

$$\frac{c}{a}=\frac{\sin. C}{\sin. A}$$

$$\log. c = \log. a + \log. \sin. C - \log. \sin. A$$

*Example* 5. Given $a=33$, $b=42.6$, $c=53.6$, required A, B, C. Taking the formula marked ($\varepsilon$) in chap. III. we have

log. sin. A

$$=\log. R+\log. 2+\tfrac{1}{2}\{\log. s+\log. (s-a)+\log.(s-b)+\log.(s-c)\}-\{\log. b+\log. c\}$$

log. sin. B

$$=\log. R+\log. 2+\tfrac{1}{2}\{\log. s+\log. (s-a)+\log.(s-b)+\log.(s-c)\}-\{\log. a+\log. c\}$$

log. sin. C

$$=\log. R+\log. 2+\tfrac{1}{2}\{\log. s+\log. (s-a)+\log.(s-b)+\log.(s-c)\}-\{\log. a+\log. b\}$$

$$\begin{array}{lll}
\text{Now} & \log. R=10. & \\
 & \log. 2=0.3010300 & \\
a=33 \therefore \log. a & =1.5185139 & \left.\begin{array}{l} \log. b+\log. c=3.3585744 \\ \therefore \log. a+\log. c=3.2476787 \\ \log. a+\log. b=3.1479235 \end{array}\right. \\
b=42.6 \therefore \log. b & =1.6294096 & \\
c=53.6 \therefore \log. c & =1.7291648 & \\
s=64.6 \therefore \log. s & =1.8102325 & \\
s-a=31.6 \therefore \log. s-a & =1.4996871 & \\
s-b=22 \therefore \log. s-b & =1.3424227 & \\
s-c=11 \therefore \log. s-c & =1.0413927 &
\end{array}$$

$$\therefore \log. s+\log.(s-a)+\log.(s-b)+\log.(s-c)=5.6937350$$

And

$$\tfrac{1}{2}\{\log. s+\log.(s-a)+\log.(s-b)+\log.(s-c)\} = 2.8468675$$

$$\therefore \log. R+\log. 2+\tfrac{1}{2}\{\log. s+\log.(s-a)+\log.(s-b)+\log.(s-c)\} - - - - - = 13.1478975$$

Subtracting from this number the values log. $b$+log. $c$; log. $a$+log. $c$; log. $a$+log. $b$; in succession we find,

$$\begin{aligned}
\log. \sin. A &= 9.7893231 \therefore A = 37^\circ\ 59'\ 53'\\
\log. \sin. B &= 9.9002188 \therefore B = 52^\circ\ 37'\ 46''\tfrac{9}{16}\\
\log. \sin. C &= 9.9999740 \therefore C = 89^\circ\ 22'\ 20''\tfrac{7}{16}
\end{aligned}$$

Having determined A and B by the above method, we find the above accurate value of C, by subtracting the sum of A and B from 180°. If, however, it had been required to determine C alone (being an angle nearly equal to 90°) we could not have found its value with sufficient accuracy from the common tables, for it will be seen, upon referring to them, that the number 9.9999740 may be the logarithm of the sine of any angle from 89° 22″ 20″ up to 89° 22′ 25″ consequently the above method cannot be applied with propriety to determine the exact value of C, unless we previously determine A and B.

The angle C may however be determined directly, and with great accuracy, from any of the three formulæ ($\sigma$), ($\zeta$), ($\eta$), in Chap. III.

Let us take these in succession, ($\sigma$).

$$\log.\cos.\frac{C}{2}=\log.R+\tfrac{1}{2}\ \{\log.s+\log.(s-c)\}-\tfrac{1}{2}(\log.a+\log.b)$$

$$\left.\begin{aligned}\log.s&=1.8102325\\ \log.(s-c)&=1.0413927\end{aligned}\right\}\ \therefore\ \tfrac{1}{2}\{\log.s+\log.(s-c)\}=1.4258126$$

$$\log.R=10$$

$$\overline{2.8516252}\qquad\qquad\overline{11.4258126}$$

$$\log.a+\log.b=3.1479235\ \therefore\ \tfrac{1}{2}\ (\log.a+\log.b)=\underline{1.5739617}$$

$$\therefore\quad\log.\cos.\frac{C}{2}=\ 9.8518509$$

$$\therefore\quad\frac{C}{2}=44^\circ\ 41'\ 10''\ \tfrac{6}{35}$$

$$C=89^\circ\ 22'\ 20''\ \tfrac{12}{35}$$

By ($\zeta$).

$$\log.\sin.\frac{C}{2}=\log.R+\tfrac{1}{2}\{\log.(c-a)+\log.(s-b)\}-\tfrac{1}{2}\ \{\log.a+\log.b\}$$

$$\left.\begin{aligned}\log.(s-a)&=1.4996871\\ \log.(s-a)&=1.3424227\end{aligned}\right\}\ \therefore\ \tfrac{1}{2}\{\log.(s-a)+\log.(s-b)\}=\ 1.4210549$$

$$\log.R=10.$$

$$\overline{2.8421098}\qquad\qquad\overline{11.4210640}$$

$$\log.a+\log.b=3.1479235\quad\therefore\quad\tfrac{1}{2}\ \{\log.a+\log.b\}=\ 1.5739617$$

$$\therefore\quad\log.\sin.\frac{C}{2}=\ 9\cdot8470932$$

$$\therefore\quad\frac{C}{2}=44^\circ\ 41'\ 10''\tfrac{6}{35}$$

$$C=89^\circ\ 22'\ 20''\tfrac{12}{35}$$

By ($\eta$).

$$\log.\tan.\frac{C}{2}=\log.R+\tfrac{1}{2}\{\log.(s-a)+\log.(s-b)\}-\tfrac{1}{2}\{\log.s+\log.(s-c)\}$$

$$\left.\begin{aligned}\log.(s-a)&=1.4996871\\ \log.(c-a)&=\underline{1.3424227}\\ &\ \ \ 2.8421098\end{aligned}\right\}\ \therefore\tfrac{1}{2}\ \left\{\begin{aligned}\log.(s-a)+\log.(s-a)\\ \log.R\end{aligned}\right\}\begin{aligned}&=1.4210549\\ &=10.\end{aligned}$$

$$\overline{11.4210549}$$

$$\left.\begin{aligned}\log.s&=1.8102325\\ \log.s-c&=1.0413927\end{aligned}\right\}\ \therefore\ \tfrac{1}{2}\ \{\log.+\log.\ (s-c)\ =\underline{1.4258126}$$

$$\overline{2.8516252}$$

$$\therefore\quad\log.\tan.\frac{C}{2}=9.9952423$$

$$\therefore\quad\frac{C}{2}=44^\circ\ 41'\ 10''\tfrac{6}{35}$$

$$C=89^\circ\ 22'\ 10\tfrac{12}{35}$$

## CHAPTER VIII.

### ON THE USE OF SUBSIDIARY ANGLES.

*Subsidiary Angles* are angles which, although not immediately connected with a given problem, are introduced by the computist in order to simplify his calculations. Their use, and the method in which they are employed, will be understood from what follows.

When the two sides of a triangle, and the included angle, are given, according to the method pursued in the last chapter, we must determine the two remaining angles before we can compute the third side. It frequently happens, however, in practice, that the side only is required, and it therefore becomes desirable to have some direct method of computing the side independently of the two angles.

Suppose that $a$, $b$, C are given, and $c$ is required. By chap. III. prop. 4,

$$c^2=a^2+b^2-2ab\cos. C.$$

the side $c$ is determined *theoretically* at once by this expression, but the formula is not adapted to logarithmic computation, and would, if employed practically, lead to a very tedious and complicated calculation. We can, however, put this expression under a form adapted to logarithmic calculation, by having recourse to an algebraical artifice, and introducing a subsidiary angle.

$$c^2=a^2+b^2-2\,ab\cos. C$$

Adding and substracting 2 $ab$ on the right hand side.

$$\begin{aligned}c^2&=a^2+b^2-2\,ab+2\,ab-2\,ab\cos. C\\&=(a-b)^2+2\,ab\,(1-\cos. C)\\&=(a-b)^2+2\,ab+2\sin.^2\frac{C}{2}\\&=(a-b)^2\left\{1+\frac{4\,ab\sin.^2\frac{C}{2}}{(a-b)^2}\right\}\end{aligned}$$

$$\text{Assume}\qquad\frac{4\,ab\sin.^2\frac{C}{2}}{(a-b)^2}=\tan.^2\varphi$$

$$\begin{aligned}c^2&=(a-b)^2\,(1+\tan.^2\varphi)\\&=(a-b)^2\sec.^2\varphi\\c&=(a-b)\sec.\varphi\end{aligned}$$

$$\therefore\quad\log. c=\log.(a-b)+\log.\sec.\varphi-\log. R$$

The angle $\varphi$ is known from the equation.

$$\tan.\ \varphi = \frac{2\sqrt{ab}\ .\ \sin.\ \frac{C}{2}}{(a-b)}$$

Whence,

$$\log.\tan.\varphi = \log.\ 2 + \tfrac{1}{2}\ (\log.a + \log.b) + \log.\ \sin.\frac{C}{2} - \log.(a-b)$$

φ being thus determined, log. sec. φ can be found from the tables, and the value of $c$ becomes known.

The angle φ, which is introduced into the above calculation, in order to render the expression convenient for logarithmic computation, is called a *subsidiary angle.*

The above transformation may be effected in a manner somewhat different, as before.

$$\begin{aligned}
c^2 &= a^2+b^2-2\,ab\cos.\ C\\
&= a^2+b^2+2\,ab-2\,ab-2\,ab\cos.\ C\\
&= (a+b)^2-2\,ab\,(1+\cos.\ C)\\
&= (a+b)^2-2\,ab\times 2\cos.^2\frac{C}{2}\\
&= (a-b)^2\left(1-\frac{4\,ab\cos.^2\frac{C}{2}}{(a+b)^2}\right)
\end{aligned}$$

Assume $\dfrac{4\,ab\cos.^2\frac{C}{2}}{(a+b)^2} = \sin.^2\varphi$

$$\begin{aligned}
\therefore\quad c^2 &= (a+b)^2\,(1-\sin.^2\varphi)\\
&= (a+b)^2\cos.^2\varphi\\
c &= (a+b)\cos.\ \varphi
\end{aligned}$$

$$\therefore\quad \log.\ c = (s+b) + \log.\ \cos.\ \varphi - \log.\ R$$

As before the angle φ must be determined from the equation.

$$\sin.\varphi = \frac{2\sqrt{ab}\ .\ \cos.\frac{C}{2}}{(a+b)}$$

In order to prove that we can always assume

$$\frac{2\sqrt{ab}\ .\ \cos.\frac{C}{2}}{(a+b)} = \sin.\ \varphi$$

we must show that $\dfrac{2\sqrt{ab}\cos.\frac{C}{2}}{(a+c)}$ is always less than unity, or, in other words, that $2\sqrt{ab}$ is always less than $(a+b)$, this is easily done.

$$
\begin{array}{lll}
\text{If} & a+b & > 2\sqrt{ab} \\
\text{Then} & a^2+2ab+b^2 & > 4ab \\
 & a^2+b^2 & > 2ab \\
 & a^2+b^2-2ab & > 0 \\
\text{Or} & (a-b)^2 & > 0
\end{array}
$$

But since $(a-b)^2$ is necessarily a positive quantity, it must always be greater than 0 (except in the particular case $a=b$, where it is$=0$), and therefore $\frac{2\sqrt{ab}\cos.\frac{C}{2}}{(a+b)}$ is always less than unity, and consequently an angle may always be found whose sine is equal to it.

In solving the same case of oblique-angled triangles, we determined the difference of the angles A, B from the equation.

$$\tan.\frac{A-B}{2}=\frac{a-b}{a+b}\cot.\frac{C}{2}$$

Whence $\log.\tan.\frac{A-B}{2}=\log.(a-b)+\log.\cot.\frac{C}{2}-\log.(a+b)$

In the solution of certain astronomical problems, the logarithms of the sides $a$, $b$ are given, but not the sides themselves, and these logarithms being given, we can very easily calculate $\frac{A-B}{2}$ without knowing the sides.

$$\tan.\frac{A-B}{2}=\frac{a-b}{a+b}\cot.\frac{C}{2}$$

$$=\frac{1-\frac{b}{a}}{1+\frac{b}{a}}\cot.\frac{C}{2}$$

Assume $\frac{b}{a}=\tan.\varphi$

$$\tan.\frac{A-B}{2}=\frac{1-\tan.\varphi}{1+\tan.\varphi}.\cot.\frac{C}{2}$$

$$=\tan.(45^\circ-\varphi)\cot.\frac{C}{2}$$

$$\therefore \log.\tan.\frac{A-B}{1}=\log.\tan.(45^\circ-\varphi)+\log.\cot.\frac{C}{2}-\log.R$$

The angle $\varphi$ is known from the equation.

$$\tan.\ \varphi = \frac{b}{a}$$

Whence $\log.\ \tan.\ \varphi = \log.\ R + \log.\ b - \log.\ a$

The angle $\frac{A-B}{2}$ thus becomes known from the logs. of $a$ and $b$, without calculating $a$ and $b$. In the same way we have

$$\cot.\ \frac{A-B}{2} = \tan.\ (45^\circ + \varphi)\ \tan.\ \frac{C}{2}$$

And $\therefore \log.\ \cot. \frac{A-B}{2} = \log.\tan.(45^\circ+\varphi) + \log.\ \tan.\frac{C}{2} - \log.R.$

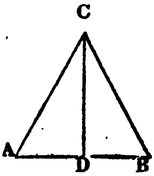
## CHAPTER IX.

### ON THE SOLUTION OF GEOMETRICAL PROBLEMS BY TRIGONOMETRY.

A great variety of geometrical problems may be solved with much elegance by the introduction of geometrical formulæ. We shall give a few examples.

### PROBLEM I.

*To express the area of a plane triangle in terms of the sides of the triangle*



Let CD be a perpendicular from C upon AB

Area of a triangle $ABC = \frac{AB \times CD}{2}$

$$= \frac{c}{2}.\ AC \sin.\ A$$

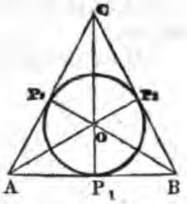
$$= \frac{bc}{2}.\ \sin.\ A$$

$$= \frac{bc}{2} \cdot \frac{2}{bc} \sqrt{s(s-a)(s-b)(s-c)} \text{..Ch. III.}$$

$$= \sqrt{s\,(s-a)\,(s-b)\,(s-c)}$$

### PROBLEM II.

*To express the radius of a circle inscribed in a given triangle, in terms of the sides of the triangle.*



Let the radius required be called $r$.

$$\text{Area of AOC}=\frac{rb}{2}$$

$$\text{..}\quad \text{AOB}=\frac{rc}{2}$$

$$\text{..}\quad \text{COB}=\frac{ra}{2}$$

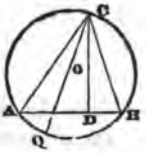
$$\therefore \text{ Whole area of triangle ABC}=\frac{r}{2}\,(a+b+c)=r\,.\,s$$

*i.e.* $\sqrt{s\,(s-a)\,(s-b)\,(s-c)}=r\,.\,s$ by last problem

$$\therefore \quad r=\frac{\sqrt{s\,(s-a)\,(s-b)\,(s-c)}}{s}$$

### PROBLEM III.

*To express the radius of a circle circumscribed about a given triangle, in terms of the sides of the triangle.*



Let fall CD perpendicular on AB
Let the radius be called R.
By Prop. XXXIX, B. IV. *El. Geom.*

$$\text{CQ}\,.\,\text{CD}=\text{AC}\,.\,\text{CB}$$

$$\therefore \quad \text{CQ}\,.\,\text{CD}\,.\,\text{AB}=\text{AC}\,.\,\text{CB}\,.\,\text{AB}$$

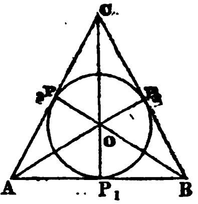
$$2\,\text{R}\times 2\text{ area}=abc$$

$$\text{R}=\frac{abc}{4\text{ area}}$$

$$=\frac{abc}{4\sqrt{s\,(s-a)(s-b)(s-c)}}$$

## PROBLEM IV.

*Given the three angles of a plane triangle, and the radius of the inscribed triangle, to find the sides of the triangle.*



Let A, B, C, be the three given angles, $r$ the radius,

$$AB \text{ or } c = AP_1 + P_1B$$

$$= r\left(\cot.\frac{A}{2} + \cot.\frac{B}{2}\right)$$

$$= r\,.\,\frac{\sin.\left(\frac{A}{2}+\frac{B}{2}\right)}{\sin.\frac{A}{2}\,\sin.\frac{B}{2}}$$

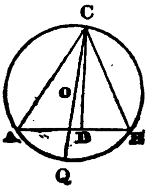
So,

$$AC \text{ or } b = r\,.\,\frac{\sin.\left(\frac{A}{2}+\frac{C}{2}\right)}{\sin.\frac{A}{2}\,\sin.\frac{C}{2}}$$

$$BC \text{ or } a = r\,.\,\frac{\sin.\left(\frac{B}{2}+\frac{C}{2}\right)}{\sin.\frac{B}{2}\,\sin.\frac{C}{2}}$$

## PROBLEM V.

*Given the three angles of a plane triangle, and the radius of the circumscribing circle, to find the sides of the triangle.*



As in Problem III.

$$CQ\,.\,CD = AC\,.\,CB$$

$$CQ\,.\,CB\sin.B = AC\,.\,CB$$

$\therefore$ $AC = 2\,R\sin.B$

So, $BC = 2\,R\sin.A$

$AB = 2\,R\sin.C$

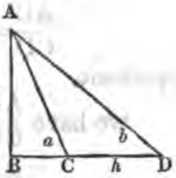
# CHAPTER X.

## PROBLEMS IN TRIGONOMETRICAL SURVEYING.

### THE DETERMINATION OF TOPOGRAPHICAL DATA BY GEOMETRICAL CONSTRUCTION AND TRIGONOMETRICAL ANALYSIS.

### PROBLEM I.

*To determine the height of an inaccessible object.*



Let AB be the object, and in a straight line towards it measure any distance DC, and observe the angles of elevation ADB, ACB at the stations D, C. Put $CD=h$, $ACB=a$, $ADB=b$: then $DAC=a-b$; hence we have

$$\frac{AB}{AC}=\sin. a;\ \frac{AC}{CD}=\frac{\sin. b}{\sin. (a-b)}$$

and, multiplying these two equations, we have

$$\frac{AB}{CD}=\frac{\sin. a \sin. b}{\sin. (a-b)}, \text{ or, } AB=h \sin. a \sin. b \operatorname{cosec.} (a-b)$$

$$\therefore \log.AB=\log.h+\log.\sin.a+\log.\sin.b+\log.\operatorname{cosec.}(a-b)-30 \quad (1)$$

*Cor.* Since $DB=AB \cot. b$, and $CB=AB \cot. a$; therefore, by subtraction, $CD=AB(\cot.b-\cot.a,)$ or $AB=\dfrac{h}{\cot.b-\cot.a}$ (2)

*Ex.* Let $DC=h=200$, $BDA=b=31°$, $BCA=a=46°$; to find AB and CB.

$\log. h$ - - - = $\log. 200 = 2.3010300$

$\log. \sin. a$ - - = $\log. \sin. 46° = 9.8569341$

$\log. \sin. b$ - - = $\log. \sin. 31° = 9.7118393$

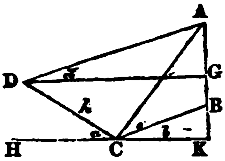
$\log. \operatorname{cosec.}(a-b) = \log. \operatorname{cosec.} 15° = 10.5870038$

$\log. AB$ - - $= \log. 286.29 = 2.4568072 \therefore AB=286.29$

Also, $BC=AB \cot. a$, $\therefore \log. BC=\log. AB+\log. \cot. a-10$.

### PROBLEM II.

*To determine the height of an inaccessible object, which has no level ground before it.*



Let AB be the object, and C, D, two stations in a vertical plane passing through AB; measure the distance CD, and at C take the angles of elevation or depression of the station D, and the top and bottom of the object. Also at D take the eleva- of the top of AB.

Put $CD=h$, $DCH=a$, $BCK=b$, $ACK=c$, $ADG=d$; then $ACB=c-b$, $ADC=a+d$, and $CAD=c-d$.

Now, $\dfrac{AB}{AC}=\dfrac{\sin. ACB}{\sin. ABC}=\dfrac{\sin.(c-b)}{\sin. ABK}=\dfrac{\sin.(c-b)}{\cos. b}$

$\dfrac{AC}{CD}=\dfrac{\sin. ADC}{\sin. CAD}=\dfrac{\sin.(a+d)}{\sin.(c-d)}$; hence, multiply these equations,

we have $\dfrac{AB}{CD}=\dfrac{\sin.(c-b)\sin.(a+d)}{\cos. b\ \sin.(c-d)}$; and, therefore,

$AB=h\sin.(c-b)\sin.(a+d)\sec. b\ \mathrm{cosec.}(c-d)$

$\therefore \log. AB=\log. h+\log.\sin.(c-b)+\log.\sin.(a+d)+\log. \sec. b+\log.\mathrm{cosec.}(c-d)-40$ - - - - - - - - - - (1)

*Cor.* When $a=90°$, and $b=0°$, then we have

$\log. AB=\log. h+\log.\sin. c+\log.\cos. d+\log.\mathrm{cosec.}(c-d)-30$ (2)

*Ex* 1. Let $h=18$ feet; $c=40°$; $d=37°30'$, $a=90°$ and $b=0°$; to find AB.

log. $h$ . . . =log. 18 . . . = 1.2552725
log. sin. $c$. . =log. sin. 40° . = 9.8080675
log. cos. $d$. . =log. cos. 37° 30′= 9.8994667
log. cosec.$(c-d)$=log. cosec.2° 30′=11.3603204

log. AB . . =log. 210.4394 = 2.3231271

$\therefore AB=210.4394$.

*Ex.* 2. The angle of elevation of the top of a tower, standing on a hill, was 33° 45′, and measuring on level ground 300 yards directly towards the tower, the angles of elevation of the top and bottom of the tower were 51° and 40° respectively. What is the height of the tower? Ans. 140 yds.

*Remark.*—When the station D is higher than A, the top of the tower, then the angle $d$ must be considered negative, and therefore we should have

$AB=h\sin.(c-b)\sin.(a-d)\sec. b\ \mathrm{cosec.}(c+d)$

### PROPOSITION I. LEMMA.

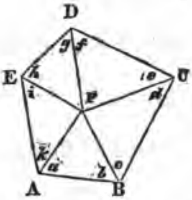
*If straight lines be drawn from any point, either within, or out of, a polygon, to all the angular points, the continued products of the sines of the alternate angles, made by the sides of the polygon, and the lines so drawn, will be equal.*

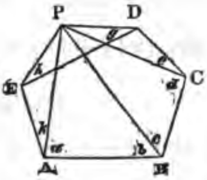
Let angle CDP$=f$, and PEA$=i$;

then $\frac{PA}{PB}=\frac{\sin. b}{\sin. a}$ $\quad\frac{PB}{PC}=\frac{\sin. d}{\sin. c}$

$\frac{PC}{PD}=\frac{\sin. f}{\sin. e}$ $\quad\frac{PD}{PE}=\frac{\sin. h}{\sin. g}$

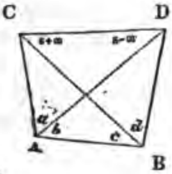
$\frac{PE}{PA}=\frac{\sin. k}{\sin. i.}$





But PA . PB . PC . PD . PE=PB . PC . PD . PE . PA; that is, the product of the numerators=product of the denominators, in the first members of these equations; hence, this being true in the second members also, sin. $b$ sin. $d$ sin. $f$ sin. $h$ sin. $k$=sin. $a$ sin. $c$ sin. $e$ sin. $g$ sin. $i$.

### PROBLEM III.



*Given* AB, *and the angles* a, b, c, d, *to find* $x$, *and* thence CD.

Put BCD+ADC$=b+c=2s$

BCD—ADC$=\ \ldots\ \ 2x$

Then BCD $= s+x$, ADC $= s-x$; also, sin. ADB $=$ sin. $(b+c+d)$, and sin. ACB $=$ sin. $(a+b+c)$; hence, by Lemma, (I) we have sin. $a$ sin $c$ sin. $(b+c+d)$ sin. $(s+x)$=sin. $b$ sin. $d$ sin. $(a+b+c)$ sin. $(s-x)$, or sin. $a$ sin. $c$ sin. $(b+c+d)$ $\{$sin. $s$ cos. $x$+cos. $s$ sin. $x\}$ $=$ sin. $b$ sin. $d$ sin. $(a+b+c)$ $\{$sin. $s$ cos. $x$—cos. $s$ sin. $x\}$.

Then, dividing by cos. $s$ cos. $x$, we have

$\sin. a \sin. c \sin. (b+c+d) (\tan. s+\tan. x)=\sin. b \sin. d \sin. (a+b+c) (\tan. s-\tan. x)$,

$$\therefore \tan. x=\frac{\sin. b \sin. d \sin.(a+b+c)-\sin. a \sin. c \sin.(b+c+d)}{\sin. b \sin. d \sin.(a+b+c)+\sin. a \sin. c \sin.(b+c+d)}\tan. s.$$

Dividing numerator and denominator by sin. $a$ sin. $c$ sin. $(b+c+d)$, and putting $\frac{\sin. b \sin. d \sin. (a+b+c)}{\sin. a \sin. c \sin. (b+c+d)} = \tan. \beta$ and $1 = \tan. 45°$; then

$$\tan. x = \frac{\tan. \beta - \tan. 45°}{\tan. \beta + \tan. 45°} \tan. s = \frac{\sin. (\beta - 45°)}{\sin. (\beta + 45°)} \tan. s;$$

hence $x$, $s+x$, $s-x$ are all known, and thence CD is known. For $\frac{CD}{BD} = \frac{\sin. d}{\sin. (s+x)}$ and $\frac{BD}{AB} = \frac{\sin. b}{\sin. (b+c+d)}$; therefore,

$$CD=AB \sin. b \sin. d \text{ cosec. } (s+x) \text{ cosec. } (b+c+d).$$

*Cor.* When CD is given, and the same angles, to find AB, we have

$$AB=CD \sin. (b+c+d) \sin. (s+x) \text{ cosec. } b \text{ cosec. } d.$$

### EXAMPLE.

Given AB=600 yards, $a=37°$, $b=58° 20'$, $c=53° 30'$, $d=45° 15'$, to find CD.

Here, $\tan. \beta=\text{cosec. } a \sin. b \text{ cosec. } c \sin. d \sin. (a+b+c) \text{ cosec. } (b+c+d)$,

| | | | | |
|---|---|---|---|---|
| log. cosec. $a$ | = log. cosec. | 37° | = 10.2205370 | |
| log. sin. $b$ | = log. sin. | 58° 20′ | = 9.9299891 | 9.9299891 |
| log. cosec. $c$ | = log. cosec. | 53 30 | = 10.0948213 | |
| log. sin. $d$ | = log. sin. | 45 15 | = 9.8513717 | 9.8513717 |
| log. sin. $(a+b+c)$ | = log. sin. | 148 50 | = 9.7139849 | |
| log. cosec. $(b+c+d)$ | = log. cosec. | 157 5 | = 10.4096131 | 10.4096131 |
| log. tan. $\beta$ | = log. tan. | 58° 56′ 39″ | = 10.2202671 | |
| log. sin. $(\beta-45°)$ | = log. sin. | 13 56 39 | = 9.3819742 | |
| log. cosec. $(\beta+45°)$ | = log. cosec. | 103 56 39 | = 10.0129906 | |

log. tan. $s$
= log. tan. 55 55 . . . = 10.1696508

log. tan. $x$
= log. tan. 20 9 3 = 9.5646156

log. cosec. $(s+x)$
= log. cosec. 76 4 3 = . . . . . . . . 10.0129687

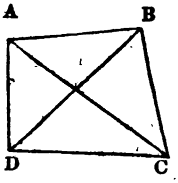
log. AB.
= log. 100 . . . . . . = . . . . . . . . 2.7781513

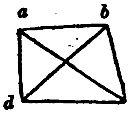
∴ CD . . . = 959.608 . . . = . . . . . . . . 2.9820939

PROBLEM IV.

*The distance of two objects at* B *being known, I find, by observation, the angles* ACD ADB BDC, *and* BCA, *taken at the stations* D, C, *required the distances* DC, AD, BD, BC, *and* CA, *both by construction and calculation.*

Assume *dc* at pleasure, and make the angles *adb*, *bdc* and *bca*, respectively equal to the angles ADB BDC and BCA; join *a*, *b*, and *abcd* will be similar to ABCD, and if *ab* represent the side AB, then will *dc* repersent DC. &c.





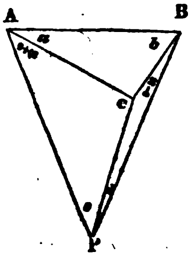
*By analysis,*

In the triangle *adc*, all the angles and the assumed side, *dc* are given to find *ad* and *ac*. Then in the triangle *bcd*, all the angles and the assumed side *dc*, are given to find *bc*, *bd*. Lastly, in the triangle *adb*, we have the sides *ad*, *db*, and the angle *adb*, to find the side *ab*. Then we have,

*ab* : AB : : *dc* : DC : : *ad* : AD : : *bc* : BC : : *ac* : AC : : *bd* : BD, whence we have the distances, DC, AD, BC, AC and BD.

### PROBLEM V.

*Given* AB, a, b, *and the angles* c, d, *taken at some point* P *in the same plane* ABC, *to find* x; *and thence* PA, PB, PC.



Put PAC+PBC$=180°-(a+b+c+d)=2s$
PAC—PBC= . . . . . . . $=2x$;
Then, PAC$=s+x$, PBC$=s-x$, and, by lemma 1, $\sin. a \sin. c \sin (s-x)=\sin. b \sin. d \sin. (s+x)$

$$\therefore \frac{\sin. b \sin. d}{\sin. a \sin. c}=\frac{\sin. (s-x)}{\sin. (s+x)}=\frac{\tan. s-\tan. x}{\tan. s+\tan. x}$$

Put $\tan \beta = \dfrac{\sin. b \sin. d}{\sin. a \ \sin c} = \text{cosec}\ a \sin b$ cosec. $c$ sin. $d$; then we have

$$\frac{\tan. s-\tan. x}{\tan. s+\tan. x}=\tan. \beta \therefore \frac{\tan. x}{\tan. s}=\frac{1-\tan. \beta}{1+\tan. \beta}=\frac{\tan. 45°-\tan. \beta}{\tan. 45°+\tan. \beta};$$

$$\therefore \tan. x=\frac{\tan. 45°-\tan. \beta}{\tan. 45°+\tan. \beta} \tan. s=\frac{\sin. (45°-\beta)}{\sin. (45°+\beta)}\tan. s.$$

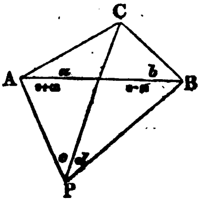
Hence $x$ is known, and thence $s+x$ and $s-x$ are known.

$$\text{Then } \frac{\text{PC}}{\text{AC}}=\frac{\sin. (s+x)}{\sin. c}, \frac{\text{AC}}{\text{AB}}=\frac{\sin. b}{\sin. (a+b)}; \text{ hence}$$

$$\text{PC} = \text{AB cosec.} (a+b) \sin. b \text{ cosec. } c \sin. (s+x).$$

### PROBLEM VI.

*When the points* P *and* C *are on opposite sides of* AB.



Put PAB+PBA$=180°-(c+d)=2s$
PAB—PBA=. . . . . . . . $=2x$;
then, PAB$=s+x$, PBA $= s-x$; and, by lemma 1, $\sin. a \sin. c \sin. (s-x) = \sin. b \sin. d \sin. (s+x)$;
hence, as in the last problem, we have

$$\tan. x=\frac{\sin. (45°-\beta)}{\sin. (45°+\beta)} \tan. s;$$

where $\tan. \beta = \text{cosec. } a \sin. b \text{ cosec. } c \sin. d$; and $2s = 180°-(c+d)$.

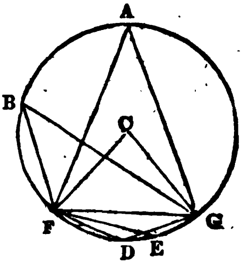
## PART II.

Many curious and highly useful problems in trigonometrical surveying, may be elegantly solved through the properties of the circle, by geometrical construction, and by analyzing this construction by trigonometrical analysis; a few of which we will give in the continuation of this chapter.

### PROPOSITION II. LEMMA.

*If two points be assumed in the circumference of a circle, they will subtend the same angle from any point whatever of the circumference on the same side of the chord joining these two points, which will be half the angle at the centre when the centre is on the same side of that chord as the point of observation, and will be equal to half its complement to 360° when on the opposite side.*

This proposition is evident from Prop. XIX. *Cors.* 1 and 3, B. III. *El. Geom.*



Hence, if F, G are two points assumed in the circumference ABFG, then will F, G appear under the same angle from any points A and B situated in the circumference, and on the same side of the chord FG; which will be half the angle C at the centre; and the points F, G will also appear under equal angles at every point D, E on the side DE of the chord FG, which will be equal to the angle measured by half the arc FBAG equal the complement of the angle FCG as enunciated.

*Cor.* 1. Hence, any two objects in the circumference of a circle will always appear under the same angle, in any point of the arc of either segment, and in no other point situated out of that circumference, on the same side of the objects, will the angle be the same.

*Cor.* 2. If the angle under which any two objects appear be less than 90 degrees, the place of observation will be some where in the arc of the greater segment; and if the angle be greater than 90 degrees, the place of observation must be some where in the arc of a segment less than a semicircle, and the angles under which the objects appear, will be the same in any point whatever of those arcs.

*Scholium.* Hence, having the angles subtended by any two objects from any two given positions, not all in the same circumference, the positions of the objects may be determined by the intersections of the circumferences of two circles, each of which is so described as to pass through the two objects and one of the given positions.

### PROBLEM VII.

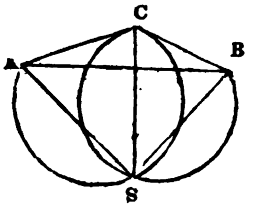
*Three points in the same plane being given in position, to determine the position of any other point or place of observation in reference to the given points.*

This problem admits of six cases.

The three given points may be the vertices of a triangle, and the required point, or station, may be without the triangle, and opposite one of its sides; it may fall in the same right line with two of the given points, it may fall directly between two of them, it may fall within the triangle or it may fall without the triangle but opposite one of the angles; and lastly, the given points may be all in the same straight line.

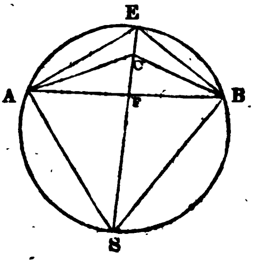
*Case* 1.

*When the given points are the vertices of a given triangle, and the station regarded, falls without the triangle and opposite one of its sides.*



Let A, B, C, be the given points whose positions in reference to each other are known, and let S be the point required. Having taken the angles ASC, ASB, describe on AC the segment of a circle that shall contain an angle equal the observed angle ASC; and on CB describe a segment that shall contain an angle equal to the angle CSB, and the point of intersection of the arcs of those segments will determine the position S.

Or,



Make the angle EBA = the observed angle ASC, and the angle BAE = the angle BSE; through A, B, and the intersection at E, describe the circle AEBS; through E and C draw EC, which produce to meet the circumference at S. Join AS, BS, and the distances AS, CS, BS will be the required distances of the station S, from the points A, B, and C.

*By trigonometrical analysis.*

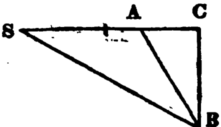
In the triangle ABC, the three sides are given to find the angle BAC. And in the triangle AEB, we have *angle* EAB = *angle* BSE, *angle* ABE = *angle* ASE, and therefore the *angle* AEB, with the side AB, to find AE and BE. Also in the triangle AEC, we have the sides AC, AE, and the included angle to find the angle AEC. Whence the sum of the angles AES, and the observed angles ASE, subtracted from 180° gives the angle SAE. Then in the triangle AES, we have all the angles and the side AE, to find the side AS. And the *angle* ACS = 180°—*angle* SAC—*angle* ASC; then in the triangle ACS we have all the angles and the side AC to find CS. *Angle* AEB—*angle* AEC = *angle* BES, whence we have the side BE of the triangle BES, and the angles E and S, to find BS; then AS, CS, and BS are the station distances required.

*Scholium.* 1st. If the angle BSC, be less than CAB, the point E will be below the point C.

2. When the points E and C fall so near each other that the production of EC toward S is attended with uncertainty, the former method of construction is preferred.

*Case* 2.

*Let it be required to determine the position of an observer at S in reference to the three objects ABC, when SAC are in the same right line.*



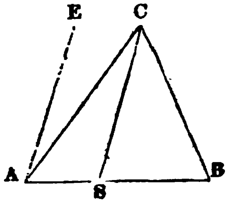
Having taken the angle at S, and calculated the angle CAB from the sides of the triangle ABC which are known by hypothesis, we have the angle ABS=*ang.* CAB—*ang.* ASB. Then at B with the side BA, construct the angle ABS, produce the side, BS till it meets the production of CA in S, and SA, SC, SB will be the several distances of S from the points A, C, B, whence having the angles S, C and B and the side BC, the sides SB, SC and SA may be obtained.

*By trigonometrical analysis.*

Angle SAB = 180°—BAC, angle SBA = CAB—ASB. Hence in the triangle SAB, we have all the angles and the side AB to find the distances AS, BS, &c.

*Case* 3.

*To determine the position of S in reference to three given objects ABC, where the required point is directly between A and B.*



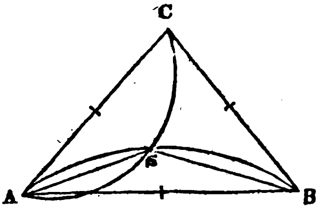
Having constructed the triangle ABC which is given by hypothesis, and having observed the angle BSC, construct from any point A on the line AB an angle BAE=the observed angle and draw CS parallel to EA, and S is the required station.

*By Analysis.*

In the triangle ABC all the sides being known, let the angle A, B be obtained. Then in the triangle BCS having the angle S and B, and the side CB we may proceed to find the distances BS, CS. Then we shall find AS=AB—SB.

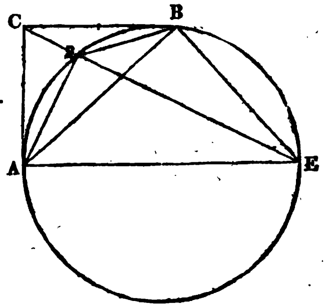
*Case 4.*

*To determine the position of a station S in reference to three given objects when the required station falls within the triangle formed by connecting those given objects.*



Let the given objects be three towns A, B and C which are all visible from a station S, which is included in the triangle formed by lines drawn from A to B, from B to C, and from C to A.

First take the angles ASB, BSC, CSA, then on either side AC describe an arc ASC, which shall contain the observed angle ASC, and one either of the other sides AB, describe an arc which will contain the angle ASB, and the point of intersection of these arcs is the station S, all of which is evident from Lemma II.



*Otherwise,* on AB make an angle ABE=the supplement of the angle ASC; and make an angle BAE equal to the supplement of the angle BSC, then will ABE=ASE and BAE = BSE, since ASE and BSE are respectively the supplements of the angles ASC and BSC, through the points A, B, and E describe a circle, join EC cutting the circumference in the point S which is the station required.

*By Analysis.*

1st. In the triangle AEB the angles B and A being the supplements of the observed angles CSA, CSB, are therefore known, and consequently the angle E and the side AB, to find the sides AE, BE.

2d. In the triangle ECA we have the sides EA, AC and their included angle EAC=BAE+CAB, to find the angle AEC.

3d. And the triangle CEB the sides CB, BE, and the included angle CBE are given to find the angle CEB.

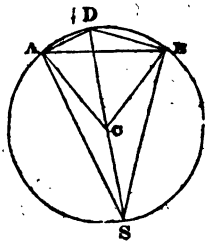
4th. Therefore in the triangle SAB the angle A, being equal the angle E, since they are angles in the same segment, is also known and also the angle ASB, hence we have all the angles and the side AB to find the sides AS, BS which are two of the station distances required.

5th. If from the angle CAB we take the angle SAB, we shall have the angle CAS. Therefore in the triangle ACS we have all the angles with the sides AC and AS to find the other station distance CS.

*Case 5.*

*Let it be required to find the distance of any station S from each of three objects A, B, C, when one of the angles C of the triangle formed by connecting the three objects falls toward the station S.*

Make the angle DAB=the observed angle CSB and the angle DBA, equal to the observed angle CSA. On AB describe a circle that shall contain in its greater segment the observed angle ASB, through D and C draw the line DC, till it intersects the circle at S, which intersection determines the position S.



*By Analysis.*

1st. In the triangle DAB, all the angles and side AB are known to find AD, DB.

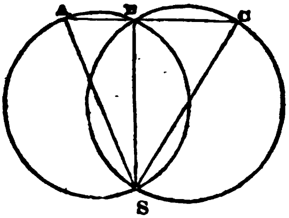
2d. In the triangle ADC are given the sides AD, AC and then included angle, to find the angle ACD.

3d. The angle CAB=ACD—ASD, since ACD, the outward angle is equal to the sum of two inward opposite angles CSA, CAS. Therefore in the triangle ACS we have all the angles and the side AC to find the distances SA, SC.

*Lastly*, in the triangle BSC the sides CB CS, and the angle BSC are given to find the distance CB.

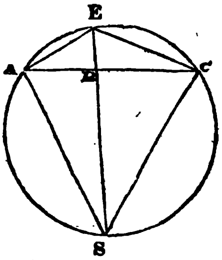
*Case 6.*

*To determine the position of any station S in reference to three objects ABC all in the same straight line.*



On AB describe an arc of a segment to contain an angle equal to the angle ASB, and on BC describe an arc containing an angle CSB=the observed angle, subtended by BC, and the point of intersection of those arcs will determine the position of S; whence if we draw the lines SA, SB, SC, those lines will the several distances of S from the objects A, B, C.

For (Lemma II) the points A, B appear under the same angle in every point in the arc BSA, and in no point out of that arc, and B, C, appears under the same angle in every part of the arc CSB and no point out of the arc. Hence the point of intersection of those arcs is the only point where both of those conditions are united, or where both of the objects appear under the observed angles.



*Otherwise,* at A and on the line AB make BAE=the observed angle CSB and at C an angle=the observed angle ASB; on AC describe a circle that shall contain an angle=the sum of the observed angles at S or which is the same, describe a circle which shall pass through the three angles A, E and C, from E through the point B, drawn EBS to cut the circle in S, and SA, SB, SC determine the relative position of the station S in reference to the three A, B and C.

*By Trigonometrical analysis.*

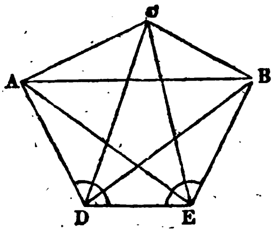
1st. In the triangle CAE all the angles are given and the side AC to find AE.

2d. In the triangle AEB the sides AE, AB, and their included angle are given to find the angles AEB and ABE.

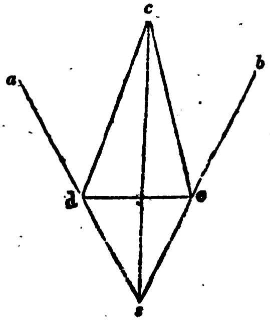
3rd. In the triangle BSC we have the angle CSB and the angle SBC=ABE, and the angle SCB=angle AEB since they are both angles in the same segment ACS, hence all the angles and the sides BC are given to find the side, SC, SB. and SA.

### PROBLEM VIII.

*The distances of three objects* A, B *and* C *being given and consequently the angles which they form with each other. There are also two stations* D, E, *such that at* D *the objects* A, C *and* E *may be seen but not* B; *at* E *the object*, B, C *and* D *may be seen but not* A. *Hence the angles* CDE ADC BED BEC *and consequently the angles* CDA *and* CEB *are given or known from observation to find the distances* DA, DC, DE, EC, *and* EB.



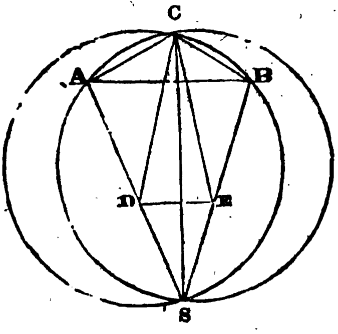
Draw $cd$ at pleasure, and at $d$ make an angle $cde$ = the given angle CDE make also the angle $dec$ = DEC, the angle $ceb$ = CEB, and $cda$ = CDA produced $ad$, $bc$, till they meet in $s$, and draw $sc$.



*By Analysis.*

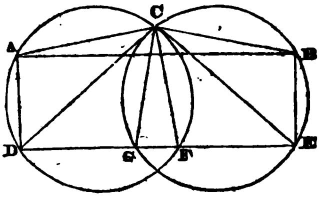
Assume any value for $de$; then in the triangle $cde$ all the angles are given and the side $de$, to find the sides $cd$, $ce$, the angle $eds=180^\circ-ade$ and the angle $des=180^\circ-bed$; hence we have in the triangle $dse$ all the angles and side $de$, to find $ds$, and $es$. In the triangle $cds$ the angle $cds=180^\circ-adc$, hence we have two sides $cd$ $ds$, and their included angle to find the angle $dsc=csa$ and side $cs$, then from the angle $dsc$ take the angle $dsc$ and we have the angle $cse=csb$.

Then with the angles $csa$, $csb$ and the triangle ABC, as data, make the following construction by case first Prop. VII, and find the station distances, SA, SB, SC. Then we shall have



*cs* : CS : : *de* DE : : *dc* : DC : : *ec* : EC : : *ds* : DS : : *se* : SE from SA take SD and there remains DA, also from SB take SE and there remains BE, hence DA, DC, DE, EC and EB are found.

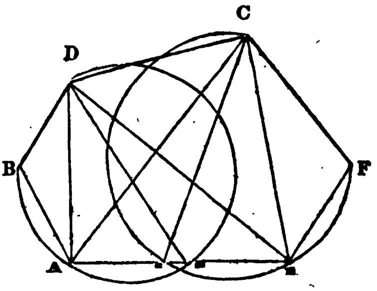
*Scholium.*



When AD and BE are parallel, the foregoing method of solution fails. In which case, on AC describe a segment to contain an angle equal to the observed angle CDE, and on CB a segment to contain an angle equal to CEB, draw the chord CF to cut off the segment CADF containing the angle CDE, and another chord CG cutting off a segment CBEG, containing the angle CED, the points F and G will be in the same right line with D and E, join GF which produce both ways till it cuts the circumference in D and E, and the points D and E will be the stations required.

PROBLEM IX.

*The relation of the four points* B,C,D,F, *to each other are known, or the four sides of a quadrilateral figure and its angles are known, there are also two stations* A, E, *such that at* A *only* B, C, E, *are visible and at* E *only the points* D, F, A, *so that the angles* BAC, BAE, AED *and* DEF *and consequently* AEF *may be known, required the distances* AB, AC, ED, EF, EC, *and* AD.



1st. On BC describe a segment to contain the angle BAC, and draw the chord C$m$ that shall cut off an angle BC$m$ = the supplement of the angle BAE.

2nd. On DF describe a segment that shall contain an angle DEF, and draw the chord D$n$, that shall cut off an angle FD$n$ = the supplement of the angle AEF, and the intersections $m$ and $n$ will be in the same right line with the stations A and E.

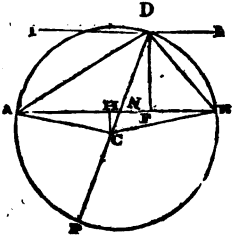
3d. Through the points of intersection $m$ and $n$, draw the line $mn$ which produce to E at its intersection with the arc of the segment and the points of intersection, A and E will be the points of station.

The analysis of this may be supplied by the student.

### PROBLEM X.

*Given the base AB, the perpendicular FD, and vertical angle ADB of a triangle to find the sides AD, DB.*

On the base AB describe an isosceles triangle ACB whose vertical angle C shall be double the given angle ADB, if that angle is less than 90°, but double its supplement if the angle ADB is greater than 90°, and from this vertice as a centre, with the radius CA or CB describe a circle, and the vertice D of the triangle will be found somewhere in the circumference (Lemma II.) At a distance FD equal to the altitude of the given triangle, draw a right line IL parallel to AB, and the point where this line cuts the circumference will determine the position of the vertical angle; hence the sides DA and DB may be drawn.



*By Analysis.*

Draw the diameter DCP, and from C draw CH perpendicular to AB. Hence having the angle ACB we have also the angles CAB and ABC each equal to $\frac{180°-ACB}{2}$; therefore in the triangle ABC having all the angles and the side AB, the two equal sides AC, BC, become known also. And in the right-angled triangle AHC, $CH=\sqrt{AC^2-AH^2}$.

And in the similar triangles DFN, CHN we have DF : DN : : CH : CN

and by composition : : DF+CH : DN+CN

or DF+CH : DC : : DF : DN

Also $HN=\sqrt{CN^2-CH^2}$;

and $BN=\frac{1}{2}AB-HN$

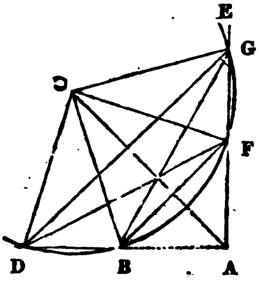
$FN=\sqrt{DN^2-DF^2}$

and BF=BN—NF

Whence we have the right-angled triangled BFD with the sides BF and DF including the right angle, to determine the third side BD which also becomes known, and consequently the side AD as required.

### PROBLEM XI.

*At the distance AB from the bottom of a tower is an object whose length is BD, how far must I ascend the tower that the object may appear under any angle T?*



Draw the line AB=the given distance, and produce it to D, then will BD represent the object; draw the indefinite line AE perpendicular to AB, which will represent the side of the tower. Then on DB make an angle BCD=2T, and from the vertex C as a centre, describe an arc of a circle passing through the points D and B, cutting the tower in F, and AF will be the distance required.

For since DCB is an angle in the centre of a circle, and AFB is an angle in the circumference subtending the same arc, hence (Prop. II.) the *angle* DFB=½ *angle* DCB=T.

*Cor.* Since by continuing the arc of the circle the perpendicular is cut also in G; this point also answers the condition of the question, for *angle* DGB is evidently=the *angle* DFB, hence, the question admits of two answers.*

*By Analysis.*

First, in the isosceles triangle CBD we have the side BD and the angle C by construction, and since the triangle is isosecles, the angles B and D are each = (180° — 2T)÷2 =90°—T. Hence, having all the angles and one side, the sides CB or CD are also known.

Second, in the triangle ABC we have the sides AB and BC, and the angle B=CBD+½ angle BCD=90°+T, to find the side AC and the angle CAB, which thereby become known.

Third, in the triangle ACF we have the sides AC and FC, and the angle CAF=90°—CAB, to find the side AF, the height required. But since the same data given to determine this triangle, apply also to the triangle ACG, the point may be also in G; hence, the problem is ambiguous both by construction and analysis, as explained in case 4th, chap. VII.

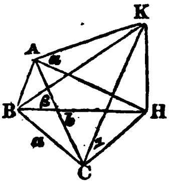
---

* This elegant construction was received from Mr. Joseph Gallup, of Norwich, Connecticut, whose mathematical talent is acknowledged to be of a high order.

*Scholium.* If the circumference which passes through the points D, B should not cut the edge of the tower or perpendicular AE, but only touch it, it would admit of only one solution, and that point which would answer the conditions would be the point of contact; but if the circle should not reach the perpendicular, the question would be impossible.

### EXAMPLES FOR PRACTICE.

Ex. 1. Given the angles of elevation of any distant object, taken at three places on a level plane, no two of which are in the same vertical plane with the object; to find the height of the object, and its distance from either station.



Let A, B, C, be the three stations, K the object, and KH perpendicular to the plane of the triangle ABC.

Put BC$=a$, AC$=b$, AB$=c$, HAK$=\alpha$, HBK$=\beta$, HCK$=\gamma$, and HK$=x$; then the angles AHK, BHK, CHK being right angles, we have AH$=x$ cot. $\alpha$, BH$=x$ cot. $\beta$, CH$=x$ cot. $\gamma$; whereby from the given data the required may be found.

Ex. 2. Given $\alpha=30° 40'$, $\beta=40° 33'$, $\gamma=50° 23'$; find $x$, when the three stations are in the same straight line, AB being$=50°$ and BC$=60$ yards. Ans. 77.7175 yards.

Ex. 3. Demonstrate that sin. $18°=$cos. $72°$ is $=\frac{1}{4}$ R $(-1+\sqrt{5})$, and sin. $54°=$cos. $36°$ is$=\frac{1}{4}$ R $(1+\sqrt{5})$.

Ex. 4. Demonstrate that the sum of the sines of two arcs which together make 60°, is equal to the sine of an arc which is greater than 60°, by either of the two arcs: Ex. gr. sin. $3'+$sin. $59° 57'=$sin. $60° 30'$; and thus that the tables may be continued by addition only.

Ex. 5. Show the truth of the following proportion: As the sine of half the difference of two arcs, which together make 60°, or 90°, respectively, is to the difference of their sines; so is 1 to $\sqrt{2}$, or $\sqrt{3}$, respectively.

Ex. 6. Demonstrate that the sum of the square of the sine and versed sine of an arc, is equal to the square of double the sine of half the arc.

Ex. 7. Demonstrate that the sine of an arc is a mean proportional between half the radius and the versed sine of double the arc.

Ex. 8. Show that the secant of an arc is equal to the sum of the tangent and the tangent of half its complement.

Ex. 9. Prove that, in any plane triangle, the base is to the difference of the other two sides, as the sine of half the sum of the angles at the base, to the sine of half their difference: also, that the base is to the sum of the other two sides as the cosine of half the sum of the angles at the base, to the cosine of half their difference.

Ex. 10. How must three trees, A, B, C, be planted, so that the angle at A may double the angle at B, the angle at B double that at C; and so that a line of 400 yards may just go round them?

Ex. 11. In a certain triangle, the sines of the three angles are as the numbers 17, 15, and 8, and the perimeter is 160. What are the sides and angles?

Ex. 12. The logarithms of two sides of a triangle are 2.2407293 and 2.5378191, and the included angle, is 37° 20′. It is required to determine the other angles, without first finding any of the sides?

Ex. 13. The sides of a triangle are to each other as the fractions $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$: what are the angles?

Ex. 14. Show that the secant of 60°, is double the tangent of 45°, and that the secant of 45° is a mean proportional between the tangent of 45° and the secant of 60°.

Ex. 15. Demonstrate that four times the rectangle of the sines of two arcs, is equal to the difference of the squares of the chords of the sum and difference of those arcs.

Ex. 16. Convert formulæ $\zeta$, Chap. III, into their equivalent logarithmic expressions; and by means of them and formulæ $\beta$, Chap. III, find the angles of a triangle whose sides are 5, 6, and 7.

Ex. 17. Being on a horizontal plane, and wanting to ascertain the height of a tower, standing on the top of an inaccessible hill, there were measured, the angle of elevation of the top of the hill 40°, and the top of the tower 51°: then measuring in a direct line 180 feet farther from the hill, the angle of elevation of the top of the tower was 33° 45′: required the height of the tower. Ans. 83.9983 feet.

Ex. 18. From a station P there can be seen three objects, A, B, and C, whose distance from each other are known, viz. AB=800, AC=600, and BC=400 yards. There are also measured the horizontal angles APC=33° 45′, BPC=22° 30′. It is required, from these data, to determine the three distances PA, PC, and PB.

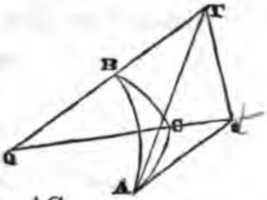
Ans. PA=710.193, PC=1042.522, PB=934.191 yards.

# SPHERICAL TRIGONOMETRY.

Having demonstrated in the treatise on Spherical Geometry, several important properties of the circle of the sphere, and of spherical triangles, we shall now proceed to deduce various relations which exist between the several parts of a spherical triangle. These constitute what is called *Spherical Trigonometry;* and enables us, when a certain number of the parts are given, to determine the rest. The first formula which we shall establish, serves as a key to the rest, and is to spherical trigonometry what the expression for the sine of the sum of two angles is to plane trigonometry.

## CHAPTER I.

### 1. *To express the cosine of an angle of a spherical triangle in terms of the sines and cosines of the sides.*



Let ABC be a spherical triangle, O the centre of the sphere.

Let the angles of the triangles be denoted by the large letters A, B, C, and the sides opposite to them by the corresponding small letters, $a$, $b$, $c$.

At the point A, draw AT a tangent to the arc AB, and A$t$ a tangent to the arc AC.

Then the spherical angle A is equal to the angle TA$t$ between the tangents, (Spher. Geom. Prop. VII.)

Join OB, and produce it to meet AT in T.

Join OC, and produce it to meet A$t$ in $t$.

Join T, $t$;

Then,

$$\frac{OT}{OC}=\text{sec. AB}=\text{sec. } c$$

$$\frac{Ot}{OC}=\text{sec. AC}=\text{sec. } b$$

$$\frac{AT}{OC}=\text{tan. AB}=\text{tan. } c$$

$$\frac{At}{OC}=\text{tan. AC}=\text{tan. } b$$

Then in triangle T*Ot*

$$Tt^2=OT^2+Ot^2-2OT\,.\,Ot\text{ cos. T}Ot$$

$$\therefore\quad \frac{Tt^2}{OC^2}=\frac{OT^2}{OC^2}+\frac{Ot^2}{OC^2}-2\cdot\frac{OT}{OC}\cdot\frac{Ot}{OC}\text{ cos. T}Ot$$

$$=\text{sec.}^2 c+\text{sec.}^2 b-2\text{ sec. } c\text{ sec. } b\text{ cos. } a, \text{ - } (1)$$

Again, in triangle TA*t*

$$Tt^2=AT^2+At^2-2AT\,.\,At\text{ cos. TA}t$$

$$\therefore\quad \frac{Tt^2}{OC^2}=\frac{AT^2}{OC^2}+\frac{At^2}{OC^2}-2\cdot\frac{AT}{OC}\cdot\frac{At}{OC}\text{ cos. TA}t$$

$$=\text{tan.}^2 c+\text{tan.}^2 b-2\text{ tan. } c\text{ tan. } b\text{ cos. A. - - } (2)$$

Equating (1) and (2)

$$\text{tan.}^2 c+\text{tan.}^2 b-2\text{ tan. } c\text{ tan. } b\text{ cos. A}$$

$$=\text{sec.}^2 c+\text{sec.}^2 b-2\text{ sec. } c\text{ sec. } b\text{ cos. } a$$

$$=1+\text{tan.}^2 c+1+\text{tan.}^2 b-2\text{ sec. } c\text{ sec. } b\text{ cos } a$$

$$\therefore\quad -2\text{ tan. } c\text{ tan. } b\text{ cos. A}=2-2\text{ sec. } c\text{ sec. } b\text{ cos. } a$$

$$\text{or, }-\frac{\text{sin. } c}{\text{cos. } c}\cdot\frac{\text{sin. } b}{\text{cos. } b}\text{cos. A}=1-\frac{1}{\text{cos. } c}\cdot\frac{1}{\text{cos. } b}\text{ cos. } a$$

$$\therefore\quad \text{cos. A}=\frac{\text{cos. } a-\text{cos. } b\text{ cos. } c}{\text{sin. } b\text{ sin. } c}$$

Similarly we shall have,

$$\text{cos. B}=\frac{\text{cos. } b-\text{cos. } a\text{ cos. } c}{\text{sin. } a\text{ sin. } c}\quad(\alpha)$$

$$\text{cos. C}=\frac{\text{cos. } c-\text{cos. } a\text{ cos. } b}{\text{sin. } a\text{ sin. } b}$$

2. *To express the cosine of a side of a spherical triangle, in terms of the sines and cosines of the angles.*

Let A, B, C, *a*, *b*, *c*, be the angles and sides of a spherical triangle; A′, B′, C′, *a*′, *b*′, *c*′, the corresponding qualities in the Polar triangle,

Then by (α),

$$\cos. A' = \frac{\cos. a' - \cos. b' \cos. c'}{\sin. b' \sin. c'}.$$

But (Spherical Geometry, Prop. X.), $A' = (180° - a')$, $a' = (180° - A)$, $b' = (180° - B)$, $c' = (180° - C)$,

$$\therefore \cos.(180° - a) = \frac{\cos.(180° - A) - \cos.(180° - B)\cos.(180° - C)}{\sin.\ (180° - B)\ \sin.\ (180° - C)}$$

$$\left.\begin{aligned} \therefore \quad \cos. a &= \frac{\cos. A + \cos. B \cos. C}{\sin. B \sin. C} \\ \text{Similarly,} \quad \cos. b &= \frac{\cos. B + \cos. A \cos. C}{\sin. A \sin. C} \\ \cos. c &= \frac{\cos. C + \cos. A \cos. B}{\sin. A \sin. B} \end{aligned}\right\} (\beta.)$$

3. *To express the sine of an angle of a spherical triangle, in terms of the sines of the sides of the triangle.*

By ($\alpha$) we have,

$$\cos. A = \frac{\cos. a - \cos. b,\ \cos. c}{\sin. b \sin. c}$$

$$\therefore 1 + \cos. A = \frac{\cos. a - \cos. b \cos. c + \sin. b \sin. c}{\sin. b \sin. c}$$

$$= \frac{\cos. a - (\cos. b \cos. c - \sin. b \sin. c)}{\sin. b \sin. c}$$

$$= \frac{\cos. a - \cos. (b + c)}{\sin. b \sin. c}$$

$$= \frac{2 \sin. \frac{a+b+c}{2} \sin. \frac{b+c-a}{2}}{\sin. b \sin. c} \text{ (Plane Trig. Ch. II.)}$$

$$\text{Let } s = \frac{a+b+c}{2}$$

$$\therefore s - a = \frac{b+c-a}{2}$$

$$s - b = \frac{a+c-b}{2}$$

$$s - c = \frac{a+b-c}{2}$$

$$\therefore 1 + \cos. A = \frac{2 \sin. s \sin. (s-a)}{\sin. b \sin. c} \quad \text{- - - - - - -} \quad (1.)$$

Again, resuming the expression for cos. A,

$$1-\cos. A = \frac{\cos. b \cos. c+\sin. b \sin. c-\cos. a}{\sin. b \sin. c}$$

$$=\frac{\cos. (b-c)-\cos. a}{\sin. b \sin. c}$$

$$=\frac{2 \sin. \frac{a+b-c}{2} \sin. \frac{a+c-b}{2}}{\sin. b \sin. c}$$

$$=\frac{2 \sin.(s-c)\sin.(s-b)}{\sin. b \sin. c} \quad . \ . \ . \ . \quad (2.)$$

Multiplying equations (1) and (2)

$$1-\cos.^2 A=\frac{4 \sin. s \sin.(s-a)\sin.(s-b) \sin.(s-c)}{\sin.^2 b \sin.^2 c.}$$

$$\left.\begin{array}{l} \therefore \sin. A=\dfrac{2}{\sin. b \sin. c}\sqrt{\sin. s \sin. (s-a) \sin.(s-b)\sin.(s-c)} \\ \text{Similarly,} \\ \sin. B=\dfrac{2}{\sin. a \sin. c}\sqrt{\sin. s \sin. (s-a)\sin.(s-b)\sin.(s-c)} \\ \\ \sin. C=\dfrac{2}{\sin. a \sin. b}\sqrt{\sin. s \sin (s-a)\sin.(s-b)\sin.(s-c)} \end{array}\right\} (\gamma.1$$

Now, by equation (1) we have,

$$1+\cos. A = \frac{2 \sin. s \sin.(s-a)}{\sin. b \sin. c}$$

or

$$2 \cos.^2\frac{A}{2} = \frac{2 \sin. s \sin. (s-a)}{\sin. b \sin. c}$$

$$\left.\begin{array}{l} \therefore \cos. \dfrac{A}{2} \sqrt{\dfrac{\sin. s \sin. (s-a)}{\sin. b \sin. c}} \\ \text{Similarly,} \\ \cos. \dfrac{B}{2} \sqrt{\dfrac{\sin. s \sin. (s-b)}{\sin. a \sin. c}} \\ \\ \cos. \dfrac{C}{2} \sqrt{\dfrac{\sin. s \sin. (s-c)}{\sin. a \sin. b}} \end{array}\right\} (\gamma.2)$$

Next, by equation (2),

$$1-\cos. A = \frac{2 \sin. (s-b)\sin. (s-c)}{\sin. b \sin. c}$$

Or,

$$2\sin.^2\frac{A}{2}=\frac{2\sin.(s-b)\sin.(s-c)}{\sin.\, b\sin.\, c};$$

$$\left.\begin{array}{l}\therefore\ \sin.\dfrac{A}{2}=\sqrt{\dfrac{\sin.\,(s-b)\sin.\,(s-c)}{\sin.\, b\sin.\, c}}\\ \text{Similarly,}\\ \sin.\dfrac{B}{2}=\sqrt{\dfrac{\sin.\,(s-a)\sin.(s-c)}{\sin.\, a\sin.\, c}}\\ \\ \sin.\dfrac{C}{2}=\sqrt{\dfrac{\sin.\,(s-a)\sin.\,(s-b)}{\sin.\, a\sin.\, b}}\end{array}\right\}(\gamma.\ 3)$$

Finally, dividing the expressions ($\gamma$. 3 by those $\gamma$. 2), we obtain,

$$\left.\begin{array}{l}\tan.\dfrac{A}{2}=\sqrt{\dfrac{\sin.\,(s-b)\sin.\,(s-c)}{\sin.\, s\sin.(s-a)}}\\ \\ \tan.\dfrac{B}{2}=\sqrt{\dfrac{\sin.\,(s-a)\sin.(s-c)}{\sin.\, s\sin.(s-b)}}\\ \\ \tan.\dfrac{C}{2}=\sqrt{\dfrac{\sin.(s-a)\sin.(s-b)}{\sin.\, s\sin.\,(s-c)}}\end{array}\right\}(\gamma.\ 4)$$

4. *To express the sine of a side of a spherical triangle in terms of the sines and cosines of the angles.*

By ($\beta$) we have,

$$\cos.\, a=\frac{\cos.\, A+\cos.\, B\cos.\, C}{\sin.\, B\sin.\, C}$$

$$\therefore\ 1+\cos.\, a=\frac{\cos.\, A+\cos.\, B\cos.\, C+\sin.\, B\sin.\, C}{\sin.\, B\sin.\, C}$$

$$=\frac{\cos.\, A+\cos.\,(B-C)}{\sin.\, B\sin.\, C}$$

$$=\frac{2\cos.\dfrac{A+B-C}{2}\cos.\dfrac{A+C-B}{2}}{\sin.\, B\sin.\, C}\ (\text{Plane Trig. Ch. II.})$$

Let

$$s'=\frac{A+B+C}{2},\ \therefore\ s'-A=\frac{B+C-A}{2},\ s'-B=\frac{A+C-B}{2},$$

and,

$$s'-C=\frac{A+B-C}{2}$$

Hence,

$$1+\text{cos. } a=\frac{2\text{ cos. }(s'-\text{C})\text{ cos. }(s'-\text{B})}{\text{sin. B sin. C}}\quad \text{. . . . . .}\quad (1.)$$

Resuming expression for cos. $a$,

$$\begin{aligned}1-\text{cos. } a &= -\frac{\text{cos. B cos. C}-\text{sin. B sin. C}+\text{cos. A}}{\text{sin. B sin C}}\\ &= -\frac{\text{cos. (B+C)}+\text{cos. A}}{\text{sin. B sin. C}}\\ &= -\frac{2\text{ cos. }\dfrac{\text{A+B+C}}{2}\text{ cos. }\dfrac{\text{B+C}-\text{A}}{2}}{\text{sin. B sin. C}}\\ &= -\frac{2\text{ cos. }s'\text{ cos. }(s'-\text{A})}{\text{sin. B sin. C}}\quad \text{. . . . . .}\quad (2.)\end{aligned}$$

Multiplying Equations (1.) and (2.).

$$1-\text{cos.}^2\, a=-\frac{4\text{ cos. }s'\text{ cos. }(s'-\text{A})\text{ cos. }(s'-\text{B})\text{ cos. }(s'-\text{C})}{\text{sin.}^2\text{ B sin.}^2\text{ C}}$$

$$\left.\begin{array}{l}\therefore\quad \text{sin. } a=\dfrac{2}{\text{sin. B sin. C}}\\ \quad\times\sqrt{-\text{cos. }s'\text{ cos. }(s'-\text{A})\text{ cos. }(s'-\text{B})\text{ cos. }(s'-\text{C})}\\ \text{Similarly,}\\ \quad\text{sin. } b=\dfrac{2}{\text{sin. A sin. C}}\\ \quad\times\sqrt{-\text{cos. }s'\text{ cos. }(s'-\text{A})\text{ cos. }(s'-\text{B})\text{ cos. }(s'-\text{C})}\\ \quad\text{sin. } c=\dfrac{2}{\text{sin. A sin. B}}\\ \quad\times\sqrt{-\text{cos. }s'\text{ cos. }(s'-\text{A})\text{ cos. }(s'-\text{B})\text{ cos. }(s'-\text{C})}\end{array}\right\}(\delta. 1.)$$

By Equation (1) we have,

$$1+\text{cos. } a=\frac{2\text{ cos. }(s'-\text{B})\text{ cos. }(s'-\text{C})}{\text{sin. B. sin. C}}$$

$$\therefore 2\text{ cos.}^2\,\frac{a}{2}=\frac{2\text{ cos. }(s'-\text{B})\text{ cos. }(s'-\text{C})}{\text{sin. B sin. C}}$$

$$\left.\begin{array}{l}\therefore\quad \text{cos. }\dfrac{a}{2}=\sqrt{\dfrac{\text{cos. }(s'-\text{B})\text{ cos. }(s'-\text{C})}{\text{sin. B sin. C}}}\\ \text{Similarly,}\\ \qquad\text{cos. }\dfrac{b}{2}=\sqrt{\dfrac{\text{cos. }(s'-\text{A})\text{ cos. }(s'-\text{C})}{\text{sin. A sin. C}}}\\ \qquad\text{cos. }\dfrac{c}{2}=\sqrt{\dfrac{\text{cos. }(s'-\text{A})\text{ cos. }(s'-\text{B})}{\text{sin. A sin. B}}}\end{array}\right\}(\delta. 2.)$$

By equation (2.)

$$1 - \cos.\ a = -\frac{2\cos.\ s' \cos.\ (s' - A)}{\sin.\ B \sin.\ C}$$

$$\therefore\ 2\sin.^2 \frac{a}{2} = -\frac{2\cos.\ s' \cos.\ (s' - A)}{\sin.\ B \sin.\ C}$$

$$\left.\begin{aligned} \therefore\ \sin.\frac{a}{2} &= \sqrt{\frac{-\cos.\ s' \cos.\ (s' - A)}{\sin.\ B \sin.\ C}} \\ \sin.\frac{b}{2} &= \sqrt{\frac{-\cos.\ s' \cos.\ (s' - B)}{\sin.\ A \sin.\ C}} \\ \sin.\frac{c}{2} &= \sqrt{\frac{-\cos.\ s' \cos.\ (s' - C)}{\sin.\ A \sin.\ B}} \end{aligned}\right\}\ (\delta.\ 3.)$$

Finally, dividing the expressions (δ. 3.) by the expressions (δ. 2.)

$$\left.\begin{aligned} \tan.\frac{a}{2} &= \sqrt{\frac{-\cos.\ s' \cos.\ (s' - A)}{\cos.\ (s' - B)\cos.\ (s' - C)}} \\ \tan.\frac{b}{2} &= \sqrt{\frac{-\cos.\ s' \cos.\ (s' - B)}{\cos.\ (s' - A)\cos.\ (s' - C)}} \\ \tan.\frac{c}{2} &= \sqrt{\frac{-\cos.\ s' \cos.\ (s' - C)}{\cos.\ (s' - A)\cos.\ (s' - B)}} \end{aligned}\right\}\ (\delta.\ 4.)$$

It is to be remarked that although the expressions (δ. 1.) (δ. 3.), (δ. 4.), appear under an impossible form, they are in reality always possible.

For by Prop. XVIII. of Spherical Geometry, the sum of the angles of a spherical triangle, is always greater than two right angles, and less than six right angles.

$$\therefore\ A+B+C \quad > 180^\circ \text{ and } < 540^\circ$$

$$\therefore\ \frac{A+B+C}{2} \text{ or } s' > \ 90^\circ \text{ and } < 270^\circ$$

Hence, cosine $s'$ is always negative, and ∴ $-\cos.\ s'$ is always positive.

Again, if $a'$, $b'$, $c'$, be the three sides of the polar triangle, since the sum of any two sides of a spherical triangle is greater than the third side:

$$b' + c' > a'$$

$$\therefore\ 180^\circ - B + 180^\circ - C > 180^\circ - A$$

$$\therefore\ B + C \ - A < 180^\circ$$

$$\frac{B + C \ - A}{2} < \ 90^\circ$$

$\therefore$ cos. $(s'-A)$ is always positive, and in like manner, cos. $(s'-B)$, cos. $(s'-C)$, are always positive; hence the above expressions are in every case possible.

5. *The sines of the angles of a spherical triangle are to each other as sines of the two sides opposite to them.*

Taking the expressions ($\gamma$. I.) and calling the common radical quantity N for the sake of brevity:

$$\text{sin. } A = \frac{2\,N}{\text{sin. } b \text{ sin. } c}$$

$$\text{sin. } B = \frac{2\,N}{\text{sin. } a \text{ sin. } c}$$

Dividing the first of these by the second:

Similarly,

$$\left.\begin{aligned} \frac{\text{sin. } A}{\text{sin. } B} &= \frac{\text{sin. } a \text{ sin. } c}{\text{sin. } b \text{ sin. } c} = \frac{\text{sin. } a}{\text{sin. } b} \\ \frac{\text{sin. } A}{\text{sin. } C} &= \frac{\text{sin. } a \text{ sin. } b}{\text{sin. } c \text{ sin. } b} = \frac{\text{sin. } a}{\text{sin. } c} \\ \frac{\text{sin. } B}{\text{sin. } C} &= \frac{\text{sin. } b \text{ sin. } a}{\text{sin. } c \text{ sin. } a} = \frac{\text{sin. } b}{\text{sin. } c} \end{aligned}\right\} \quad (\epsilon.)$$

6. *To express the tangent of the sum and difference of two angles of a spherical triangle, in terms of the sides opposite to these angles, and the third angle of the triangle.*

By ($\alpha$) we have,

$$\text{cos. } A = \frac{\text{cos. } a - \text{cos. } b \text{ cos. } c}{\text{sin. } b \text{ sin. } c} \quad \text{- - - - - - - - -} \quad (1.)$$

And,

$$\text{cos. } C = \frac{\text{cos. } c - \text{cos. } a \text{ cos. } b}{\text{sin. } a \text{ sin. } b}$$

$$\therefore \text{ cos. } c = \text{cos. } a \text{ cos. } b + \text{sin. } a \text{ sin. } b \text{ cos. } C \quad \text{- - - -} \quad (2.)$$

Substituting this value of cos. $c$ in Equation (1.):

$$\text{cos. } A = \frac{\text{cos. } a - \text{cos. } a \text{ cos.}^2\, b - \text{cos. } b \text{ sin. } a \text{ sin. } b \text{ cos. } C}{\text{sin. } b \text{ sin. } c}$$

$$= \frac{\text{cos. } a\ (1 - \text{cos.}^2\, b) - \text{cos. } b \text{ sin. } a \text{ sin. } b \text{ cos. } C}{\text{sin. } b \text{ sin. } c}$$

$$= \frac{\text{cos. } a \text{ sin. } b - \text{cos. } b \text{ sin. } a \text{ cos. } C}{\text{sin. } c} \quad \text{- - - -} \quad (3.)$$

In like manner, substituting the value of cos. $c$ in Equation (2), in the expression for cos. B, we shall find,

$$\cos. \mathrm{B} = \frac{\cos. b \sin. a - \cos. a \sin. b \cos. \mathrm{C}}{\sin. c} \quad - \; - \; - \quad (4.)$$

Adding equations (3) and (4):

$$\cos. \mathrm{A} + \cos. \mathrm{B}$$

$$= \frac{\sin. a \cos. b + \sin. b \cos. a - (\sin. a \cos. b + \sin. b \cos. a) \cos. \mathrm{C}}{\sin. c}$$

$$= \frac{\sin. (a+b) - \sin. (a+b) \cos. \mathrm{C}}{\sin. c}$$

$$= \frac{\sin. (a+b)\,(1 - \cos. \mathrm{C})}{\sin. c} \quad - \; - \; - \; - \; - \; - \; - \quad (5.)$$

Again, by Equation (s) we have,

$$\frac{\sin. \mathrm{A}}{\sin. \mathrm{B}} = \frac{\sin. a}{\sin. b}$$

$$\therefore \frac{\sin. \mathrm{A} + \sin. \mathrm{B}}{\sin. \mathrm{B}} = \frac{\sin. a + \sin. b}{\sin. b}$$

$$\therefore \sin. \mathrm{A} \pm \sin. \mathrm{B} = (\sin. a \pm \sin. b) \frac{\sin. \mathrm{B}}{\sin. b}$$

$$= (\sin. a \pm \sin. b) \frac{\sin. \mathrm{C}}{\sin. c} \quad - \; - \; - \quad (6.)$$

Dividing Equation (6) by Equation (5), and taking first the positive sign:

$$\frac{\sin. \mathrm{A} + \sin. \mathrm{B}}{\cos. \mathrm{A} + \cos. \mathrm{B}} = \frac{\sin. a + \sin. b}{\sin. (a+b)} \; \frac{\sin. \mathrm{C}}{1 - \cos. \mathrm{C}}$$

$$\therefore \frac{2 \sin. \frac{\mathrm{A}+\mathrm{B}}{2} \cos. \frac{\mathrm{A}-\mathrm{B}}{2}}{2 \cos. \frac{\mathrm{A}+\mathrm{B}}{2} \cos. \frac{\mathrm{A}-\mathrm{B}}{2}} = \frac{2 \sin. \frac{a+b}{2} \cos. \frac{a-b}{2}}{2 \sin. \frac{a+b}{2} \cos. \frac{a+b}{2}} \cot. \frac{\mathrm{C}}{2}$$

$$\therefore \quad \tan. \frac{\mathrm{A}+\mathrm{B}}{2} = \frac{\cos. \frac{a-b}{2}}{\cos. \frac{a+b}{2}} \cot. \frac{\mathrm{C}}{2}$$

Again, dividing Equation (6) by Equation (5) and taking the negative sign.

$$\frac{\sin. \mathrm{A} - \sin. \mathrm{B}}{\cos. \mathrm{A} + \cos. \mathrm{B}} = \frac{\sin. a - \sin. b}{\sin. (a+b)} \; \frac{\sin. \mathrm{C}}{1 - \cos. \mathrm{C}}$$

$$\frac{2\ \text{sin.}\ \frac{A-B}{2}\ \text{cos.}\ \frac{A+B}{2}}{2\ \text{cos.}\ \frac{A-B}{2}\ \text{cos.}\ \frac{A+B}{2}} = \frac{2\ \text{sin.}\ \frac{a-b}{2}\ \text{cos.}\ \frac{a+b}{2}}{2\ \text{sin.}\ \frac{a+b}{2}\ \text{cos.}\ \frac{a+b}{2}}\ \frac{\text{sin.}\ C}{1-\text{cos.}\ C}$$

$$\therefore\ \text{tan.}\ \frac{A-B}{2} = \frac{\text{sin.}\ \frac{a-b}{2}}{\text{cos.}\ \frac{a+b}{2}}\ \text{cot.}\ \frac{C}{2}$$

We have thus obtained the required expression, viz.

$$\left.\begin{aligned}
\text{tan.}\ \frac{A+B}{2} &= \frac{\text{cos.}\ \frac{a-b}{2}}{\text{cos.}\ \frac{a+b}{2}}\ \text{cot.}\ \frac{C}{2} \\
\text{tan.}\ \frac{A-B}{2} &= \frac{\text{sin.}\ \frac{a-b}{2}}{\text{sin.}\ \frac{a+b}{2}}\ \text{cot.}\ \frac{C}{2} \\
&\text{Similarly,} \\
\text{tan.}\ \frac{B+C}{2} &= \frac{\text{cos.}\ \frac{b-c}{2}}{\text{cos.}\ \frac{b+c}{2}}\ \text{cot.}\ \frac{A}{2} \\
\text{tan.}\ \frac{B-C}{2} &= \frac{\text{sin.}\ \frac{b-c}{2}}{\text{sin.}\ \frac{b+c}{2}}\ \text{cot.}\ \frac{A}{2} \\
\text{tan.}\ \frac{A+C}{2} &= \frac{\text{cos.}\ \frac{a-c}{2}}{\text{cos.}\ \frac{a+c}{2}}\ \text{cot.}\ \frac{B}{2} \\
\text{tan.}\ \frac{A-C}{2} &= \frac{\text{sin.}\ \frac{a-c}{2}}{\text{sin.}\ \frac{a+c}{2}}\ \text{cot.}\ \frac{B}{2}
\end{aligned}\right\}\ (\zeta)$$

7. *To express the tangent of the sum and difference of two sides of a spherical triangle, in terms of the angles opposite to them and the third side of the triangle.*

Let A, B, C, *a*, *b*, *c*, be the sides and angles of a spherical triangle, A′, B′, C′, *a*′, *b*′, *c*′, the corresponding parts of the polar triangle then by expression (ζ),

$$\tan.\frac{A'+B'}{2}=\frac{\cos.\frac{a'-b'}{2}}{\cos.\frac{a'+b'}{2}}\cot.\frac{C'}{2}$$

Therefore,

$$\tan.\frac{180^\circ-a+180^\circ-b}{2}=\frac{\cos.\frac{(180^\circ-A)-(180^\circ-B)}{2}}{\cos.\frac{(180^\circ-A)+(180^\circ-B)}{2}}\cot.\frac{(180^\circ-c)}{2}$$

$$\tan.\left(180-\frac{a+b}{2}\right)=\frac{\cos.\left(-\frac{A-B}{2}\right)}{\cos.\left(180^\circ-\frac{A+B}{2}\right)}\cot.\left(90^\circ-\frac{c}{2}\right.$$

$$\therefore\quad\tan.\frac{a+b}{2}=\frac{\cos.\frac{A-B}{2}}{\cos.\frac{A+B}{2}}\tan.\frac{c}{2}$$

$$\tan.\frac{A'-B'}{2}=\frac{\sin.\frac{a'-b'}{2}}{\sin.\frac{a'+b'}{2}}\cot.\frac{C'}{2}$$

Therefore,

$$\tan.\frac{(180^\circ-a)-(180^\circ-b)}{2}=\frac{\sin.\frac{(180^\circ-A)-(180^\circ-B)}{2}}{\sin.\frac{(180^\circ-A)+(180^\circ-B)}{2}}\cot.\frac{(180^\circ-c)}{2}$$

$$\therefore\quad\tan.\frac{a-b}{2}=\frac{\sin.\frac{A-B}{2}}{\sin.\frac{A+B}{2}}\tan.\frac{c}{2}$$

We shall thus obtain another group of formulæ analogous to the last.

$$\left.\begin{array}{l}
\left.\begin{array}{l}
\tan.\frac{a+b}{2}=\dfrac{\cos.\frac{A-B}{2}}{\cos.\frac{A+B}{2}}\tan.\frac{c}{2} \\[2ex]
\tan.\frac{a-b}{2}=\dfrac{\sin.\frac{A-B}{2}}{\sin.\frac{A+B}{2}}\tan.\frac{c}{2}
\end{array}\right\} \\[6ex]
\left.\begin{array}{l}
\tan.\frac{b+c}{2}=\dfrac{\cos.\frac{B-C}{2}}{\cos.\frac{B+C}{2}}\tan.\frac{a}{2} \\[2ex]
\tan.\frac{b-c}{2}=\dfrac{\sin.\frac{B-C}{2}}{\sin.\frac{B+C}{2}}\tan.\frac{a}{2}
\end{array}\right\} \\[6ex]
\left.\begin{array}{l}
\tan.\frac{a+c}{2}=\dfrac{\cos.\frac{A-C}{2}}{\cos.\frac{A+C}{2}}\tan.\frac{b}{2} \\[2ex]
\tan.\frac{a-c}{2}=\dfrac{\sin.\frac{A-C}{2}}{\sin.\frac{A+C}{2}}\tan.\frac{b}{2}
\end{array}\right\}
\end{array}\right\}(\zeta')$$

8. *To express the cotangent of an angle of a spherical triangle, in terms of the side opposite one of the other sides and the angle contained between these two sides.*

By ($\alpha$)

$$\cos.A=\frac{\cos.a-\cos.b\cos.c}{\sin.b\sin.c}\quad\cdots\cdots\cdots\quad(1)$$

and,

$$\cos.C=\frac{\cos.c-\cos.a\cos.b}{\sin.a\sin.b}$$

Hence,

$$\cos. c=\cos. b+\sin. a \sin. b \cos. C.$$

Substituting this value of cos. $c$ in equation (1), it becomes

$$\cos. A=\frac{\cos. a-\cos. a \cos.^2 b-\sin. a \sin. b \cos. b \cos. C}{\sin. b \sin. c}$$

$$=\frac{\cos. a\,(1-\cos.^2 b)-\sin. a \sin. b \cos. b \cos. C}{\sin. b \sin. c}$$

$$\therefore\ \cos. A=\frac{\cos. a\,(1-\cos.^2 b)-\sin. a \sin. b \cos. b \cos. C}{\sin. b \sin. c}$$

$$\therefore\ \cos. A \sin. c=\cos. a \sin. b-\sin. a \cos. b \cos. C$$

But,

$$\sin. c=\frac{\sin. C}{\sin. A}\sin. a, \text{ by } (\varepsilon),$$

$$\therefore\quad \cos. A\,\frac{\sin. C}{\sin. A}\sin. a=\cos. a \sin. b-\sin. a \cos. b \cos C$$

$$\therefore\qquad \cot. A=\cot. a \sin. b \operatorname{cosec.} C-\cos. b \cot. C.$$

In which the cotangent of A is expressed in the required manner.

If in Equation (1), instead of substituting for cos. $c$, we had substituted for cos. $b$, the value derived from the Equation.

$$\cos. B=\frac{\cos. b-\cos. a \cos. c}{\sin. a \sin. c},$$

we should have found a value for cot. A in terms of $a$, $c$, B, or

$$\cot. A=\cot. a \sin. c \operatorname{cosec.} B-\cos. c \cot. B.$$

Proceeding in like manner for the other angles, we shall obtain similar results, and presenting them at one view, we have

$$\left.\begin{array}{l}
\left.\begin{array}{rl}\cot. A= & \cot. a \sin. b \operatorname{cosec.} C-\cos. b \cot. C\\ = & \cot. a \sin. c \operatorname{cosec.} B-\cos. c \cot. B\end{array}\right\}\\
\left.\begin{array}{rl}\cot. B= & \cot. b \sin. a \operatorname{cosec.} C-\cos. a \cot. C\\ = & \cot. b \sin. c \operatorname{cosec.} A-\cos. c \cot. A\end{array}\right\}\\
\left.\begin{array}{rl}\cot. C= & \cot. c \sin. a \operatorname{cosec.} B-\cos. a \cot. B\\ = & \cot. c \sin. b \operatorname{cosec.} A-\cos. b \cot. A\end{array}\right\}
\end{array}\right\}(\eta)$$

9. *To express the cotangent of a side of a spherical triangle, in terms of the opposite angle, one of the other angles, and the side interjacent to those two angles.*

Let A, B, C, $a$, $b$, $c$, be the angles and sides of a spherical triangle, and A′, B′, C′, $a'$, $b'$, $c'$, the corresponding parts in the polar triangle.

Then by $(\eta)$

$$\cot.\ A' = \cot.\ a' \sin.\ b' \operatorname{cosec.} C' - \cos.\ b' \cot.\ C'$$

$$\therefore \cot.\,(180^\circ - a) = \cot.(180^\circ - A)\sin.\,(180^\circ - B)\operatorname{cosec.}(180^\circ - c) - \cos.\,(180^\circ - B)\cot.\,(180^\circ - c)$$

$$-\cot.\ a = -\cot.\ A \sin.\ B \operatorname{cosec.} c - \cos.\ B \cot.\ c$$

$$\therefore \cot.\ a = \cot.\ A \sin.\ B \operatorname{cosec.} c + \cos.\ B \cot.\ c.$$

Applying the same process to each of the expressions in $(\eta)$, we shall obtain analogous results, and thus have a new set of formulæ:

$$\left.\begin{array}{rl}
\cot.\ a & = \cot.\ A \sin.\ B \operatorname{cosec.} c + \cos.\ B \cot.\ c \\
 & = \cot.\ A \sin.\ C \operatorname{cosec.} b + \cos.\ C \cot.\ b \\
\cot.\ b & = \cot.\ B \sin.\ A \operatorname{cosec.} c + \cos.\ A \cot.\ c \\
 & = \cot.\ B \sin.\ C \operatorname{cosec.} a + \cos.\ C \cot.\ a \\
\cot.\ c & = \cot.\ C \sin.\ A \operatorname{cosec.} b + \cos.\ A \cot.\ b \\
 & = \cot.\ C \sin.\ B \operatorname{cosec.} a + \cos.\ B \cot.\ a
\end{array}\right\} (\theta)$$

By aid of the nine groups of formulæ marked, $(\alpha)$, $(\beta)$, $(\gamma)$, $(\delta)$, $(\varepsilon)$, $(\zeta)$, $(\zeta')$, $(\eta)$, $(\theta)$, we shall be enabled to solve all the cases of spherical triangles, whether right-angled, or oblique-angled; and we shall proceed in the next chapter to apply them.

## CHAPTER II.

### ON THE SOLUTION OF RIGHT-ANGLED SPHERICAL TRIANGLES.

Spherical triangles, that have one right angle only, are the subject of the investigation of this chapter; those that have two or three right angles are excluded.

A spherical triangle consists of 6 parts, the 3 sides and 3 angles, and any 3 of these being given, the rest may be found. In the present case, one of the angles is by supposition a right angle; if any other two parts be given, the other three may be determined. Now the combination of 5 quantities taken, 3 and $3 = \frac{5.\ 4.\ 3}{1.\ 2.\ 3} = 10$; therefore ten different cases present themselves in the solution of right-angled triangles.

The manner in which each case may be solved individually, by applying the formulæ already deduced, will be pointed out at the conclusion of this chapter; but we shall in the first place explain two rules, by aid of which the computist is enabled to solve every case of right-angled triangles. These are known by the name of *Napier's Rules for Circular Parts;* and it has been well observed by the late Professor Wood-house, that, in the whole compass of mathematical science, there cannot be found rules which more completely attain that which is the proper object of all rules, namely, facility and brevity of computation.

The rules and their descriptions are as follow:

### *Description of the Circular parts.*

The right angle is thrown altogether out of consideration. The two sides, the complements of the two angles, and the complement of the hypothenuse, are called *the circular parts.* And one of these circular parts may be called a *middle part* (M), and then the two circular parts immediately adjacent to the right and left of M are called *adjacent parts;* the other two remaining circular parts, each separated from M the middle part by an adjacent part, are called *opposite parts,* or *opposite extremes.*

This being premised, we now give

### *Napier's Rules.*

1. *The product of* sin. M *and tabular radius=product of the tangents of the adjacent parts.*

2. *The product of* sin. M *and tabular radius=product of the cosines of the opposite parts.*

These rules will be clearly understood if we show the manner in which they are applied in various cases.

Let A, B, C, be a spherical triangle, right angle at C.
Let $a$ be assumed as the middle part.
Then (90°— B) and $b$ are the adjacent parts.
And (90°— $c$) and (90°—A) are the opposite parts.
Then by rule (1)

$$R\times \sin. a = \tan. (90^\circ - B) \tan. b$$
$$= \cot. B \tan. b \quad - - - - - - - - - \quad (1)$$

By Rule (2)

$$R. \sin. a = \cos. (90^\circ - A) \cos. (90^\circ - c)$$
$$= \sin. A \sin. c, \quad - - - - - - - - - \quad (2)$$

2. Let $b$ be the middle part,
Then (90°—A) and $a$ are adjacent parts,
Then (90°—$c$)and (90° —B) are opposite parts.
Then by Rule I,

$$\therefore \; R. \sin. b = \tan. (90^\circ - A) \tan. a$$
$$= \cot. A \tan. a \quad - - - - - - - - - \quad (3)$$

And Rule II,
and $R. \sin. b = \cos. (90° - B) \cos. (90° - c)$
$= \sin. B \sin. c$ - - - - - - - - - - (4)

3. Let $(90° - c)$ be the middle part.
Then $(90° - A)$, and $(90° - B)$ are adjacent parts,
And $b$ and $a$ are opposite parts.
Then,
$R \sin. (90° - c) = \tan. (90° - A) \tan. (90° - B)$
$R. \cos. c = \cot. A \cot. B$ - - - - - - - - (5)
And,
$R. \sin. (90° - c) = \cos. a \cos. b.$
$R. \cos. c = \cos. a \cos. b$ - - - - - - - - - - (6)

4. Let $(90° - A)$ be the middle part.
Then $(90° - c)$ and $b$ are adjacent parts,
And $(90° - B)$ and $a$ are opposite parts.
Then Rule I.
$R. \sin. (90° - A) = \tan. (90° - c) \tan. b.$
$\therefore$ $R. \cos. A = \cot. c \tan. b$ - - - - - - - - - - (7)
And Rule II.
$R. \sin. (90° - A) = \cos. (90° - B) \cos. a,$
$\therefore$ $R. \cos. A = \sin. B \cos. a$ - - - - - - - - - (8)

5. Let $(90° - B)$ be the middle part.
Then $(90° - c)$ and $a$ are the adjacent parts,
And $(90° - A)$ and $b$ are the opposite parts.
Then Rule I.
$\cos. B = \tan. (90° - c) \tan. a,$
$= \tan. a \cot. c$ - - - - - - - - - - (9)
$\cos. B = \cos. (90° - A) \cos. b,$
$= \sin. A \cos. b$ - - - - - - - - - - (10)

Collecting the above results, and making $R = 1$, we shall have
$\sin. a = \cot. B \tan. b$ - - - - - - - - - - - - (1)
$\sin. a = \sin. A \sin. c$ - - - - - - - - - - - - (2)
$\sin. b = \cot. A \tan. a$ - - - - - - - - - - - - (3)
$\sin. b = \sin. B \sin. c$ - - - - - - - - - - - - (4)
$\cos. c = \cot. A \cot. B$ - - - - - - - - - - - (5)
$\cos. c = \cos. a \cos. b$ - - - - - - - - - - - - (6)
$\cos. A = \tan. b \cot. c$ - - - - - - - - - - - (7)
$\cos. A = \sin. B \cos. a$ - - - - - - - - - - - (8)
$\cos. B = \tan. a \cot. c$ - - - - - - - - - - - (9)
$\cos. B = \sin. A \cos. b$ - - - - - - - - - - - - (10)

It now remains for us to show that these conclusions are accurate, and in accordance with the formulæ already deduced.

Now by ($\alpha$).

$$\cos. C = \frac{\cos. c - \cos. a \cos. b}{\sin. a \sin. b}$$

But when C=90°, then cos. C=0.

$$\therefore \quad 0 = \frac{\cos. c - \cos. a \cos. b}{\sin. a \sin. b}$$

$\therefore$ $\cos. c = \cos. a \cos. b$, which is formula (6) in the above table.

Again by ($\varepsilon$)

$$\frac{\sin. a}{\sin. c} = \frac{\sin. A}{\sin. C}$$

But when C=90° sin. C=1

$\therefore$ $\sin. a = \sin. A \sin. c$, which is formula (2) above.

Similarly,

$$\frac{\sin. b}{\sin. c} = \frac{\sin. B}{\sin. C}$$

$\therefore$ $\sin. b = \sin. B \sin. c$, which is formula (4).

Next since by ($\alpha$)

$\cos. A = \dfrac{\cos. a - \cos. b \cos. c}{\sin. b \sin. c}$, substitute for cos. $c$ its value in (6).

$$= \frac{\cos. a - \cos. a \cos.^2 b}{\sin. b \sin. c}$$

$= \dfrac{\cos. a \sin. b}{\sin. c}$, substitute for sin. $c$, its value as found in (2.)

$$= \frac{\cos. a \sin. b}{\dfrac{\sin. a}{\sin. A}}$$

$\therefore$ $\sin. b = \cot. A \tan. a$, which is formula (3.)

Again,

$\cos. A = \dfrac{\cos. a - \cos. b \cos. c}{\sin. b \sin. c}$ substitute for cos. $a$, its value in (6.)

$$= \frac{\dfrac{\cos. c}{\cos. b} - \cos. b \cos. c}{\sin. b \sin. c}$$

$$= \frac{\cos. c \sin. b}{\sin. c \cos. b}$$

$= \tan. b \cot. c$, which is formula (7.)

Again by ($\alpha$.) cos. B

$$=\frac{\cos. b-\cos. a \cos. c}{\sin. a \sin. c}$$ substitute for cos. *c* its value from (6)

$$=\frac{\cos. b-\cos. b \cos.^2 a}{\sin. a \sin. c}$$

$$=\frac{\cos. b \sin. a}{\sin. c}$$ substitute for sin. *c*, its value from (4.)

$$=\frac{\cos. b \sin. a}{\frac{\sin. b}{\sin. B}}$$

Sin. $a$=cot. B tan. $b$, which is formula (1.)

Again, cos. B

$$=\frac{\cos. b-\cos. a \cos. c}{\sin. a \sin. c},$$ substitute for cos. *b*, its value in (6.)

$$=\frac{\frac{\cos. c}{\cos. a}-\cos. a \cos. c}{\sin. a \sin. c}$$

$$=\frac{\cos. c \sin. a}{\sin. c \cos. a}$$

=tan. $a$ cot. $c$, which is formula (9.)

Next by ($\beta$.)

$$\cos. a=\frac{\cos. A+\cos. B \cos. C}{\sin. B \sin. C}$$

But C=90° ∴ cos. C=0, and sin. C=1.

$$\therefore \cos. a=\frac{\cos. A}{\sin. B}$$

∴ cos. A=sin. B cos. $a$, which is formula (8.)

Again,

$$\cos. b=\frac{\cos. B+\cos. A \cos. C}{\sin. A \sin. C}$$ and when C=90°.

$$=\frac{\cos. B}{\sin. A}$$

∴ cos. B=sin. A cos. $b$, which is formula (10.)

Lastly,

$$\cos. c=\frac{\cos. C+\cos. A \cos. B}{\sin. A. \sin. B}$$ and in this case,

$$= \frac{\cos. A \cos. B}{\sin. A \ \sin. B}$$

$$= \cot. A \cot. B, \text{ which is formula (5.)}$$

We have thus proved the truth of the results derived from the application of Napier's rules, and may therefore apply these rules without scruple to the solution of various cases of right-angled triangles.

Let us then take each combination of the two data, and determine in each case the other three quantities, adapting our formulæ to computation by tables.

1. Given A, B, required *a*, *b*, *c*.

$$R \cos. A = \sin. B \cos, a \ \therefore\ \cos. a = R\frac{\cos. A}{\sin. B} \qquad (1)$$

$$R \cos. B = \sin. A \cos. b \ \therefore\ \cos. b = R\frac{\cos. B}{\sin. A} \qquad (2)$$

$$R \cos. c = \cot. A \cot. B \qquad (3)$$

2. Given *a*, *b*, required A, B, *c*,

$$R \sin. a = \cot. B \tan. b \ \therefore\ \cot. B = R \sin. a \cot. b \qquad (4)$$

$$R \sin. b = \cot. A \tan. a \ \therefore\ \cot. A = R \sin. b \cot. a \qquad (5)$$

$$R \cos. c = \cos. a \cos. b \qquad (0)$$

3. Given *a*, *c*, required A, B, *b*

$$R \sin. a = \sin. A \sin. c \ \therefore\ \sin. A = R\frac{\sin. a}{\sin. c} \qquad (7)$$

$$R \cos. c = \cos. a \cos. b \ \therefore\ \cos. b = R\frac{\cos. c}{\cos. a} \qquad (8)$$

$$R \cos. B = \tan. a \cot. c \qquad (9)$$

4. Given *b*, *c*, required A, B, *a*.

$$R \sin. b = \sin. B \ \sin. c \ \therefore\ \sin. B = R\frac{\sin. b}{\sin. c} \qquad (10)$$

$$R \cos. c = \cos. a \cos. b \ \therefore\ \cos. a = R\frac{\cos. c}{\cos. b} \qquad (11)$$

$$R \cos. A = \tan. b \cot. c \qquad (12)$$

5. Given A, *c*, required B, *a*, *b*.

$$R \cos. A = \tan. b \cot. c \ \therefore\ \tan. b = R \cos. a \tan. c \qquad (13)$$

$$R \cos. c = \cot. A \cot. B \ \therefore\ \cot. B = R \tan. A \cos. c \qquad (14)$$

$$R \sin. a = \sin. A \sin. c \qquad (15)$$

6. Given B, *c*, required A, *a*, *b*.

$$R \cos. B = \cot. c \tan. a \ \therefore\ \tan. a = R \cos. B \tan. c \qquad (16)$$

R cos. $c$ = cot. A cot. B ∴ cot. A = R tan. B cos. $c$ - (17)
R sin. $b$ = sin. B sin. $c$ - - - - - - - - - - - (18)

7. Given A, $b$, required B, $c$, $a$.
R cos. A = cot. $c$ tan. $b$ ∴ cot. $c$ = R cos. A cot. $b$ - (19)
R sin. $b$ = cot. A tan. $a$ ∴ tan. $a$ = R tan. A sin. $b$ - (20)
R cos. B = sin. A cos. $b$ - - - - - - - - - - - (21)

8. Given B, $a$, required A, $c$, $b$.
R cos. B = cot. $c$ tan. $a$ ∴ cot. $c$ = R cos. B cot. $a$ - (22)
R sin. $a$ = cot. B tan. $b$ ∴ tan. $b$ = R tan. B sin. $a$ - (23)
R cos. A = sin. B cos. $a$ - - - - - - - - - - - (24)

9. Given A, $a$, required B, $b$, $c$.
R cos. A = sin. B cos. $a$ ∴ sin. B = R $\frac{\text{cos. A}}{\text{cos. } a}$ - - (25)
R sin. $a$ = sin. A sin. $c$ ∴ sin. $c$ = R $\frac{\text{sin. } a}{\text{sin. A}}$ - - - (26)
R sin. $b$ = cot. A tan. $a$ - - - - - - - - - - - (27)

10. Given B, $b$, required A, $a$, $c$.
R cos. B = sin. A cos. $b$ ∴ sin. A = R $\frac{\text{cos. B}}{\text{cos. } b}$ - - - (28)
R sin. $b$ = sin. B sin. $c$ ∴ sin. $c$ = R $\frac{\text{sin. } b}{\text{sin. B}}$ - - - (29)
R sin. $a$ = cot. B tan. $b$ - - - - - - - - - - - (30)

## CHAPTER III.

### ON THE SOLUTION OF OBLIQUE-ANGLED SPHERICAL TRIANGLES.

The different cases which present themselves are contained in the following enumerations.

1. When two sides and the included angle are given.
2. When two angles and the side between them are given.
3. When two sides and the angle opposite to one of them are given.
4. When two angles and the side opposite to one of them are given.
5. When three sides are given.
6. When three angles are given.

I. **When two sides and the included angle are given.** The remaining angles may be determined from the formula ($\sigma$.)

Thus, let $a$, $b$, C, be given, A, B, $c$, required.

$$\tan.\frac{A+B}{2}=\frac{\cos.\frac{a-b}{2}}{\cos.\frac{a+b}{2}}\cot.\frac{C}{2}$$

$$\tan.\frac{A-B}{2}=\frac{\sin.\frac{a-b}{2}}{\sin.\frac{a+b}{2}}\cot.\frac{C}{2}$$

Whence $\frac{A+B}{2}$ and $\frac{A-B}{2}$ are known from the tables.

Let
$$\frac{A+B}{2}=\theta$$
$$\frac{A-B}{2}=\phi$$
$$\therefore \quad A=\theta+\phi$$
$$B=\theta-\phi$$

A and B being known, $c$ may be obtained from ($\varepsilon$.)

For
$$\frac{\sin.c}{\sin.a}=\frac{\sin.C}{\sin.A}$$
$$\therefore \quad \sin.c=\sin.a\frac{\sin.C}{\sin.A}$$

And, in like manner, if any two other sides and the included angle be given, the remaining parts may be determined.

II. When two angles and the side between them are given. The remaining sides may be determined from the formula($\zeta'$.)

Thus, let A, B, $c$, be given; $a$, $b$, C, required.

$$\tan.\frac{a+b}{2}=\frac{\cos.\frac{A-B}{2}}{\cos.\frac{A+B}{2}}\tan.\frac{c}{2}$$

$$\tan.\frac{a-b}{2}=\frac{\sin.\frac{A-B}{2}}{\sin.\frac{A+B}{2}}\tan.\frac{c}{2}$$

Whence $\frac{a+b}{2}$ and $\frac{a-b}{2}$ are known from the tables.

Let
$$\frac{a+b}{2}=\theta'$$
$$\frac{a-b}{2}=\varphi'$$
$$\therefore \quad a=\theta'+\varphi'$$
$$b=\theta'-\varphi'$$

$a$ and $b$ being known, C may be obtained by ($\varepsilon$.)

For
$$\frac{\sin. C}{\sin. A}=\frac{\sin. c}{\sin. a}$$
$$\therefore \quad \sin. C=\sin. A\ \frac{\sin. c}{\sin. a}$$

And, in like manner, if any two other angles and the included side are given, the remaining parts may be determined.

III. **When two sides and the angle opposite to one of them are given.**

The angle opposite to the other side may be found from formula ($\varepsilon$.)

Thus, let $a$, $b$, A be given, B, C, $c$, required.
$$\frac{\sin. B}{\sin. A}=\frac{\sin. b}{\sin. a}$$
$$\therefore \quad \sin. B=\sin. A\ \frac{\sin. b}{\sin. a}$$

The angle B being determined, the remaining angle C will be found from ($\sigma$.)

For
$$\tan. \frac{A+B}{2}=\frac{\cos. \frac{a-b}{2}}{\cos. \frac{a+b}{2}} \cot. \frac{C}{2}$$
$$\cot. \frac{C}{2}=\frac{\cos. \frac{a+b}{2}}{\cos. \frac{A-B}{2}} \tan. \frac{A+B}{2}$$

The angle C being determined, the remaining side $c$ will be found from ($\varepsilon$.)

For
$$\frac{\sin. c}{\sin. a}=\frac{\sin. C}{\sin. A}$$
$$\therefore \quad \sin. c=\sin. a\ \frac{\sin. C}{\sin. A}$$

or $c$ may be found from ($\zeta'$.)

And, in like manner, if any other two sides and the angle opposite to one of them be given, the remaining parts may be determined.

**IV. When two angles and the side opposite to one of them are given.**

The side opposite to the other angle may be found from formula (ε.)

Thus, let A, B, $a$, be given; $b$, $c$, C, required.

$$\frac{\sin. b}{\sin. a} = \frac{\sin. B}{\sin. A}$$

$$\therefore \qquad \sin. b = \sin. a \, \frac{\sin. B}{\sin. A}$$

The side $b$ being determined, the remaining side $c$ will be found from (ζ′)

$$\text{For} \qquad \tan. \frac{a+b}{2} = \frac{\cos. \frac{A-B}{2}}{\cos. \frac{A+B}{2}} \tan. \frac{c}{2}$$

$$\therefore \qquad \tan. \frac{c}{2} = \frac{\cos. \frac{A+B}{2}}{\cos. \frac{A-B}{2}} \tan. \frac{a+b}{2}$$

The side $c$ being determined, the remaining angle C will be found from (ε.)

$$\text{For} \qquad \frac{\sin. C}{\sin. A} = \frac{\sin. c}{\sin. a}$$

$$\therefore \qquad \sin. C = \sin. A \, \frac{\sin. c}{\sin. a}$$

or $c$ may be found from (σ.)

And, in like manner, any other two sides being given and the angle opposite to one of them, the remaining parts may be determined.

V. When three sides are given.

The three angles may be immediately determined from any one of the formulæ (γ 1,) (γ 2,) (γ 3,) (γ 4.)

The choice of the formula, which it will be advantageous to employ in practice, will depend upon the consideration already noticed in the solution of the analogous case in plane trigonometry.

VI. When three angles are given.

The three sides may be immediately determined from any of the groups of formulæ (δ 1,) (δ 2,) (δ 3,) (δ 4.)

## CHAPTER IV.

### ON THE USE OF SUBSIDIARY ANGLES.

We have already explained in plane Trigonometry, the meaning of Subsidiary Angles, and the purpose for which they are introduced; we shall now proceed to point out under what circumstances they may be employed with advantage, in Spherical Trigonometry.

In the solution of case I., where two sides and the included angle were given, we first determined the two remaining angles, and having found these, we were enabled to find the side also. It frequently happens, however, that the side alone is the object of our investigations, and it is therefore convenient to have a method of determining it, independently of the angle.

Thus, for example, let $b$, $c$, A be given, and let it be required to determine $a$, independently of the angles B, $c$.

By ($\alpha$,) we have

$$\cos. A=\frac{\cos. a-\cos. b\cos. c}{\sin. b\sin. c}$$

Whence $\cos. a=\cos. A\sin. b\sin. c+\cos. b\cos. c.$

From which equation $a$ is determined, but the expression is not in a form adapted to the logarithmic computation; we can, however, effect the necessary transformation by the introduction of a subsidiary angle.

Add and subtract $\sin. b\sin. c$ on the right hand side of the equation.

Then $\cos. a$

$$=\cos. A\sin. b\sin. c+\cos. b\cos. c+\sin. b\sin. c-\sin. b\sin. c$$
$$=\cos. b\cos. c+\sin. b\sin. c+\sin. b\sin. c\cos. A-\sin. b\sin. c$$
$$=\cos. (b-c)-\sin. b\sin. c\text{ vers. }A$$
$$1-\cos. a=1-\cos. (b-c)+\sin. b\sin. c\text{ vers. }A$$
$$\text{vers. }a=\text{vers. }(b-c)+\sin. b\sin. c\text{ vers. }A$$
$$=\text{vers. }(b-c)\left\{1+\frac{\sin. b\sin. c\text{ vers. }A}{\text{vers. }(b-c)}\right\}$$

Let $\tan.^2\theta=\dfrac{\sin. b\sin. c\text{ vers. }A}{\text{vers. }(b-c)}$

$$\therefore\ \text{vers. }a=\text{vers. }(b-c)\{1+\tan.^2\theta\}$$
$$=\text{vers. }(b-c)\sec.^2\theta$$

from which $a$ may be determined by the tables, $\theta$ being known from the equation

$$\tan.^2\theta=\frac{\sin. b\sin. c\text{ vers. }A}{\text{vers. }(b-c)}$$

In like manner in case II, where two angles and the included side were given, we first determined the remaining sides, and then we were enabled to find the remaining angle. Now, let us suppose that A, B, $c$, are given, and that we are required to find C independently of $a$ and $b$.

From $(\beta)$ $$\text{cos. } c=\frac{\text{cos. C}+\text{cos. A cos. B}}{\text{sin. A sin. B}}$$

$$\therefore \quad \text{cos. C}=\text{cos. } c \text{ sin. A sin. B}-\text{cos. A cos. B}$$

$$\therefore \; 1-\text{cos. C}=1-\text{sin. A sin. B}\,(1-\text{vers. } c)+\text{cos. A cos. B.}$$

$$=1+\text{cos. (A+B)}+\text{sin. A sin. B vers. } c.$$

or $$2 \text{ sin.}^2 \frac{\text{C}}{2}=2 \text{ cos.}^2 \frac{\text{A+B}}{2}+\text{sin. A sin. B vers. } c$$

$$=2 \text{ cos.}^2 \frac{\text{A+B}}{2}\left\{1+\frac{\text{sin. A sin. B vers. } c}{2 \text{ cos.}^2 \frac{\text{A+B}}{2}}\right\}$$

$$\therefore \text{ sin.}^2 \frac{\text{C}}{2} = \text{cos.}^2 \frac{\text{A+B}}{2} \text{sec.}^2\, \theta$$

If we assume

$$\text{tan.}^2\, \theta=\frac{\text{sin. A sin. B vers. } c}{2 \text{ cos}^2 \frac{\text{A+B}}{2}}$$

In case III, where two sides and the angle opposite to one of them were given, we first determined the angle opposite to the other side, and then the remaining angles and the remaining side in succession. Now, let us suppose the side $c$, independently of the angle B and of each other, under a form adapted for logarithmic computation.

To find C, we have $(\eta.)$

$$\text{cot. A}=\text{cot. } a \text{ sin. } b \text{ cosec. C}-\text{cos. } b \text{ cot. C}$$

or $$\text{cot. A sin. C}=\text{cot. } a \text{ sin. } b-\text{cos. } b \text{ cos. C}$$

or $$\text{sin. C}=\text{cot. } a \text{ sin. } b \text{ tan. A}-\text{cos. } b \text{ cos. C tan. A}$$

$$\therefore \text{ sin. C}+\text{cos. C cos. } b \text{ tan. A}=\text{cot. } a \text{ sin. } b \text{ tan. A.}$$

Let $$\text{cos. } b \text{ tan. A}=\text{tan. } \theta=\frac{\text{sin. } \theta}{\text{cos. } \theta}$$

$$\therefore \quad \text{sin. C}+\frac{\text{sin. } \theta}{\text{cos. } \theta} \text{ cos. C}=\text{cot. } a \text{ sin. } b \text{ tan. A}$$

$$\therefore \text{ sin. C cos. } \theta+\text{cos. C sin. } \theta=\text{cot. } a \text{ sin. } b \text{ tan. A cos. } \theta$$

$$\text{sin. (C}+\theta)=\text{cot. } a \text{ sin. } b \text{ tan. A}\frac{\text{sin. } \theta}{\text{cos. } b \text{ tan. A}}$$

$$=\text{cot. } a \text{ tan. } b \text{ sin. } \theta$$

whence C is known, $\theta$ being previously determined from equation

$$\tan.\ \theta = \cos.\ b \tan.\ A.$$

To find $c$, we have from $(\alpha.)$

$$\cos.\ A = \frac{\cos.\ a - \cos.\ b \cos.\ c}{\sin.\ b \sin.\ c}$$

$$\therefore \quad \sin.\ c \sin.\ b \cos.\ A = \cos.\ a - \cos.\ b \cos.\ c$$

$$\sin.\ c \tan.\ b \cos.\ A = \frac{\cos.\ a}{\cos.\ b} - \cos.\ c$$

$$\sin.\ c \tan.\ b \cos.\ A = \frac{\cos.\ a}{\cos.\ b} - \cos.\ c$$

Let $\quad \tan.\ b \cos.\ A = \tan.\ \theta = \dfrac{\sin.\ \theta}{\cos.\ \theta}$

$$\therefore \quad \sin.\ c \frac{\sin.\ \theta}{\cos.\ \theta} + \cos.\ c = \frac{\cos.\ a}{\cos.\ b}$$

$$\cos.\ (c - \theta) = \frac{\cos.\ a \cos.\ \theta}{\cos.\ b}$$

whence $c$ may be found, $\theta$ being previously determined from the equation

$$\tan.\ \theta = \tan.\ b \cos.\ A.$$

In like manner, in case IV, when two angles and the side opposite to one of them were given, we first determined the side opposite to the other angle, then the remaining side and the remaining angle in succession. Now, let A, B, $a$, be given, and let it be required to determine $c$ and C, independently of $b$ and of each other, and under a form adapted to logarithmic computations. If we take the formula $(\theta.)$

$$\cot.\ a = \cot.\ A \sin.\ B \operatorname{cosec.} c + \cos.\ B \cot.\ c$$

or $\quad \cot.\ a \sin.\ c = \cot.\ A \sin.\ B + \cos.\ B \cos.\ c$

or $\quad \sin.\ c = \cot.\ A \sin.\ B \tan.\ a + \cos.\ B \cos.\ c \tan.\ a$

$\therefore \ \sin.\ c - \cos.\ c \cos.\ B \tan.\ a = \cot.\ A \sin.\ B \tan.\ a.$

Let $\quad \cos.\ B \tan.\ a = \tan.\ \theta = \dfrac{\sin.\ \theta}{\cos.\ \theta}$

$$\sin.\ c - \frac{\sin.\ \theta}{\cos.\ \theta} \cos.\ c = \cot.\ A \sin.\ B \tan.\ a$$

$$\begin{aligned} \sin.\ (c - \theta) &= \cot.\ A \sin.\ B \tan.\ a \cos.\ \theta \\ &= \cot.\ A \sin.\ B \tan.\ a \frac{\sin.\ \theta}{\cos.\ B \tan.\ a} \\ &= \cot.\ A \tan.\ B \sin.\ \theta \end{aligned}$$

whence $c$ may be determined, $\theta$ being previously known from equation $\quad \tan.\ \theta = \cos.\ B \tan.\ a.$

To find C, we have from ($\beta$)

$$\cos. a=\frac{\cos. A+\cos. B\cos. C}{\sin. B\sin. C.}$$

$$\therefore\quad \sin. B\sin. C\cos. a=\cos. A+\cos. B\cos. C$$

$$\therefore\quad \sin. C\tan. B\cos. a=\frac{\cos. A}{\cos. B}+\cos. C$$

$$\therefore\ \sin. C\tan. B\cos. a-\cos. C=\frac{\cos. A}{\cos. B}$$

$$\text{Let}\qquad \tan. B\cos. a=\tan. \theta=\frac{\sin. \theta}{\cos. \theta}$$

$$\therefore\quad \sin. C\times\frac{\sin. \theta}{\cos. \theta}-\cos. C=\frac{\cos. A}{\cos. B}$$

$$\therefore\qquad -\cos. (C+\theta)=\frac{\cos. A\cos. \theta}{\cos. B}$$

whence C may be found, $\theta$ being known from equation

$$\tan. \theta=\tan. B\cos. a.$$

In the fifth and sixth cases, any one of the angles or sides required, may be found independently of the rest by the formulæ referred to.

## EXAMPLES IN SPHERICAL TRIGONOMETRY.

Ex. 1. In the right-angled spherical triangle ABC, the hypothenuse AB is 65° 5′, and the angle A is 48° 12′; find the sides AC, CB, and the angle B.

Ans. AC=55° 7′ 32″
BC=42 32 19
angle B=64 46 14

Ex. 2. In the oblique-angled spherical triangle ABC, given AB=76° 20′, BC=119° 17′, and angle B=52° 5′; to find AC and the angles A and C.

Ans. AC=66° 5′ 36″
angle A=131 10 42
angle C= 56 58 58

Ex. 3. In an oblique spherical triangle the three sides are

$$a=81^\circ\ 17',\ b=114^\circ\ 3',\ c=59^\circ\ 12';$$

required the angles A, B, C.

Ans. A= 62° 39′ 42″
B=124 50 50
C= 50 34 42.

# APPLICATION OF ALGEBRA TO GEOMETRY.

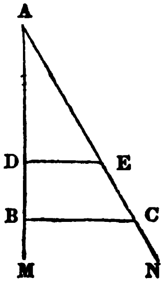
## ON THE GEOMETRICAL CONSTRUCTION OF ALGEBRAICAL QUANTITIES.

As lines, surfaces, and solids are quantities which admit of increase and decrease, like other quantities, they may, like others, be made the subjects of algebraical operations, either by their numerical representatives, or by symbols expressing such quantities. It is only necessaryfor this purpose, that their representatives should possess values in relation to each other corresponding to the magnitudes of the quantities which they represent; and that those values should be expressed according to the properties or relations of the lines, surfaces, or solids, to each other; subject to geometrical construction and algebraical notation.

We are enabled, also, to express by lines, surfaces, and solids, the solutions furnished by Algebra. This is founded on the known properties of geometrical figures, corresponding to similar properties of the quantities algebraically expressed. In this view of the subject, all quantities under algebraic expressions, may be conceived to be susceptible of some kind of geometrical construction.

2. We will proceed to explain the manner of constructing those expressions, and representing, under a geometrical form, the conditions of an equation. This is called *constructing* the algebraic quantities.

Ex. 1. Let it be proposed to construct such a quantity as the following: $\frac{ab}{c}$ the value of the letters composing the quantity being known.



From any point A draw two indefinite lines AM, AN, making any angle with each other; upon one of these lines AM take AB$=c$, and AD$=a$; then upon the line AN take AC$=b$. Having drawn the line BC, draw also DE parallel to BC, this will determine AE as the value of $\frac{ab}{c}$. For the parallels DE, BC, give this proportion AB : AD : : AC : AE, (Prop. XIV. Cor. 2 B. IV. *El. Geom.*) or $c : a :: b :$ AE.

Therefore, $AE=\frac{ab}{c}$ - - - - - - - - - - - - - $\therefore$ 1

Hence, the line AE being the fourth proportional to the three lines represented by $c$, $a$, $b$, it may be used for the construction of the quantity $\frac{ab}{c}$.

2d. Hence, also if it were proposed to construct the quantity $\frac{a^2}{c}$ it may evidently be done in a similar manner, since in this case the lines $b$ and $a$ would be equal, for if $\frac{a^2}{c}=\frac{ab}{c}$ then $a=b$.

3d. If it were proposed to construct $\frac{ab+bd}{cd}$; it may be observed that the quantity may be resolved into the expression $\frac{(a+d)\,b}{c+d}$, hence representing $a+d$ by $m$, and $c+d$ by $n$, we shall have $\frac{mb}{n}$ to be constructed, which may be referred to the former case.

4th. Let the quantity to be constructed be $\frac{a^2-b^2}{c}$; it may be observed that $a^2-b^2$ is equivalent to $(a+b)\times(a-b)$; hence, $\frac{c^2-b^2}{c}$ may be represented under the form $\frac{(a+b)\times(a-b)}{c}$; and we have only to find the fourth proportional to $c$, $a+b$, $a-b$.

5th. If the quantity to be constructed be $\frac{abc}{de}$, it may be put under the form $\frac{ab}{d}+\frac{c}{e}$ and having constructed $\frac{ab}{d}$ in the manner just explained, we call the line given by this construction, $m$; then $\frac{ab}{d}+\frac{c}{e}$ becomes $\frac{mc}{e}$ which may also be constructed as above shown.

6th. It will be presumed, therefore, that in order to construct $\frac{a^2b}{c^2}$ it may be represented under the form $\frac{a^2}{c}+\frac{b}{c}$; whence if we construct $\frac{a^2}{c}$ and represent its value by $m$, we may proceed to construct $\frac{mb}{c}$

Thus the whole art consists in decomposing the quantity into portions, each of which retains the form $\frac{ab}{c}$ or $\frac{a^2}{c}$; and although this process may appear sometimes difficult, yet we may easily arrive at the object proposed by employing transformations.

7th. If, for example, $\frac{a^3+b^3}{a^2+c^2}$ is to be constructed, we may take $b^3=a^2m$, and $c^2=an$; then, $\frac{a^3+b^3}{a^2+c^2}$ becomes $\frac{a^3+a^2m}{a^2+an}$ which may be reduced to $\frac{a^2+am}{a+n}$ or $\frac{(a+m)\,a}{a+n}$, a quantity easy to be constructed after what has been said, when $m$ and $n$ are known.

Now to determine $m$ and $n$, the equations $b^3=a^2m$, $c^2=an$, give, $m=\frac{b^3}{a^2}$ and $n=\frac{c^2}{a}$ which may be constructed by the methods already explained.

Thus, while the quantity is rational, that is, without radical expressions, if the dimensions of the numerator do not exceed those of the denominator except by unity, we may always reduce the construction to the finding of a fourth proportional to three given lines.

It sometimes happens, that quantities present themselves under a form, that renders recourse to transformations of no use; this is when the quantity is not *homogeneous*, that is, when each of the terms of the numerator and denominator is not composed of the same number of factors; when the quantity, for example, is such as $\frac{a^3+b}{c^2+d}$.

But it should be observed, that we never arrive at a result of this kind, except when, in the course of an investigation, we suppose, with a view of simplifying the calculation, some one of the quantities equal to unity. If, for example, in $\frac{a^3+b^2c}{a^2+c^2}$, we suppose $b$ equal to 1, we shall have $\frac{a^3+c}{a^2+c^2}$. But, as we never undertake to construct a quantity without knowing the elements which we are to use for this construction, we always know in each case what is the quantity which is supposed equal to unity. We can always therefore restore it, and the above difficulty cannot occur; because, as the number of dimensions must be the same in each term of the numerator, and also of the denominator, although the number of terms may be different in the one from what it is in the other, we restore in each term a power of the line, which is taken for unity, sufficiently raised to complete the number of dimensions; thus, if we construct $\frac{a^3 + b + c^2}{a + b^2}$; $d$ being supposed to be the line which is taken for unity, we may write formula $\frac{a^3 + b\,d^2 + c^2\,d}{a\,d + b^2}$, which may be constructed by making $b^2 = d\,m$, $c^2 = d\,n$, and $a^3 = d^2\,p$, which will change it into

$$\frac{d^2\,p + b\,d^2 + d^2\,n}{a\,d + d\,m}.$$

or $\frac{d\,p + b\,d + d\,n}{a + m}$, or $\frac{(p + b + n)\,d}{a + m}$, a quantity easily constructed, when we have constructed the value of $m$, $n$, and $p$; namely, $m = \frac{b^2}{d}$, $n = \frac{c^2}{d}$, $p = \frac{a^3}{d^2}$, which is readily done after what has been said.

Hitherto we have supposed that the number of factors, or the dimensions of each term of the numerator exceeds the number of factors, or the dimensions of the denominator only by unity. It may exceed that number by two or even three, but never by more than three, unless some line has been supposed equal to unity, or some of the factors do not represent numbers.

3. When the dimensions of the numerator of the proposed quantity exceed by two the dimensions of the denominator, the quantity expressed is a surface, the construction of which can always be referred to that of a rhomboid. and consequently to that of a square. If, for example, the quantity to be constructed be $\frac{a^3 + a^2\,b}{a + c}$

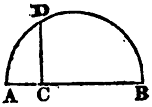
it may be considered as $a \times \frac{a^2 + a\,b}{a + c}$. Now $\frac{a + a\,b}{a + c}$ is easily constructed, after what has been laid down, by considering it as $a \times \frac{a + b}{a + c}$. Let us suppose therefore that $m$ is the value of the line thus obtained; then $a \times \frac{a^2 + a\,b}{a + c}$ will become $a \times m$. Now if we make $a$ the altitude and $m$ the base of a rhomboid, we shall have $a \times m$ for the surface of this rhomboid, (Prop. VI, B. IV, *El. Geom.*) therefore, reciprocally, this surface will represent $a \times m$, or $\frac{a^3 + a^2b}{a + c}$.

In like manner, the quantity $\frac{a^3 + b\,c^2 + d^3}{a + c}$ may be reduced to a similar construction by making $b c = a m$, and $d^2 = a n$; for it will then become $\frac{a^3 + a m c + a n d}{a + c}$ or $a\left(\frac{a^2 + m c + n d}{a + c}\right)$

Now the factor $\frac{a^2 + m c + n d}{a + c}$ refers itself to the preceding constructions, as also the values of $m$, $n$. Having found the value of this factor, if we represent it by $p$, we have only to construct $a \times p$, that is, to make a rhomboid whose altitude is $a$ and base $p$.

4. Lastly, if the dimensions of the numerator exceed the dimensions of the denominator by three, the quantity expresses a solid, the construction of which may always be reduced to a parallelopiped. If, for example, we were to construct $\frac{a^3 b + a^2 b^2}{a + c}$, we might consider this quantity as the same as $a b \times \frac{a^2 + a b}{a + c}$; and, having constructed $\frac{a^2 + a b}{a + c}$ in the manner already explained, if we represent by $m$, the line given by this construction, the question will be reduced to this, namely, to construct $a b \times m$. Now $a b$ represents, as we have seen, a rhomboid; if, therefore, we conceive a parallelopiped, having for its base this rhomboid, and for its altitude the line $m$, the solidity of this parallelopiped will represent $a b \times m$, that is, $\frac{a^3 b + a^2 b^2}{a + c}$.

5. What has been said will suffice for constructing any rational quantity; we proceed now to rational quantities of the second degree.



1st. In order to construct $\sqrt{a b}$, let us draw an indefinite line AB, upon which we may take the part CA, equal to $a$, and the part BC, equal to $b$; upon the whole AB as a diameter, describe a semicircle, cutting in D, the perpendicular CD, raised upon AB at the point C; then CD will be the value of $\sqrt{a b}$; that is, the value of $\sqrt{a b}$ is obtained by finding a mean proportional between the two quantities represented by $a$, $b$. Indeed, we have

$$\text{AC} : \text{CD} :: \text{CD} : \text{CB},$$

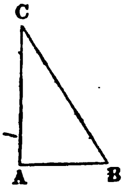
or $$a : \text{CD} :: \text{CD} : b;$$

whence $$\text{CD}^2 = ab, \text{ or } \text{CD} = \sqrt{ab}$$

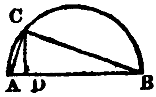
2nd. If we were to construct $\sqrt{3\, ab + b^2}$, or which is the same thing, $\sqrt{(3\, a + b)\, b}$, we should find a mean proportional between $3\, a + b$ and $b$.

3d. In like manner, if the quantity to be constructed were
$$\sqrt{a^2 - b^2},$$
we might consider this the same as $\sqrt{(a+b)(a-b)}$; and then find a mean proportional between $a + b$ and $a - b$. If the quantity were $\sqrt{a^2 + bc}$, and we make $bc = am$, then we shall have $\sqrt{a^2 + am}$, or $\sqrt{(a + m)\,a}$, which is constructed by finding a mean proportional between $a + m$ and $a$ after having constructed the value of $m = \frac{bc}{a}$ by the rules already given.

4th. To construct $\sqrt{a^2 + b^2}$, we can in like manner make $b^2 = am$, and construct $\sqrt{a^2 + am}$, in the manner just explained. But the property of a right angled triangle furnishes a more simple construction. If we draw the line AB, equal to $a$, and at its extremity A erect a perpendicular AC, equal to $b$, joining BC, we shall have $BC^2 = AB^2 + AC^2 = a^2 + b^2$, and, consequently, $BC = \sqrt{a^2 + b^2}$.



5th. We can also, by means of a right angled triangle, construct $\sqrt{a^2 - b^2}$ in a manner different from that above given; draw a line, AB, equal to $a$, and having described upon AB, as a diameter, the semicircle ACB, draw from the point A a chord AC, equal to $b$; then, if we draw BC, this line will be the value of $\sqrt{a^2 - b^2}$; for the triangle ABC being right angled, we shall have $AB^2 = AC^2 + BC^2$; consequently, $BC^2 = AB^2 - AC^2 = a^2 - b^2$; therefore $BC = \sqrt{a^2 - b^2}$.



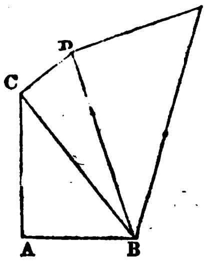
6th. Hence, also, $\sqrt{a^2 + bc}$ admits of a different construction from the above. Make $bc = m^2$, and construct $\sqrt{a^2 + m^2}$, as just shown, first finding for $m$ a mean proportional between $b$ and $c$, as indicated by the equation $bc = m^2$, which gives $m = \sqrt{bc}$.

7th. If there are more than two terms under the radical sign, the construction is to be reduced to one of the preceding methods by means of transformations. If, for example, we have $\sqrt{a^2 + bc + ef}$ we may make $bc = am$, $ef = an$, and we have $\sqrt{a^2 + am + an}$, or $\sqrt{(a + m + n)\,a}$, which may be constructed by finding a mean proportional between $a$ and $a + m + n$, after having constructed the values of $m$ and $n$, namely, $m = \frac{bc}{a}$, $n = \frac{ef}{a}$. We might, moreover, make
$$bc = m^2,\ ef = n^2,$$

and then we should have to construct $\sqrt{a^2+m^2+n^2}$. Now, when there are several positive squares contained under the radical sign, as $\sqrt{a^2+m^2+n^2+p^2+\&c.}$ we may make

$$\sqrt{a^2+m^2}=h,$$

$\sqrt{h^2+n^2}=i$, $\sqrt{i^2+p^2}=k$, and so on; and, as each of the quantities is determined by the preceding, the last will give the value of $\sqrt{a^2+m^2+n^2+p^2+\&c.}$ In order to construct these quantities in the most simple manner, each hypothenuse is to be regarded successively as a side; having, for example, taken AB $= a$, and raised the perpendicular AC $= c$, we may join BC, which will be $h$; then at the point C if we raise upon BC the perpendicular CD $= n$; and having drawn BD, which will be $i$, at the extremity D, we may raise upon BD the perpendicular DE $= p$, and BE will be $k$, and equal to $\sqrt{a^2+m^2+n^2+p^2}$.



If some of the squares are negative, we may combine the method just given with that for constructing $\sqrt{a^2-b^2}$

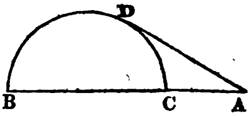
8th. Lastly, if the quantity to be constructed be of this form

$$a\frac{\sqrt{b+c}}{\sqrt{d+e}},$$

multiplying by $\sqrt{d+e}$, will change it into $a\,\dfrac{\sqrt{(b+c)\,(d+e)}}{d+e}$; then, by finding a mean proportional between $b+c$ and $d+e$, and calling it $m$, we have $\dfrac{a\,m}{d+e}$, which is easily constructed.

The construction often becomes much more simple by setting out always from the same principles; but these simplifications are derived from certain considerations which are peculiar to each question, and consequently can be made known only as the occasion presents itself. We will merely remark, in concluding, that although the construction of the radical quantities, which we have been considering, reduces itself to finding fourth proportionals, mean proportionals, and constructing right-angled triangles, still we can arrive at constructions more or less simple or elegant by the method employed for finding these mean proportionals; we shall now, therefore, introduce two other methods of finding a mean proportional between two given lines.

The first consists in describing upon the greater AB (see 3d diagram to Art. 5) of two given lines a semicircle ACB, and, having taken a part AD equal to the less, raising a perpendicular DC and drawing the chord AC, which will be a mean proportional between AB and AD; for, by drawing CB, the triangle ACB is right angled, (Prop. XIX, *Cor.* 2, B. III *El. Geom.*) and consequently AC is a mean proportional between the hypothenuse AB and the segment AD. (Prop. XVII, *Cor.* 5, B. IV, *El. Geom.*)



The second method consists in drawing a line AB, equal to the greater given line, and having taken a part AC equal to the less, describing upon the remainder BC a semicircle CDB, to which if we draw the tangent AD; this tangent is a mean proportional between AB and AC. (Prop. XXVII, B. IV, *El. Geom.*)

It is evident, therefore, that rational quantities may always be constructed by means of straight lines, and radical quantities of the second degree may be constructed by means of the circle and straight line united.

As to radical quantities of higher degrees, their construction depends upon the combination of different curved lines.

We will now proceed to the consideration of questions, the solution of which depends either upon rational quantities or radical quantities of the second degree.

### *Geometrical questions, and modes of forming equations therefrom, and their solutions.*

6. The precepts usually given in algebra for putting questions into equations, are equally applicable to questions in geometry. Here, also, the thing sought is to be represented by some symbol; and the equation is to be constructed in such manner, as to express the relations of the quantity represented by such symbol, in quantities that are known, or in those whose values are attainable; and the reasoning is to be conducted by the aid of this symbol, and of those which represent the other quantities, algebraically, as if the whole were known, and we were proceeding to verify it; this method of proceeding is called *analysis.*

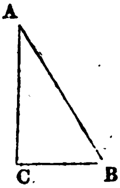
Although in expressing geometrical questions by algebraic equations, we have more resources and more facilities according as we are acquainted with a greater number of the properties of lines, surfaces, &c., still, as algebra itself furnishes the means of discovering these properties, the number of propositions really necessary is very limited. The two propositions that *similar triangles have their homologous sides proportional;* and, that, *in a right angled triangle the square of the* *hypothenuse is equivalent to the sum of the squares of the two other sides*, are the fundamental propositions and the basis of the application of algebra to geometry.

But there are many ways of making use of these propositions, according to the nature of the question, and there is always a discretion to be exercised in the choice of the means, and manner of applying them; and this disecrtion can only be acquired by practice.

When a geometrical question is to be resolved algebraically, it will be necessary to construct a figure that shall represent the several parts or conditions of the problem under consideration, and, if possible, get such expressions for the unknown quantities in terms of those that are known, as may easily be determined, according to the known properties of the figure. But if it so happens, that the required quantity can have no expression which will render it available under the present construction, we may, frequently, by drawing lines having certain relations to the known parts of the figure and also to the unknown, so connect the known to those that are required, as to get available expressions for their values. Having proposed a figure as above, we may, by means of the proper geometrical theorems, proceed to make out as many independent equations as there are unknown quantities; and the resolutions of these will give the solution.

### PROBLEM I.

*The base* BC, *and the sum of the hypothenuse* AB *and perpendicular* AC, *of a right angled triangle being given, to determine the triangle.*



Let BC = $b$, and AC = $x$, and if AB + AC be represented by $s$, then will the hypothenuse AB be represented by $s - x$.

Therefore, by the properties of the right angled triangle (Prop. XXIV, B. IV, *El. Geom.*) we have

$$AC^2 + BC^2 = AB^2$$

Or, $$x^2 + b^2 = s^2 - 2\,sx + x^2,$$

omitting $x^2$ which is common to both sides of the equation, and transposing the other numbers we have,

$$2\,sx = s^2 - b^2$$

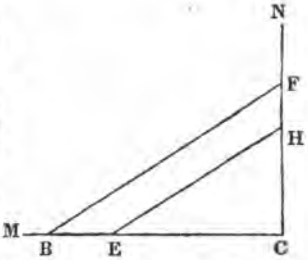
Or, $$x = \frac{s^2 - b^2}{2s}$$

which is the value of the perpendicular AC; where $s$ and $b$ may be any numbers whatever, provided $s$ be greater than $b$. (Prop. X, B. II, *El. Geom.*)

If this quantity is to be constructed, it may be resolved (Art. 2, *Ex.* 4,) into the form

$$x = \frac{(s+b)\ (s-b)}{2s}$$

and constructed as follows. From any point C, draw an indefinite line, CM, and perpendicular thereto another indefinite line, CN,



Set off on CM, CB = $2s$, and take also CE = one of the factors of the numerator, as $s+b$, and take CF = the other factor, viz: $s-b$, draw BF and also EH parallel thereto, and CH will be the value of $x$ required.

In like manner, if the base and the difference of the hypothenuse and perpendicular be given, we shall have by putting $d$ for the difference and the other letters $b$ and $x$ as before; $d+x$ for the hypothenuse. Whence we have,

$$x^2 + 2dx + d^2 = b^2 + x^2$$

$$x = \frac{b^2 - d^2}{2d}$$

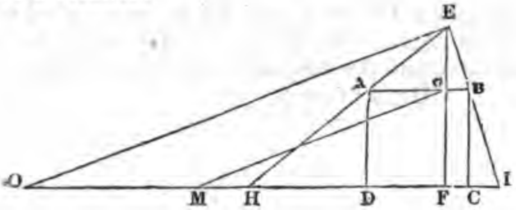
Which may be constructed as before

### PROBLEM II.

*To desribe a square in a given triangle.*

(By *a given triangle* is understood a triangle in which the construction is known, viz: one whose sides, angles, altitude, &c., are known.)

It will be perceived that this question resolves itself into the determination of some point, G, in the altitude EF, through which a line AB, drawn parallel to HI, shall be equal to GF; we may, therefore, determine in algebraic expression, for AB, and also for GF, and put them equal to each other and we shall have a solution.



Let us therefore designate the known altitude EF by $a$, the known base HI by $b$, and the unknown line, GF by $x$; then will $EG = a - x$.

Since AB is parallel to HI, we shall have

EF : EG : : FI : GB : : EI : EB : : HI : AB

consequently, EF : EG : : HI : AB

Or, $a : a - x : : b : AB$,

whence $AB = \frac{ab - bx}{a}$

But, $AB = GF = x$,

therefore, $\frac{ab - bx}{a} = x$

and $ab - bx = ax$

Or, $ab = ax + bx = (a + b)\,x$,

hence, $x = \frac{a + b}{ab}$

In order to construct this quantity, it is necessary to find a fourth proportional to $a+b$, $b$, and $a$ (Art. 2,) which may be done as follows:

From F to O apply a line $FO = a+b$, that is, $= EF + HI$, and join EO; then, having taken $FM = HI = b$, draw MG parallel to EO, which, by its meeting with EF, gives the determination of GF, or the value of $x$; for the similar triangles EFO, GFM, give FO : FM : : FE : FG

or, $a + b : b : : a : FG$

therefore, $FG = \frac{ab}{a + b}$

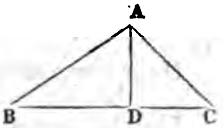
### PROBLEM III.

*Given the base* BC, *and the angles* B *and* C *of the triangle* ABC, *to determine the altitude* AD.

(Angles are made to enter into an algebraic expression by the aid of lines employed in trigonometry, viz, sines, tangents, &c. Thus when it is said that an angle is given, it may be understood that the value of its sine or tangent is given.)

If we designate BC by $a$, and AB by $y$, we shall have CD : AD : : *radius* : *tan.* ACD, (Trigonometry) or if we designate the radius by $r$, and the tangent of the angle C by $t$, we have,

$CD ; y : : r : t$



**whence** $\quad CD = \frac{ry}{t}$

In like manner designating the tangent of the angle B by $t'$ we shall have,

$$BD : y :: r : t'$$

whence $\quad BD = \frac{ry}{t'}$

But, $\quad CD + BD = BC = a$

therefore, $\frac{ry}{t} + \frac{ry}{t'} = a$

whence by changing the construction we may present

$$y = \frac{at\,t'}{rt' + rt}$$

This expression is susceptible of greater simplicity, by introducing, instead of the tangents of the angles C, B, their cotangents; which, let us designate by $q$ and $q'$; observing that the tangent is to radius, as radius to the cotangent, (Trigonometry) and we shall have

$$t : r :: r\,q, \text{ and } t' : r :: r : q';$$

$$\text{whence } t = \frac{r^2}{q}, \text{ and } \frac{t' = r^2}{q'}$$

substituting these values for $t$ and $t'$ in the former equations we have,

$$y = \frac{ar^4 \div qq'}{r^3 \div q + r^3 \div q'} = \frac{ar^4}{qq'} \times \frac{qq'}{qr^3 + q'r^3} = \frac{ar}{q + q'}$$

*Scholium*

From the above it may be perceived that when among quantities that are given those employed do not lead to results so simple as may be desired, it is not always necessary to commence the work anew in order to arrive at a more simple result; but it may be sufficient to express, by equations, the ratios of the quantities first employed to those which we would introduce, as we have expressed $t$ and $t'$ by the equations $t = \frac{r^2}{q}$ and $t' = \frac{r^2}{q'}$ by which a solution dependent upon $q$ and $q'$ is obtained.

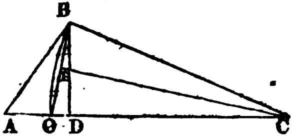
### PROBLEM IV.

*Given the three sides of a triangle* ABC, *to find the segments* AD, DC, *formed by the perpendicular* BD, *and also to find the perpendicular* BD.

Let BD $= y$, CD $= x$, BC $= a$, AB $= b$, AC $= c$, then AD or AC — CD $= c - x$.

Hence, we have

$$x^2 + y^2 = a^2, \text{ and } c^2 - 2\,cx + x^2 + y^2 = b^2$$



Let the second equation be subtracted from the first, and we have

$$2\,cx - c^2 = a^2 - b^2;$$

whence, we have

$$x = \frac{a^2 - b^2 + c^2}{2c} = \frac{a^2 - b^2}{2c} + \tfrac{1}{2}c,$$

which may be resolved into

$$x = \tfrac{1}{2}\,\frac{(a+b)\,(a-b)}{c} + \tfrac{1}{2}c$$

By reference to article 2 it will be perceived that to obtain the value of $x$, we have to find a fourth proportional to $c, a+b$ and $a-b$, to take one half of this and add it to $\frac{1}{2}c$, or one half the side AC.

*Scholium.*

Several important conclusions may be drawn from this solution, some of which we will notice; showing at once, different modes of putting geometrical questions into equations, and how, by varying the propositions of these equations, new propositions may be discovered.

1st. The equation $2\,cx - c^2 = a^2 - b^2$ is resolveable into $c\,(2x - c) = (a+b)\,(a-b)$.

Now since the product of the first two factors is equal to the product of the last two, we may consider the first two as the extremes, and the last two as the means of a proportion; hence we have

$$c : a+b :: a-b : 2x - c, \text{ or } x - (c - x);$$

or, $\quad$ AC : BC + AB : : BC — AB : CD — AD

2nd. If from the point C as a centre, and with a radius BC, we describe the arc BO, and draw the chord BO we have

$$\text{BD}^2 + \text{DO}^2 = \text{BO}^2;$$

now $\quad$ DO = CO — CD = BC — CD $= a - x$,

therefore $\quad \text{BO}^2 = y^2 + a^2 - 2\,ax + x^2$;

but we have found above $\quad y^2 + x^2 = a^2$;

consequently $\quad \text{BO}^2 = 2\,a^2 - 2\,ax = 2\,a\,(a - x)$.

Putting for $x$ its value $\dfrac{a^2 - b^2 + c^2}{2c}$,

since $2\,ac - a^2 - c^2 = -(a^2 - 2\,ac + c^2) = -(c - a)^2$,

we shall have

$$\text{BO}^2 = 2\,a\left(a + \frac{b^2 - a^2 - - b^2}{2\,c}\right) = 2\,a\left(\frac{2\,ac - a^2 - c^2 + b^2,}{2c}\right)$$

$$= \frac{a}{c}\left(b^2 - (c - a)^2\right).$$

Now, by considering $c - a$ as a single quantity, we find
$$b^2 - (c - a)^2 = (b + c - a)\,(b - c + a),$$
hence
$$\text{BO}^2 = \frac{a}{c}\,(b + c - a)\,(b - c + a),$$
which may be put under this form,
$$\text{BO}^2 = \frac{a}{c}\,(a + b + c - 2\,a)\,(a + b + c - 2\,c).$$

If, therefore, we designate the sum of the three sides by $2s$, we shall have
$$\text{BO}^2 = \frac{a}{c}\,(2\,s - 2\,a)\,(2\,s - 2\,c) = 4\frac{a}{c}\,(s - a)\,(s - c).$$

Letting fall from the point C upon OB the perpendicular CI, we obtain from the right angled triangle CIO this proportion,
$$\text{CO} : \text{OI} :: \text{R} : \text{sin. OCI}, \quad \text{(Trigonometry,)}$$
that is,
$$a : \tfrac{1}{2}\,\text{BO} :: \text{R} : \text{sin. OCI},$$
whence
$$\tfrac{1}{2}\,\text{BO} = \frac{a \text{ sin. OCI},}{\text{R}}, \text{ or BO} = \frac{2\,a \text{ sin. OCI}}{\text{R}};$$
consequently
$$\text{BO}^2 = \frac{4\,a^2\,(\text{sin. OCI})^2}{\text{R}^2}$$
Putting these two valves of BO² equal to each other we have
$$\frac{4\,a^2\,(\text{sin. OCI})^2}{\text{R}^2} = \frac{4\,a}{c}\,(s - a)\,(s - c),$$
or, dividing by $4\,a$, and making the denominators to disappear,
$$a\,c\,(\text{sin. OCI})^2 = \text{R}^2(s - a)\,(s - c):$$
that is, dividing by $ac$, putting R equal to 1, and extracting the square root,
$$\text{sin. OCI} = \sqrt{\frac{(s - a)\,(s - c,)}{ac}}$$
which agrees with a formula, in Trig.

3d. We may, from the equation $y^2 + x^2 = a^2$, deduce the following: $y^2 = a^2 - x^2 = (a + x)\,(a - x)$, putting for $x$ its value, as found in the problem we have
$$y^2 = \left(a + \frac{a^2 - b^2 + c^2}{2c}\right)\left(a + \frac{b^2 - a^2 - c^2}{2c}\right)$$
$$= \left(\frac{2\,ac + a^2 + c^2 - b^2}{2c}\right)\left(\frac{2\,ac - a^2 - c^2 + b^2}{2c}\right)$$
$$= \left(\frac{(a + c)^2 - c^2}{2c}\right)\left(\frac{b^2 - (c - a)^2}{2c}\right)$$

$$= \left(\frac{(a+c+b)(a+c-b)}{2c}\right)\left(\frac{(b+c-a)(b-c+a)}{2c}\right);$$

consequently,

$$4c^2y^2 = (a+c+b)(a+c-b)(b+c-a)(b-c+a)$$
$$=(a+b+c)(a+b+c-2b)(a+b+c-2a(a+b+c-2c);$$

or, designating the sum of the three sides $a+b+c$ by $2s$,

$$4c^2y^2=2s(2s-2b)(2s-2a)(2s-2c)$$
$$=16s(s-b)(s-c)(s-a),$$

or, dividing by 16 and taking the square root,

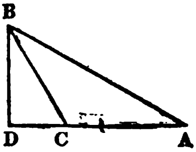
$$\frac{cy}{2} = \sqrt{s(s-b)(s-c)s-a)}$$

But $\frac{cy}{2}$, or $\frac{AC \times BD}{2}$ is the surface of the triangle ABC.

Hence, *to find the surface of a triangle by means of the three sides, we must subtract each side successively from the half sum, multiply the half sum and the three remainders continually together, and take the square root of this product; which agrees with* Prop. XL, B. IV, El. Geom.

4th. The equations $2cx - c^2 = a^2 - b^2$ may be resolved as follows,

$$b^2 = a^2 + c^2 - 2cx$$



but if the perpendicular fall without the triangle as in the present diagram, AD will then be $c+x$ instead of $c-x$, hence, designating the sides as before, we have $y^2+x^2 = a^2$, and $y^2+c^2+2cx+x^2=b^2$, the first subtracted from the second gives $c^2+2cx = b^2-a^2$, or $c(c+2x) = (b+a)(b-a)$;

whence, $c : b+a :: b-a : c+2x$

Now, $c+2x$, or $x+c+x$ = CD + AD;

consequently, AC : AB + BC : : AB — BC : CD + AD

5th. The same equation $c^2+2cx = b^2 - a^2$, may also be put under the following form

$b^2 = a^2 + c^2 + 2cx$, which answers to the last figure. comparing this with the equation,

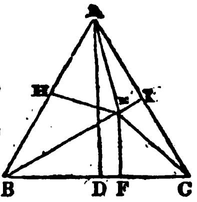
$b^2 = a^2 + {}^2c - 2cx$, which answers to the former figure, we observe that $b^2$ the square of the side AB opposite to the acute angle C, is less than the sum of the squares of the other two sides $a^2 + c^2$ by $2cx$; on the contrary, the square of the side AB opposite the obtuse angle, (see last figure,) is equal to $a^2 + c^2 + 2cx$, that is, greater than the sum of the squares of the other two sides, by $2cx$, which agrees with propositions XXVI and XXVII, B. IV, *El. Geom*; by these propositions we may determine when the angles of a triangle are to be calculated by means of the sides, whether the angle sought, be acute, or obtuse.

6th. The two equations $b^2 = a^2 + c^2 - 2\,cx$
and $b^2 = a^2 + c^2 + 2\,cx$
confirms the theory of positive and negative quantities, for it is plain that the segment CD takes different directions, according as the perpendicular BD falls within the triangle or without it. In these two equations the term $2\,cx$ has, in fact, contrary signs. Hence, whatever result we obtain with regard to one of these triangles, we obtain that which belongs to the analogous case of the other by merely changing its sign of that part which takes a different direction on the same line.

Now, since in the above theorem, respecting the surface of a triangle the segment CD does not come into consideration; therefore, the proposition is equally applicable to all kinds of plane triangles.

### PROBLEM. V.

*Having the lengths of the three perpendiculars,* EF, EI, EH, *drawn from a certain point* E, *within an equilateral triangle* ABC, *to its three sides, to determine the sides.*



Draw the perpendicular AD, and having joined EA, EB, and EC, put $EF=a$ $EI=b$, $EH=c$, and BD (which is $\frac{1}{2}$BC)$=x$.

Then, since AB, BC, or CA, are each$=2x$, we shall have, Prop. XXIV. B. IV. *El. Geom.*
$AD=\sqrt{(AB^2-BD^2)}=\sqrt{(4x^2-x^2)}=\sqrt{3x^2}$ $=x\sqrt{3}$.

And because the area of any plane triangle is equal to half the rectangle of its base and perpendicular, it follows that

$$\begin{aligned} \text{triangle } ABC &= \tfrac{1}{2}BC\times AD = x\times x\sqrt{3} = x^2\sqrt{3}, \\ BEC &= \tfrac{1}{2}BC\times EF = x\times a \quad = ax, \\ AEC &= \tfrac{1}{2}AC\times EI = x\times b \quad = bx, \\ AEB &= \tfrac{1}{2}AB\times EH = x\times c \quad = cx. \end{aligned}$$

But the last three triangles BEC, AEC, AEB, are together, equal to the whole triangle ABC, whence

$$x^2\sqrt{3}=ax+bx+cx.$$

And, consequently, if each side of this equation be divided by $x$, we shall have

$$x\sqrt{3}=a+b+c, \text{ or}$$

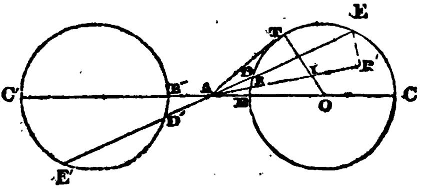
$$x=\frac{a+b+c}{\sqrt{3}}$$

Which is therefore, half the length of either of the three equal sides of the triangle.

*Cor.* Since, from what is above shown, AD is$=x\sqrt{3}$, it follows, that the sum of all the perpendiculars, drawn from any point in an equilateral triangle to each of the sides, is equal to the whole perpendicular of the triangle.

### PROBLEM VI.

*From a given point* A, *without a circle* BDC, *to draw a straight line* AE *in such a manner that the part* DE, *intercepted in the circle, shall be equal to a given line.*



Since the circle BDEC is given, its diameter is supposed to be known; and, since the point A is given, we draw through the centre O the straight line AOC, the line AB is to be considered as known, and consequently the line AC. In order to know how the line AE is to be drawn, we have only to determine what ought to be the magnitude of AD, that, when produced, the part DE should be equal to the given line. We will designate AD by $x$, AB by $a$, AC by $b$, and the given line, to which DE is to be made equal, by $c$.

Since the figure BDEC is a circle, the secants AC, AE, must be reciprocally proportional to the parts without the circle; that is,

AC : AE : : AD : AB (Prop. XXXVI. B. IV. *El. Geom.*), or $b : x+c :: x : a$;

whence $x^2+cx=ab$,

an equation of the second degree, which, being resolved, gives

$$x=-\tfrac{1}{2}c\pm\sqrt{\tfrac{1}{4}c^2+ab}.$$

of which the first value only, $-\frac{1}{2}c+\sqrt{\frac{1}{4}c^2+ab}$, satisfies the question under consideration.

In order to finish the solution, it is necessary to construct this quantity, which can be done without employing the transformations made known, art. 2. For this purpose, we draw from the point A the tangent AT, which, being a mean proportional between AB and AC, gives $\text{AT}^2=ab$; the value of $x$ therefore becomes

$$x=-\tfrac{1}{2}c+\sqrt{\tfrac{1}{4}c^2+\text{AT}^2}.$$

The radius TO being drawn, becomes a perpendicular at AT;

if then we take TI equal to $\frac{1}{2}c$, by drawing AI, we shall have $AI=\sqrt{\frac{1}{4}c^2+AT^2}$; therefore, in order to obtain $x$, we have only to apply TI from I to R, and to describe from the point A, as a centre, and with the radius AR, the arc RD, which will determine D, the point sought; for

$$AD, \text{ or } AR=AI-IR=AI-TI=\sqrt{\tfrac{1}{4}c^2+AT^2}-\tfrac{1}{2}c=x$$

In order now to know what the second value of $x$ signifies, namely,

$$x=-\tfrac{1}{2}c-\sqrt{\tfrac{1}{4}c^2+ab},$$

it must be observed that, as it is wholly negative, it can only fall in the direction opposite to that toward which AD tends. Let us see, then, if there be a question depending upon the same quantities and the same reasoning, which fulfils this condition. If now we suppose $a$ and $b$ negative, the equation $x^2+cx=ab$, undergoes no change; since, therefore, when the circle BDEC becomes B′D′E′C′, situated toward the left in the same manner that BDEC is toward the right, it follows that the solution of this case is contained in the same equation; the second value of $x$, or $-\frac{1}{2}c-\sqrt{\frac{1}{4}c^2+ab}$, belongs to the same case, and satisfies the same conditions; if, therefore, in the preceding construction, we apply IT from I to R′ on AI produced, and from the point A, as a centre, and with a radius equal to AR′, we describe an arc cutting the circumference B′D′E′C′ in E′, the point E′ will be such that the part intercepted, E′D′, will be equal to $c$. Indeed,

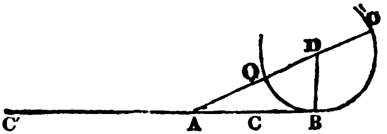
$$AE'=AR'=AI+IR'=\sqrt{\tfrac{1}{4}c^2+AT^2}+\tfrac{1}{2}c,$$

that is, AE′ is equal to the second value of $x$, the signs being changed. Now, since we apply this quantity in a direction opposite to that in which $x$ extends, it follows that AE′ is in reality the second value of $x$.

Hence, as the two circles are equal and situated in the same manner, the two solutions may both belong to the same circle, so that if we describe from the point A, as a centre, and with a radius AR′, the arc R′E, the line AE will also resolve the question; indeed, it is evident that the point E, determined in this manner, is in the line AD, (obtained by the first construction,) produced. But of the two solutions, furnished by algebra, the first falls on the right of the point A, and appertains to the point D of the convex circumference, while the second falls on the left, and appertains to the point E′ of the concave part of the circumference.

### PROBLEM VII.

*Let it be required to find the direction of a given line* AB *from a point* C, *such, that its distance from the point* A, *shall be a mean proportional between its distance from the point* B *and the whole line.*



Let the given line AB be designated by $a$, and the distance AC required by $x$; then BC will be $a-x$; and, since the proportion required is

$$AB : AC :: AC : CB,$$

or

$$a : x :: x : a-x,$$

we shall have

$$x^2=a^2-ax, \text{ or } x^2+ax=a^2,$$

an equation of the second degree, which, being resolved, gives

$$x=-\tfrac{1}{2}a\pm\sqrt{\tfrac{1}{4}a^2+a^2}$$

In order to construct the first value of $x$, we must, according to what has been said, (Art. 5,) raise the point B the perpendicular BD$=\frac{1}{2}a$; and, having drawn AD, we shall have

$$AD=\sqrt{BD^2+AB^2}=\sqrt{\tfrac{1}{4}a^2+a^2};$$

we have then only to subtract from this line the quantity $\frac{1}{2}a$, which is done by applying BD from D to O; then we shall have AO$=\sqrt{\frac{1}{4}a^2+a^2}-\frac{1}{2}a$, that is, it will be equal to $x$. We then apply AO from A to C toward B, and C will be the point sought.

As to the second value of $x$, namely,

$$x=-\tfrac{1}{2}a-\sqrt{\tfrac{1}{4}a^2+a^2},$$

if we apply BD from D to O′ on AD produced, then we shall have

$$AO'=\tfrac{1}{2}a+\sqrt{\tfrac{1}{4}a^2+a^2};$$

and, as the value of $x$ is this quantity taken negatively, we apply AO′ from A to C′ on AB produced in a direction opposite to that toward which $x$ is supposed in the solution to extend; and we shall have a second point C′, which will also be such, that its distance from the point A, will be a mean proportional between its distance from the point B and the whole line AB.

*Scholium.* 1. We may observe that this question contains that of *dividing a line in extreme and mean ratio;* also the construction which we have obtained, is the same as that given in the *Elements of Geometry,* (Prop. IV. B. IV.) But it will be perceived, that we are made acquainted with this construction by algebra, whereas in the *Elements of Geometry* we supposed the construction, and only demonstrated its truth.

2. With a little attention to the course pursued in the preceding questions, it will be evident that we have always taken for the unknown quantity a line, which being once known, serves, by observing the conditions of the question, to determine all the others. This is the course to be pursued in all cases, but there is a choice with regard to the line to be used; there are often several, each of which has the property of determining all the others, if once known. Among these some would lead to more simple equations than others. The following rule is given to aid in such cases.

3. *If among the lines or quantities, which would, when taken each for the unknown quantity, serve to determine all the other quantities, there are two which would in the same way answer this purpose, and it would be foreseen that such would lead to the same equation, (the signs + and — excepted); then we ought to employ neither of these, but take for the unknown quantity one which depends equally on both;* that is, their half sum, or their half difference, or a mean proportional between them, or &c., and we shall always arrive at an equation more simple than by employing either the one or the other.

4. The question we have resolved, (Prob. VI,) may be used to illustrate what is here said. In this question there is no reason for taking AD rather than AE, for the unknown quantity; by taking AD for the unknown quantity $x$, we have $x+c$ for AE; and, by taking AE for the unknown quantity $x$, we should have $x-c$ for AD; and, as to the rest, the mode of proceeding is the same for each case; so that the equations differ only in the signs. If, therefore, instead of taking either for the unknown quantity, we take their half sum, and designate it by $x$, since their half difference DE$=c$ is given, we shall have

$$AE=x+\frac{1}{2}c, \text{ and } AD=x-\frac{1}{2}cx,$$

whence, according to the proposition adopted in the first solution,

$$(x+\frac{1}{2}c)(x-\frac{1}{2}c)=ab$$

or

$$x^2-\frac{1}{4}c^2=ab,$$

a more simple equation than the former, and which gives

$$x=\sqrt{\frac{1}{4}c^2+ab};$$

and, since AE$=x+\frac{1}{2}c$, we have immediately

$$AE=\frac{1}{2}c+\sqrt{\frac{1}{4}c^2+ab},$$

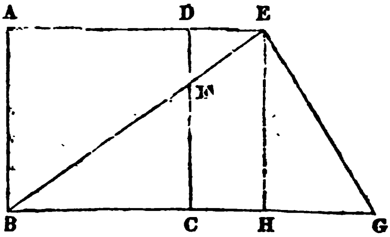
and

$$AD=-\frac{1}{2}c+\sqrt{\frac{1}{4}c^2+ab},$$

as before found.

### PROBLEM VIII.

*Let it be required to draw a right line* BFE *from one of the angles* B *of a given square* BC, *so that the part* FE *intercepted by* DE *and* DC, *shall be of a given length.*



Draw EG perpendicular to BE to meet BC produced in G, and from the angle E draw EH perpendicular to BG.

Let BC or DC$=a$, FE$=b$, BF$=y$, and CG$=x$.

Since the triangle EHG is similar to the triangle BCF and the side EH=the side BC, hence the hypothenuse EG= the hypothenuse BF.

But $BE^2+EG^2=BG^2$,

or $2y^2+2by+b^2=a^2+2ax+x^2$

and because the triangle BCF and BEG are similar,

$BF : BC :: BG : BE$

or $y : a :: a+x : y+b$

hence, $y^2+by=a^2+ax$

multiplying this equation by 2, we have

$$2y^2+2by=2a^2+2ax.$$

Subtracting the last from the former equation, we have

$$b^2=-a^2+x^2,$$

or $b^2+a^2=x^2$,

hence, $x=\sqrt{b^2+a^2}$

having the value of $x$, $y$ may be found in the equation

$$y^2+by=a^2+ax$$

completing the square $y^2+by+\frac{1}{4}b=a^2+ax+\frac{1}{4}b$

hence $y=\sqrt{a^2+ax+\frac{1}{4}b}-\frac{1}{2}b$

Let DE$=z$,

then $y : a :: b : z$

and $yz=ab$

hence, $z=\dfrac{ab}{y}$

*Scholium.* This problem is susceptible of several modes of solution, but perhaps none more simple than the one here given, for most of the modes of which it is susceptible, involve powers and equations higher than quadratics.

### EXAMPLES FOR PRACTICE.

Ex. 1. To find the side of a square, inscribed in a given semicircle, whose diameter is $d$.

Ans. $\frac{1}{5}d\sqrt{5}$

Ex. 2. To find the side of an equilateral triangle inscribed in a circle whose diameter is $d$; and that of another circumscribed about the same circle. Ans. $\frac{1}{2}d\sqrt{3}$, and $d\sqrt{3}$

Ex. 3. To find the sides of a rectangle, the perimeter of which shall be equal to that of a square, whose side is $a$, and its area half that of a square. Ans. $a+\frac{1}{2}a\sqrt{2}$ and $a-\frac{1}{2}a\sqrt{2}$

Ex. 4. Having given the perimeter (12) of a rhombus, and the sum (8) of its two diagonals, to find the diagonals.

Ans. $\frac{1}{2}\sqrt{2} + \sqrt{5}$ $4+\sqrt{2}$ and $4-\sqrt{2}$

Ex, 5. Required the area of a right angled triangle, whose hypothenuse is $x^{3x}$ and the base and perpendicular $x^{2x}$ and $x^{x}$,

Ans. 1.029085

Ex. 6. Having given the two contiguous sides ($a$, $b$) of a parallelogram, and one of its diagonals ($d$), to find the other diagonal. Ans. $\sqrt{(2a^2+2b^2-d^2)}$

Ex. 7. Given the base (194) of a plane triangle, the line that bisects the vertical angle (66), and the diameter (200) of the circumscribing circle, to find the other two sides.

Ans. 81.36587 and 157, 43865

Ex. 8. The lengths of two lines that bisect the acute angles of a right angled plane triangle, being 40 and 50 respectively, it is required to determine the three sides of the triangle.

Ans. 35.80737, 47.40728, and 59.41143

Ex. 9. Given the hypothenuse (10) of a right angled triangle, and the difference of two lines drawn from its extremities to the centre of the inscribed circle (2), to determine the base and perpendicular. Ans. 8.08004 and 5.87447

Ex. 10. Having given the lengths ($a$, $b$) of two chords, cutting each other at right angles, in a circle, and the distance ($c$) of their point of intersection from the centre, to determine the diameter of a circle.

Ans. $\frac{1}{2}\sqrt{\left\{\frac{1}{2}(a^2+b^2)+2c^2\right\}}$

Ex. 11. Two trees, standing on a horizontal plane, are 120 feet asunder; the height of the highest of which is 100 feet, and that of the shortest 80; where in the plane must a person place himself, so that his distance from the top of each tree, and the distance of the tops themselves, shall be all equal to each other?

Ans. $20\sqrt{21}$ feet from the bottom of the shortest
and $40\sqrt{3}$ feet from the bottom of the other

Ex. 12. Having given the sides of a trapezium, inscribed in a circle, equal to 6, 4, 5, and 3, respectively, to determine the diameter of the circle. Ans. $\frac{1}{20}\sqrt{(130\times 133)}$ or 6.574572

Ex. 13. Supposing the town A to be 30 miles from B, B 25 miles from C, and C 20 miles from A; where must a house be erected that it shall be at an equal distance from each of them?

Ans. 15.118578 miles from each, viz: in the centre of a circle whose circumference passes through each of the three towns.

Ex. 14. In a plane triangle, having given the perpendicular ($p$), and the radii ($r$ R) of its inscribed and circumscribing circles, to determine the triangle.

Ans. The base $\dfrac{2r\sqrt{(2p\text{R}-4r\text{R}-r^2)}}{p-2r}$

## DETERMINATION OF ALGEBRAICAL EXPRESSIONS FOR SURFACES AND SOLIDS.

7. We have seen in the *Elements of Geometry*, that surfaces depend upon the product of two dimensions, and solids upon the product of three dimensions; so that, if the several dimensions of one or two solids, or two surfaces, which we would compare, have to the several dimensions of the other, each the same ratio, the two surfaces will be to each other as the squares, and the two solids as the cubes, of the homologous dimensions; and more generally still, if any two quantities of the same nature are expressed each by the same number of factors, and if the several factors of the one have to the several factors of the other, each the same ratio, the two quantities will be to each other as their homologous factors, raised to a power whose exponent is equal to the number of factors. If, for example, the two quantities were $a\,b\,c\,d$, $a'\,b'\,c'\,d'$, and we had

$$a : a' :: b : b' :: c : c' :: d : d',$$

then we should have

$$b' = \frac{a'\,b}{a},\ c' = \frac{a'\,c}{a},\ d' = \frac{a'\,d}{a},$$

and consequently,

$$\begin{aligned} a\,b\,c\,d : a'\,b'\,c'\,d' &:: a\,b\,c\,d : \frac{a'^4\,b\,c\,d}{a^3}, \\ &:: a \quad : \frac{a'^4}{a^3}, \\ &:: a^4 \quad : a'^4. \end{aligned}$$

What is here said is true not only of simple quantities; the same may be shown with respect to compound quantities. Let the quantities whose dimensions are proportional be

$$a\,b + c\,d,\ a'\,b' + c'\,d';$$

since, by supposition,

$$a : a' :: b : b' :: c : c' :: d : d',$$

we shall have

$$b' = \frac{a'\,b}{a},\ c' = \frac{a'\,c}{a},\ d' = \frac{a'\,d}{a},$$

and consequently

$$\begin{aligned} a\,b + c\,d : a'\,b' + c'\,d' &:: a\,b + c\,d : \frac{a'^2\,b}{a} + \frac{a'^2\,c\,d}{a^2} \\ &:: a\,b + c\,d : \frac{a'^2\,ab + a'^2\,cd}{a^2} \\ &:: a^2\,(a\,b + c\,d) : a'^2\,(a\,b + c\,d), \\ &:: \quad a^2 \quad : \quad a'^2. \end{aligned}$$

It follows, from what is here proved, that the surfaces of similar figures are as the squares of their homologous dimensions, and that the solidities of similar solids are as the cubes of their homologous dimensions; for, whatever these figures and these solids may be, the former may always be considered as composed of similar triangles, having their altitudes and bases proportional, (Prop. XXIII. B. IV. *El. Geom.*,) and the latter as composed of similar pyramids, having their three dimensions also proportional. (Prop. XXXII, *Cor.* 5, B. II, *El. Sol. Geom.*)

It will hence be perceived, that quantities may be readily compared, when they are expressed algebraically; and this may be done, whether the quantities be of the same or of a different species, as a cone and a sphere, a prism and a cylinder, provided only that they are of the same nature, that is, both solids, or both surfaces.

*Let it be required to investigate the properties of a pyramid and and also of a frustum of a pyramid.*

Let $h$ = the altitude, $s$ = the greater base and $s'$ the smaller base, and $h'$ = the altitude of the vertical pyramid taken from the top of the frustum,

then we shall have $\sqrt{s'} : \sqrt{s} :: h' : h + h'$ or the altitude of the whole pyramid, and consequently,

$$(h + h')\,\sqrt{s'} = h'\,\sqrt{s} = h\,\sqrt{s'} + h'\,\sqrt{s'}$$

and $$h'\sqrt{s} - h'\,\sqrt{s'} = h\,\sqrt{s'}$$

dividing by $\sqrt{s} - \sqrt{s'}$

we have $$h' = \frac{h\sqrt{s'}}{\sqrt{s} - \sqrt{s'}}$$
whence $h'$ becomes known.

Let now $h + h'$ be represented by $k$, and we may have from the first equation $\frac{h'\sqrt{s}}{\sqrt{s'}} = k$

then we shall have for the solidity of the whole pyramid $\frac{s\,k}{3}$ (1)

the solidity of the small pyramid $\frac{s'\,h'}{3}$

substituting for $k$ its value, we have for the solidity of the whole pyramid $\frac{h'\,s\sqrt{s}}{3\sqrt{s'}}$

hence the solidity of the frustum will be $\frac{h'\,s\sqrt{s}}{3\sqrt{s'}} - \frac{s'\,h'}{3}$

$$\text{or } \frac{h'}{3}\left(\frac{s\sqrt{s}}{\sqrt{s'}} - s'\right) \text{ or } \frac{h'}{3}\left(\frac{s\sqrt{s} - s'\sqrt{s'}}{\sqrt{s'}}\right) \text{ - - - - } (2)$$

putting for $h'$ its value found above we have,

$$\frac{h\,\sqrt{s'}}{3\,(\sqrt{s} - \sqrt{s'})} \times \frac{s\,\sqrt{s} - s'\sqrt{s'}}{\sqrt{s'}}$$

which being reduced gives,

$$\frac{h}{3}\,(s + \sqrt{ss'} + s') \text{ - - - - - - - - } (3)$$

that is, the solidity of the frustum is equal to the sum of the greater base, the smaller base, and a mean proportional between the two bases, multiplied by the altitude of the frustum; which agrees with the proposition in geometry.

And if the two bases are equal, viz: if $s = s'$ then the solid becomes a prism, and the expression will become

$$\frac{h}{3}(s + s + s) \text{ or } \frac{1}{3}\,h\,(3s) = hs \text{ - - - } (4)$$

that is, the solidity of the prism is equal to its base multiplied by its altitude.

Let the lateral surface of the pyramid be used as an element in its investigation.

To find the lateral surface of a frustum of a regular pyramid, having the two bases and slant height given, as well as the radius of the circle inscribed in the larger base.

Let the larger base be called $s$, the smaller $s'$, the slant height $h$, and the radius of the circle inscribed in the larger base, $r$.

The perimeter of the larger base will be $\frac{s}{\frac{1}{2}r}$ and the perimeter of the smaller base may be found from the following

$\sqrt{s} : \sqrt{s'} :: \frac{s}{\frac{1}{2}r} : \frac{s\sqrt{s'}}{\frac{1}{2}r\sqrt{s}}$ the perimeter of the smaller base: whence $\left(\frac{s}{r} + \frac{s\sqrt{s'}}{r\sqrt{s}}\right) h =$ the lateral surface.

Or we may investigate the surface of the frustum in connection with the whole pyramid of which it is a part.

Thus $\frac{s}{\frac{1}{2}r} : \frac{s\sqrt{s'}}{\frac{1}{2}r\sqrt{s}} :: h : \frac{h\sqrt{s'}}{\sqrt{s}}$ the slant height of the whole pyramid, which make $=k$, and the vertical pyramid cut from the frustum will be $k-h$.

Hence, we have $r : k :: s : \frac{sk}{r}$ the lateral surface of the whole pyramid. - - - - - - - - - - - - - - (5)

And since the lateral surfaces of similar pyramids are proportional to their bases, we may make $s : s' :: \frac{sk}{r} : \frac{s'k}{r}$ the surface of the vertical pyramid cut from the frustum.

Hence $\frac{sk}{r} - \frac{s'k}{r}$ - - - - - - - - - - - (6)

or $(s - s')$ the difference of the two bases, multiplied by $\frac{k}{r}$ the ratio of the slant height of the pyramid to the radius of the base, is equal to the lateral surface of the frustum.

It may be observed that the ratio $\frac{k}{r}$ is constant whether applied to the whole pyramid, to the pyramid cut off, or to the frustum; and is such as would be represented by the sine of the angle formed by its slant side with the plane of the base.

*Cor.* Whence we have for the lateral surface of the frustum of a pyramid, this rule:

*Multiply the difference of the bases by the sine of the angle which the slant side makes with the base, or by the ratio of the whole slant height of a perfect pyramid on the same base to the radius of the base, which will give the lateral surface.*

8. If $\pi$ represent the ratio of the circumference of a circle to the diameter, a ratio which is known with sufficient accuracy for practical purposes (Prop. XIX. B. V. *El. Geom.*,) the circumference of any circle whose radius is $r$, will be $2\pi r$ - (1,)
and its surface $\pi r^2$ - - - - - - - - (2.)
Hence it is evident that the areas of circles increase as the squares of their radii, $\pi$ being always of the same value, the quantity $\pi r^2$ depends on, and is proportional to $r^2$ - (3.)

If $h$ be the altitude of a cylinder the, radius of whose base is $r$, for its convex surface we shall have $\pi rh$ - - (4)
for its solidity $\pi r^2h$ - - - - - - - - (5)
and for the same reason we shall have $\pi r'^2h'$ for the solidity of another cylinder, whose altidude is $h'$, and the radius of whose base is $r'$, hence the solidities of the two cylinders are to each other as the altitudes multiplied by the squares of the radii of the bases. If their altitudes and radii of their bases are proportional, in which case the cylinders will be similar, we shall have

$$h : h' : : r : r'$$

consequently $= h' \frac{hr'}{r}$

and the ratio $r^2 h : r'^2h$

becomes $\frac{r^2h : r^3h}{r}$

or multiplying by $r$ and dividing by $h$, $r^2 : r'^3$ - - (6,)
that is the solidities are as the cubes of the radii of their bases, as before shown in geometry.

Also if the altitude of a cone is $h$, and $r$ the radius of the base, its convex surface may be expressed by

$$\sqrt{r^2 + h^2} \times \frac{2r\pi}{2} = r\pi\sqrt{r^2 \times h^2} \quad - \quad - \quad - \quad (7)$$

Or let $k =$ the slant height of the cone, then will its lateral surface be

$$r\pi \times k = rk\pi \quad - - - - - - - - \quad (8)$$

which result will be also obtained if we take $\pi r^2$ the area of the base, and increase it in the ratio of $r : k$, viz. :

$$r : k : : \pi r^2 : \frac{r^2k\pi}{r} \text{ or } rk\pi$$

The solidity of the cone will be

$$\pi r^2 \times \frac{h}{3} = \frac{\pi r^2h}{3} \quad - - - - - - - - \quad (9)$$

or if we multiply the convex surface of the cone by one-third of its distance from the centre of the base, (Prop. IX. B. III. *El. S. Geom.*) we shall obtain the same result.

The distance from the centre of the base to the surface may be expressed $k : h : : r : \frac{rh}{k}$ the distance, - - (10.)

hence the solidity will also be

$$rk\pi \times \frac{rh}{3k} = \frac{\pi r^2h}{3} \text{ as before.}$$

If $h' =$ the altitude of a frustum of the cone, then may the slant height of the frustum be $h : k : h' : \frac{h'k}{h}$ the slant height

required, which call $k'$; let the radius of the smaller base of the frustum be $r'$, then will the lateral surface be $\frac{2r\pi+2r'\pi}{2}\times k'$ $= rk'\pi+r'k'\pi$ - - - - - - - - - (11)
$\pi r^2 =$ the greater base, and $\pi r'^2 =$ the smaller base (Prop. X, B. III. *El. S. Geom.*) the solidity$=\frac{1}{3}h'(\pi r^2+\pi r'^2+\sqrt{\pi r^2 r'^2})$ (12)

And since $\frac{\pi r^2 h'}{3}$ and $\frac{\pi r'^2 h}{3}$ each expresses the solidity of a cone on one or the other of the bases, and whose altitude is equal to that of the frustum, hence if one of those expressions is taken from that of the frustum, the remainder will express a conesected frustum.

Thus $\frac{1}{3}h(\pi r^2+\pi r'^2+\sqrt{\pi r^2 r'^2})-\frac{\pi r^2 h}{3}=\frac{1}{3}h(\pi r'^2+\sqrt{\pi r^2 r'^2})$ - (13)
which is a conesected frustum, having a conical cavity on its larger base and $\frac{1}{3}h(\pi r^2+\sqrt{\pi r^2 r'^2})$ - - - - (14)
expresses a conesected frustum the cavity of which is formed on the smaller base.

Or if we multiply the convex surface by $\frac{1}{3}$ its distance from the centre of either base, we shall have the solidity of a conesected frustum, whose cavity is taken from the opposite base (Prop. XI, Cor. B. III, *El. Sol. Geom.*)

Thus $rk\pi + r'k'\pi$, formula (11) the expression of the lateral surface, multiplied $\frac{rh}{k}$ formula (10,) the distance of the surface from the centre of the larger base gives

$$\frac{r^2h\pi + rr'hk'\pi}{k} \quad - \; - \; - \; - \; - \; - \quad (15)$$

When the two bases of the frustum are equal, the coinsected frustum becomes a conesected cylinder, and $r$ and $r'$, $k$ and $h'$ becomes identical. Hence the expression becomes

$$= 2r^2k\pi \quad - \; - \; - \; - \; - \; - \; - \quad (16)$$

where $k$ represents the altitude.

9. Applying the same notation to express the sphere, we have for the surface of any sphere whose radius is $r$, $4\pi r^2$; and $4\pi r^2\times\frac{1}{3}r=\frac{4}{3}\pi r^3$, will be its solidity, (Prop. XXI. B. III. *El. Sol. Geom.*) - - - - - - - - - - - - - - - - (1)

If the surface of a spherical zone is required, it may be expressed by the product of the altitude of the zone multiplied by the circumference of the sphere; let $h=$the altitude, and we have $2\pi rh$ for the spherical surface of the zone, - - (2.)

The solidity of the sector of which this is the spherical base, is $2\pi rh\times\frac{1}{3}r=\frac{2}{3}\pi r^2h$, - - - - - - - - - - - - (3.)

Now, since the sector CBAD (see the diagram to the Problem on the 187 page,) may be considered to be made up of a cone CBD and a segment BDA; in order to express these portions, it will be necessary to find the area of the circular section BD; for this purpose, since CP=CA — AP $=r-h$, and CB$=r$, we have in the right angled-triangle BPC, BP$=\sqrt{\text{CB}^2-\text{PC}^2}=\sqrt{r^2-r^2+2rh-h^2}=\sqrt{2rh-h^2}$, - (4.)

The radius of the circular section BPD, hence the area of the circle will be $2\pi rh-\pi h^2$, - - - - - - - - - - (5.)

And the solidity of the cone will be

$$(2\pi rh-\pi h^2)\frac{\times r-h}{3}=\frac{2\pi r^2h-3\pi r^2h+\pi h^3}{3} \quad \text{- (6.)}$$

Hence the solidity of the segment will be

$$\tfrac{2}{3}\pi r^2h-\frac{2\pi r^2h+3\pi rh^2-\pi h^3}{8}=\pi rh^2-\tfrac{1}{3}\pi h^3, \quad \text{- - (7.)}$$

Which may be resolved into $\pi h^2(r-h,)$ - - - - (8.)

Hence, *the solidity of the segment is equal to the product of a circle, whose radius is the altitude of the segment multiplied by the radius of the sphere, minus a third of this altitude.*

10. Let $r'$ be the radius of a sphere inscribed in a vertical polyedroid, and $r$ the radius of the circumscribed sphere. Then (Prop. XVIII. B. III, *El. Sol. Geom.*) the surface of the polyedroid will be $2r'\pi\times 2r=4r'r\pi$, - - - - - - - (1.)

And since the surface of the circumscribed sphere is$=4r^2\pi$, formula (1, Art. 9,) it follows that the surface of the polyedroid and that of its circumscribed sphere, are to each other as $r' : r$, since those are the only variable quantities which enter into their expressions.

If $h$ be the height of any zone of the sphere, its surface, formula (2, Art. 9,) will be $2r'\pi\times h=2rh\pi$, - - - - (2)

The surface of the corresponding zone of the polyedroid, will be $2r'\pi\times h=2r'h\pi$, (P. XVIII. Cor. B. III. *El. S. G.*) - (3.)

And hence the surface of any zone or segments of the polyedroid and sphere made by the same perpendicular to their common axis, will be in the ratio of their whole surface, viz.

$r' : r$, or $\frac{r}{r'}$. - - - - - - - - - - - - - - - (4.)

The base of any segment of the polyedroid made by a plane passing through a circle common to the polyedroid and sphere, may be found from formula (5, Art. 9.)

Let $2\pi rh-\pi h^2$, be the base of acommon segment of the sphere and polyedroid, and $h$ the common altitude of the polyedroidal and spherical sector.

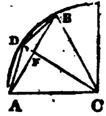
The solidity of the polyedroidal sector will be

$$2r'h\pi\times\frac{r'}{3}=\frac{2}{3}r'^2h\pi \quad \text{- - - - - (5.)}$$

And since the corresponding spherical sector is $\frac{1}{3}r^2h\pi$, formula (3, Art. 9) it follows that the solidities of the corresponding sector of these solids are as the ratio of $r : r'$, as has been shown with regard to the convex surface. The cone whose base is the base of the segment, and whose vertice is the centre of the polyedroid or sphere, formula (6, Art. 9,) may be expressed $\frac{2\pi r^2h — 3\pi rh^2+\pi h^3}{3}$ hence the segment will be

$$\tfrac{2}{3}r'^2h\pi — \tfrac{2}{3}\pi r^2h+\pi rh^2-\pi h^3$$
$$=\pi h(\tfrac{2}{3}r'^2+\tfrac{2}{3}r^2+rh — h^2) \quad - \; - \; - \; - \quad (6.)$$

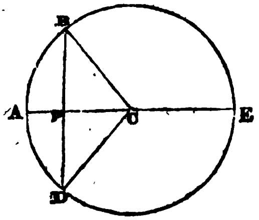
If the polyedroidal segments consists only of a vertical cone, its solidity will be $2\pi rh — \pi h^2)\times\frac{1}{3}h=\pi rh^2 — \frac{1}{3}\pi h^3$, subtract this from the spherical segment on the same base, formula (7, Art. 9,) $\pi rh^2 — \frac{1}{3}\pi h^3$, and we have $\frac{1}{3}\pi rh^2$. - - - - - - - - - (7.) which is the value of that portion of the segment of the sphere not included in that of the inscribed polyedroid, which is such a portion of the sphere as would be generated by the revolutions of a circular segment as BD, about the axis DC, passing through the centre of curvature, and perpendicular to the arc of the segment at the point of contact D.



### PROBLEM.

*It is required to find when the spherical segment and the cone composing a spherical sector are equal to each other.*

Let ABED represent a sphere generated by the revolution of a semicircle ABE about its diameter AE. The sector ABC, by this revolution, generates a spherical sector, which is composed of a spherical segment generated by the revolution of a semisegment ABP, and of a cone generated by the revolution of the right-angled triangle BPC.



The solidity of the sector, formula (3, Art. 9.) will be $\frac{2}{3}\pi r^2h$.

The solidity of the cone, $\frac{2}{3}\pi r^2h — \pi rh^2+\frac{\pi h^3}{3}$, formula (6.)

Now, in order that the cone may be equal to the segment, the sector, which is the sum of both, must be double the cone: hence, $\frac{2}{3}\pi r^2h=\frac{4}{3}\pi r^2h=2\pi rh^2+\frac{2}{3}\pi h^3$,
dividing by 2, transposing, &c.,

$$\pi rh^2=\frac{1}{3}\pi r^2h+\frac{1}{3}\pi h^3$$

dividing by $\pi h$ $\quad rh=\frac{1}{2}r^2+\frac{1}{3}h^2$

transposing $\quad \frac{1}{3}h^2-rh=\frac{1}{3}r^2$

$$h^2-3rh=-r^2$$

from which we obtain

$$h=\frac{2}{3}r\pm\sqrt{\frac{5}{4}r^2}$$

Of these two solutions it is evident that only $h=\frac{3}{2}r-\sqrt{\frac{5}{4}r^2}$ can satisfy the conditions of the question, since $\frac{3}{2}r+\sqrt{\frac{5}{4}r^2}$ is more than $2r$, or more than the diameter of the sphere.

### EXAMPLES FOR EXERCISE.

1. What is the solidity of the spherical segments of which the frigid zones are the convex surfaces, the altitude of each segment being 327 miles, and the radius of the base 1575,28 miles? Ans. 1282921583 solid miles nearly.

2. What is the solidity of the spherical segments of which the temperate zones are the convex surface, the radius of the superior base being 1575,28 miles, that of the inferior 3628,86 miles, and the altitude 2053,7 miles? Ans. 55021192817 solid miles nearly.

3. What is the solidity of the spherical segment of which the torrid zone is the convex surface, the radii of the bases being 3628,86 miles, and the altitude 3150,6? Ans. 146715018499 solid miles nearly.

4. Having two vats or two tubs in the form of conical frusta, whose dimensions are as follows, viz.: the first has a base whose diameter is 3 feet, its altitude is 3½ feet, and the slant height of its side is 4 feet: the diameter of the base of the second is 3½ feet, its altitude is 5 feet, and the curve surface is 60 square feet, what must be the dimensions of one capable of containing as much as the other two, if the diameter of the bottom and top, and the altitude are in the proportion of 2 2½ and 3.

5. What is the difference in surface of a vertical hexedroid circumscribing a sphere whose diameter is 10, and the whole surface of a conesected frustum of a cone inscribed in the same sphere, and whose wanting base is 6, and perfect base 4?

# CONIC SECTIONS.

There are three curves, whose properties are extensively applied in mathematical investigations, which, being the sections of a cone made by a plane in different positions, as will be shown in another place, are called the *Conic Sections.* These are,

1. The *Parabola.* 2. The *Ellipse.* 3. The *Hyperbola.*

## PARABOLA.

### DEFINITIONS.

1. A Parabola is a plane curve, such, that if from any point in the curve two straight lines be drawn; one to a given fixed point, the other perpendicular to a straight line given in position: these two straight lines will always be equal to one another.

2. The given fixed point is called the *focus* of the parabola.

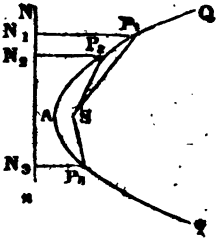
3. The straight line given in position, is called the *directrix* of the parabola.

Thus, let QA$q$ be a parabola, S the focus, N$n$ the directrix;

Take any number of points, $P_1$, $P_2$, $P_3$, .... in the curve;

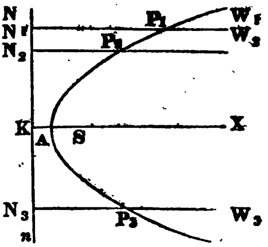
Join S, $P_1$ ; S, $P_2$ ; S, $P_3$ ; ... and draw $P_1N_1$, $P_2N_2$, $P_3N_3$, .... perpendicular to the directrix ; then

$SP_1 = P_1N_1$, $SP_2 = P_2N_2$, $SP_3 = P_3N_3$, ....



4. A straight line drawn perpendicular to the directrix, and cutting the curve, is called a *diameter*; and the point in which it cuts the curve is called the *vertex of the diameter.*

5. The diameter which passes through the focus is called the *axis,* and the point in which it cuts the curve is called the *principal vertex.*



Thus: draw $N_1P_1W_1$, $N_2P_2W_2$, $N_3P_3W_3$, KASX, through the points $P_1$, $P_2$, $P_3$, S, perpendicular to the directrix; each of these lines is a diameter; $P_1$, $P_2$, $P_3$, A, are the vertices of these diameters; ASX is the axis of the parabola, A the principal vertex.

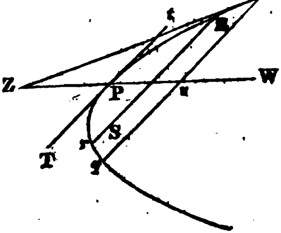
6. A straight line which meets the curve in any point, but which, when produced both ways, does not cut it, is called a *tangent* to the curve at that point.

7. A straight line drawn from any point in the curve, parallel to the tangent at the vertex of any diameter, and terminated both ways by the curve, is called an *ordinate* to that diameter.

8. The ordinate which passes through the focus, is called the *parameter* of that diameter.

9. The part of a diameter intercepted between its vertex and the point in which it is intersected by one of its own ordinates, is called the *abscissa* of the diameter.

10. The part of a diameter intercepted between one of its own ordinates and its intersection with a tangent, at the extremity of the ordinate, is called the *sub-tangent* of the diameter.



Thus: let TP*t* be the tangent at P, the vertex of the diameter PW.

From any point Q in the curve, draw Q*q* parallel to T*t* and cutting PW in *v*. Through S draw RS*r* parallel to T*t*.

Let QZ, a tangent at Q, cut WP, produced in Z.

Then Q*q* is an ordinate to the diameter PW; R*r* is the parameter of PW.

P*v* is the abscissa of PW, corresponding to the point Q.

*v*Z is the sub-tangent of PW, corresponding to the point Q.

11. A stright line drawn from any point in the curve, perpendicular to the axis, and terminated both ways by the curve, is called an *ordinate to the axis*.

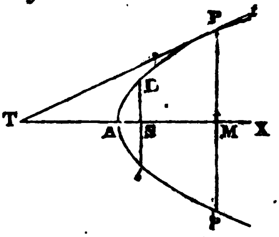
12. The ordinate to the axis which passes through the focus is called the *principal parameter*, or *latus rectum* of the parabola.

13. The part of the axis intercepted between its vertex and the point in which it is intersected by one of its ordinates, is called the *abscissa of the axis*.

14. The part of the axis intercepted between one of its own

ordinates, and its intersection with a tangent at the extremity of the ordinate, is called the *sub-tangent of the axis.*

Thus: from any point P in the curve draw P*p* perpendicular to AX and cutting AX in M. Through S draw LS*l* perpendicular to AX.



Let PT, a tangent at P, cut XA produced in T.

Then, P*p* is an ordinate to the axis; L*l* is the latus rectum of the curve.

AM is the abscissa of the axis corresponding to the point P.

MT is the subtangent of the axis corresponding to the point P.

It will be proved in Prop. III, that the tangent at the principal vertex is perpendicular to the axis; hence, the four last definitions are in reality included in the four which immediately precede them.

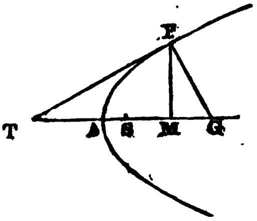
*Cor.* It is manifest from Def. 1, that the parts of the curve on each side of the axis are similar and equal, and that every ordinate P*p* is bisected by the axis.

15. If a tangent be drawn at any point, and a straight line be drawn from the point of contact perpendicular to it, and terminated by the curve, that straight line is called a *normal.*

16. The part of the axis intercepted between the intersections of the normal and the ordinate, is called the *sub-normal.*

Thus: Let TP be a tangent at any point P.

From P draw PG perpendicular to the tangent, and PM perpendicular to the axis.



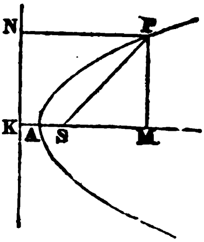
Then PG is the normal corresponding to the point P; MG is the sub-normal corresponding to the point P.

### PROPOSITION I. THEOREM.

*The distance of the focus from any point in the curve, is equal to the sum of the abscissa of the axis corresponding to that point, and the distance from the focus to the vertex.*

That is, $SP=AM+AS$.

For



$SP=PN$ by Def. (1,)

$=KM$ ∵ NM is a parallelogram.

$=AM+AK$

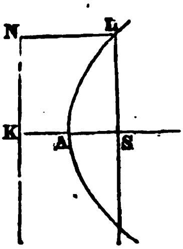
$=AM+AS$ ∵ $AK=AS$, by Def. (1.)

### PROPOSITION II. THEOREM.

*The latus rectum is equal to four times the distance from the focus to the vertex.*

That is $Ll=4\ AS$.

For,



$Ll=2\ LS$, Def. (14.) cor

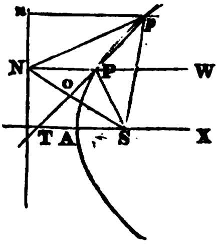
$=2\ LN$

$=2\ SK$

$=4\ AS$ ∵ $AS=AK$.

### PROPOSITION III. PROBLEM.

*To draw a tangent to the parabola at any point.*



Let P be the given point.

Join S, P; draw PN perpendicular to the directrix.

Bisect the angle SPN by the straight line T*t*.

T*t* is a tangent at the point P.

For if T*t* be not a tangent, let T*t* cut the curve in some other point *p*.

Join S, *p*; draw *pn* perpendicular to the directrix; join S, N.

Since SP=PN, PO common to the triangles SPO, NPO, and angle SPO=angle NPO by construction,

$\therefore$ SO=NO, and angle SOP=angle NOP.

Again, since SO=NO, O*p* common to the triangles SO*p*, NO*p*, and angle SO*p*=angle NO*p*.

$\therefore$ S*p*=N*p*.

But since *p* is a point in the curve, and *pn* is drawn perpendicular to the directrix,

$$Sp=pn$$
$$\therefore \quad pN=pn.$$

That is, the hypothenuse of a right-angled triangle equal to one of the sides, which is impossible, $\therefore$ *p* is not a point in the curve; and in the same manner it may be proved that no point in the straight line T*t* can be in the curve, except P.

$\therefore$ T*t* is a tangent to the curve at P.

*Cor.* 1. A tangent at the vertex A, is a perpendicular to the axis.

*Cor.* 2. SP=ST,

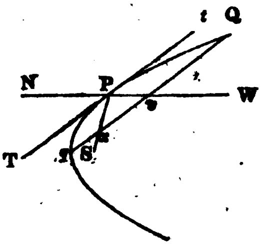
For, since NW is parallel to TX

$\therefore$ angle STP= angle NPT

= angle SPT by construction,

$\therefore$ SP=ST

*Cor.* 3. Let Q*q* be an ordinate to the diameter PW, cutting SP in *x*.



Then, P*x*=P*v*

For, since Q*q* is parallel to T*t*

$\therefore$ angle P*xv*=angle *x*PT

= NPT by construction,

= P*vx* interior opposite angle,

$\therefore$ P*x*=P*v*

*Cor.* 4. Draw the normal PG, (see diagram Prop. V.)

Then, SP=SG,

For since angle GPT is a right angle,

angle GPT=PGT+PTG=PGT+SPT

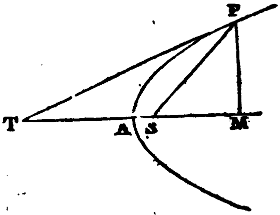
Take away the common angle SPT and there remains angle SPG=angle SGP

$\therefore$ SP=SG.

### PROPOSITION IV. THEOREM.

*The subtangent to the axis is equal to twice the abscissa.*

That is, $MT=2\ AM$



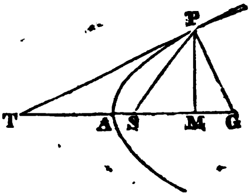
For, $MT=MS+ST$
$=MS+SP$. Prop. III. cor. 2.
$=MS+SA+AM$. Prop. I.
$=2\ AM$.

*Cor.* MT is bisected in A.

### PROPOSITION V. THEOREM.

*The subnormal is equal to one half of the latus rectum.* That is,

$MG=\frac{L}{2}$ if we denote the latus rectum by L.



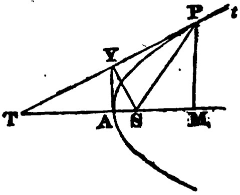
For, $MG=SG-SM$
$=SP-SM$. Prop. III, cor. 4.
$=AS+AM-SM$. Prop. I.
$=AS+AS+SM-SM$
$=2\ AS$
$=\frac{L}{2}$ Prop. II.

### PROPOSITION VI. THEOREM.

*If a straight line be drawn from the focus perpendicular to the tangent at any point, it will be a mean proportional between the distance from the focus to that point, and the distance from the focus to the vertex.*

That is, if SY be a perpendicular let fall from S upon T*t* the tangent at any point P

$SP : SY :: SY : SA.$



Join A, Y.

Since $SP=ST$, and SP is drawn perpendicular to the line PT,

$\therefore TY=YP$,

Also by Prop. IV.,

$TA=AM$

$\therefore$ Since AY cuts the sides of a triangle TPM proportionally; AY is parallel to MP,

$\therefore$ AY is perpendicular to AM.

Hence the triangles SYA, SYT, are similar,

$\therefore$ ST : SY : : SY : SA

or, SP : SY : : SY : SA $\therefore$ SP=ST by Prop. III, cor. 2.

*Cor.* 1. Multiplying extremes and means,

$$SY^2=SP\,.\,SA.$$

*Cor.* 2. SP : SA : : $SP^2 : SY^2$

*Cor.* 3. By Cor. 1,

$$SP=\frac{SY^2}{SA}.$$

And since SA is constant for the same parabola,

SP proportional to $SY^2$.

*Cor.* 4. By Cor. 1.,

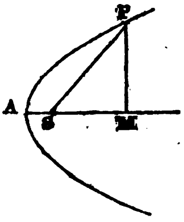
$$SY^2 = AS\,.\,SP$$
$$\therefore 4\,SY^2 = 4\,AS\,.\,SP$$
$$= L\,.\,SP. \quad \text{Prop. II.}$$

### PROPOSITION VII. THEOREM.

*The square of any semi-ordinate to the axis is equal to the rectangle under the latus rectum and the abscissa.*

That is, if P be any point in the curve

$$PM^2 = L\,.\,AM.$$



For,

$PM^2 = SP^2 - SM^2$ (Prop. XXIV. B. IV. *El. Geom.*)

$= (AM=AS)^2 - (AM-AS)^2$

$\therefore SP=AM+AS$ (Prop. 1,) & $SM=AM-AS$

$= 4\,AS\,.\,AM.$ (Prop. X. and XI, B. IV. *El. Geom.*)

$= L\,.\,AM.$ Prop. I.

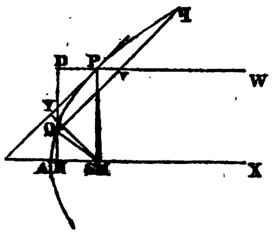
*Cor.* 1. Since L is constant for the same parabola $PM^2$ proportional to AM,

That is, *The abscissæ are propotional to the squares of the ordinates.*

### PROPOSITION VIII. THEOREM.

*If Qq be an ordinate to the diameter* PW *and* Pv, *the corresponding abscissa, then,*

$$Qv^2 = 4SP \times Pv.$$



Draw PM an ordinate to the axis. Join S, Q; and through Q draw DQN perpendicular to the axis.

From S let fall SY perpendicular on the tangent at P.

The triangles SPY, QD$v$, are similar.

$$Qv^2 : QD^2 :: SP^2 : SY^2$$
$$:: SP : SA, \text{ Prop, VI. Cor. 2.}$$

The triangles PTM, QD$v$, are also similar;

$$\therefore \quad QD : Dv :: PM : MT$$
$$:: PM^2 : PM \,.\, MT$$
$$:: 4AS \,.\, AM : 2PM \,.\, AM$$
$$:: 4AS : 2PM$$
$$\therefore \quad 2PM \,.\, QD = 4AS \,.\, Dv$$

But,

$$PM^2 - QN^2 = 4AS \,.\, AM - 4AS \,.\, AN = 4AS(AM - AN)$$
$$= 4AS \,.\, MN$$

And, $PM^2 - QN^2 = PM+QN)\,(PM-QN)$

$$= (PM+QN) \,.\, QD$$
$$\therefore (PM+QN) \,.\, QD = 4AS \,.\, MN = 4AS \,.\, DP$$

But, $2\,PM \,.\, QD = 4AS \,.\, Dv$

$$\therefore (PM-QN) \,.\, QD = 4AS \,.\, Pv$$

Or, $QD^2 = 4AS \,.\, Pv$

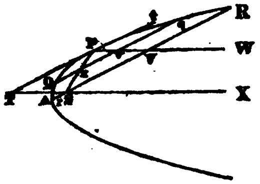
$$\therefore \quad Qv^2 : 4AS \,.\, Pv :: SP : SA.$$
$$\therefore \quad Qv^2 = 4SP \,.\, Pv.$$

*Cor.* 1. In like manner it may be proved, that

$$qv^2 = 4SP \times Pv.$$

Hence, $Qv = qv$; and since the same may be proved for any ordinate, it follows, that

*A diameter bisects all its own ordinates.*



*Cor.* 2. Let R$r$ be the parameter to the diameter PW.

Then, by Prop. III. *Cor.* 3.

$$Px = Pv.$$
$$\therefore \quad PS = PV$$

Now, by the Proposition,

$$RV^2 = 4SP \,.\, PV$$
$$= 4SP^2$$
$$\therefore 4RV^2 \text{ or } Rr^2 = 16SP^2$$
$$Rr = 4SP.$$

Hence the proposition may be thus enunciated:

*The square of the semi-ordinate to any diameter is equal to the rectangle under the parameter and abscissa*

It will be seen, that Prop. VII. is a particular case of the present proposition.

# ELLIPSE.

## DEFINITIONS.

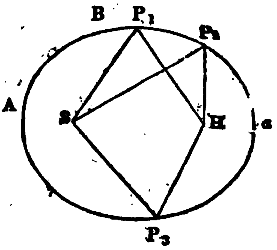
1. An ellipse is a plane curve, such that, if from any point in the curve two straight lines be drawn to two given fixed points, the sum of these straight lines will always be the same.

2. The two given fixed points are called the *foci.*

Thus, let AB*a* be an ellipse, S and H the foci.

Take any number of points in the curve $P_1$, $P_2$, $P_3$, - - - -

Join $S,P_1$, $H,P_1$; $S,P_2$, $H,P_2$; $S,P_3$, $H,P_3$; - - - - then,
$SP_1 + HP_1 = SP_2 + HP_2 = SP_3 + HP_3 = - - - -$



3. If a straight line be drawn joining the foci and bisected, the point of bisection is called the *centre.*

4. The distance from the centre to either focus is called the *eccentricity.*

5. Any straight line drawn through the centre, and terminated both ways by the curve, is called a *diameter.*

6. The points in which any diameter meets the curve are called the *vertices* of that diameter.

7. The diameter which passes through the foci is called the *axis major*, and the points in which it meets the curve are called the *principal vertices.*

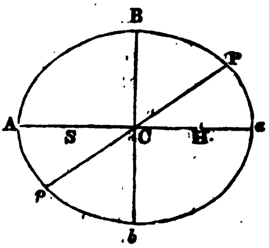
8. The diameter at right angles to the axis major is called the *axis minor.*

Thus, let AB*a* be an ellipse, S and H the foci.

Join S,H; bisect the straight line SH in C, and produce it to meet at the curve in A and *a*.

Through C draw any straight line P*p*, terminated by the curve in the points P, *p*.

Through C draw B*b* at right angles to A*a*.



Then, C is the centre, CS or CH the eccentricity. P*p* is a diameter, P and *p* its vertices, A*a* is the major axis, B*b* is the minor axis.

9. A straight line which meets the curve in any point, but which, being produced both ways, does not cut it, is called a *tangent* to the curve at that point.

10. A diameter drawn parallel to the tangent at the vertex of any diameter, is called the *conjugate diameter* to the latter, and the two diameters are called a *pair of conjugate diameters.*

11. Any straight line drawn parallel to the tangent at the vertex of any diameter and terminated both ways by the curve, is called an *ordinate* to that diameter.

12. The segments into which any diameter is divided by one of its own ordinates are called the *abscissæ* of the diameter.

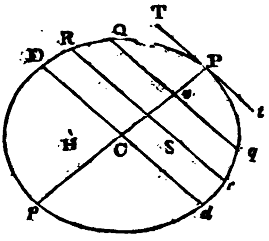
13. The ordinate to any diameter, which passes through the focus, is called the *parameter* of that diameter.

Thus, let P*p* be any diameter, and T*t* a tangent at P.

Draw the diameter D*d* parallel to T*t*.

Take any point Q in the curve, draw Q*q* parallel to T*t*, cutting P*p* in *v*.

Through S draw R*r* parallel to T*t*.



Then, D*d* is the conjugate diameter to P*p*.

Q*q* is the ordinate to the diameter P*p*, corresponding to the point Q.

P*v*, *vp* are the abscissæ of the diameter P*p*, corresponding to the point Q.

R*r* is the parameter of the diameter P*p*.

14. Any straight line drawn at right angles to the major axis, and terminated both ways by the curve, is called an *ordinate to the axis.*

15. The segments into which the major axis is divided by one of its own ordinates are called the *abscissæ to the axis.*

16. The ordinate to the axis which passes through either focus is called the *latus rectum.*

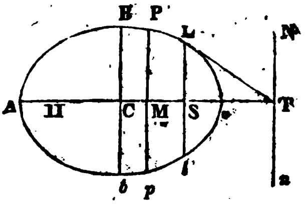
(It will be proved in Prop. IV., that the tangents at the principal vertices are perpendicular to the major axis; hence, definitions 14, 15, 16, are in reality included in the three which immediately precede them.)

17. If a tangent be drawn at the extremity of the latus rectum and produced to meet the major axis, and if a straight line be drawn through the point of intersection at right angles to the major axis, the tangent is called the *focal tangent,* and the straight line the *directrix.*

Thus, from P any point in the curve, draw PM$p$ perpendicular to A$a$, cutting A$a$ in M.

Through S draw L$l$ perpendicular to A$a$.

Let LT, a tangent at L, cut A$a$ produced in T.



Through T draw N$n$ perpendicular to A$a$.

Then, P$p$ is the ordinate to the axis, corresponding to the point P.

AM, M$a$ are the abscissæ of the axis, corresponding to the point P.

L$l$ is the latus rectum.

LT is the focal tangent.

N$n$ is the directrix.

18. A straight line drawn at right angles to a tangent from the point of contact, and terminated by the major axis, is called a *normal.*

The part of the major axis intercepted between the intersections of the normal and the ordinate, is called the *subnormal.*

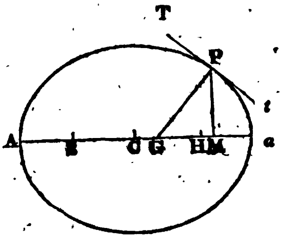
Let T$t$ be a tangent at any point P.

From P draw PG perpendicular to T$t$ meeting A$a$ in G.

From P draw PM perpendicular to A$a$.

Then PG is the normal corresponding to the point P.

MG is the subnormal corresponding to the point P.



PROPOSITION I. THEOREM.

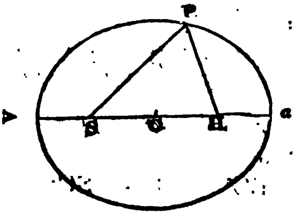
*The sum of two straight lines drawn from the faci to any point in the curve is equal to the major axis.* That is, if P be any point in the curve.

$$SP + HP = Aa.$$

For,

$$\left.\begin{aligned} SP + HP &= AS + AH \\ &= AS + SH, \\ \text{And,} & \\ SP + HP &= aS + aH \\ &= 2\,aH + SH, \end{aligned}\right\} \text{Def. 1.}$$

$\therefore 2\,(SP+HP) = 2\,(AS + SH + Ha)$



Or,

$SP + HP = Aa$.

*Cor.* The centre bisects the axis major, for

$2\,AS + SH = 2\,aH + SH \therefore AS = aH$

And, $SC = CH$ by definition 3. $\therefore AC = aC$.

*Cor.* 2. $SP + HP = 2\,AC \therefore SP = 2\,AC - HP$

$HP = 2\,AC - SP$

hence $SP - HP = 2\,AC - 2\,HP$,

### PROPOSITION II. THEOREM.

*The centre bisects all diameters.*

Take any point P in the curve.

Join S,P ; H,P ; S,H ;

Complete the parallelogram SPH$p$

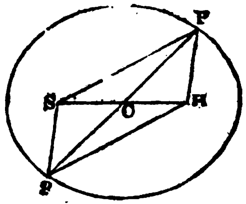
Join C,$p$ ; CP ;

Then, since the opposite sides of a parallelogram are equal,

$SP = Hp$, $HP = Sp \therefore SP + PH = Sp + pH$

$\therefore$ $p$ is a point in the curve.

Again, since the diagonals of parallelogram bisect each other, and since SH is bisected in C,



$\therefore$ P$p$ is a straight line, and a diameter, and is bisected in C.

And in like manner, it may be proved that every other diameter is bisected in C.

### PROPOSITION III. THEOREM.

*The distance of either focus from the extremity of the axis minor is equal to the semi-axis major.*

That is,

$$SB \text{ or } HB = AC.$$

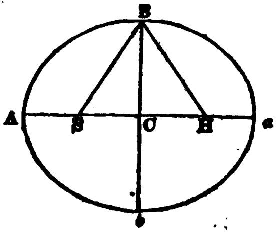
Since $SC = HC$, and CB is common to the two right-angle triangles SCB, HCB,

$\therefore SB = HB$.

But,

$SB + HB = 2\,AC$. Prop. I.

$\therefore SB = HB = AC$.



*Cor.* 1. $BC^2 = AS\,.\,Sa$.

For,

$BC^2 = SB^2 - SC^2$

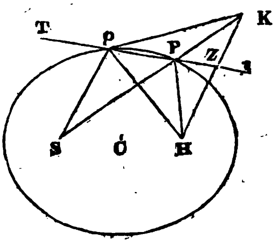
$$= AB^2 - SC^2$$
$$= (AC + SC) . (AS - SC)$$
$$= AS . Sa.$$

*Cor.* 2. The square of the eccentricity is equal to the difference of the squares of the semi-axes;

For, $SC^2 = SB^2 - BC^2$
$= AC^2 - BC^2.$

PROPOSITION IV. PROBLEM.

*To draw a tangent to the ellipse at any point.*



Let P be the given point.

Join S,P; H,P produce SP.

Bisect the exterior angle HPK by the straight line T*t*.

T*t* is a tangent to the curve at P.

For, if T*t* be not a tangent, let T*t* cut the curve in some other point *p*.

Join S,*p*; H,*p*; make PK=PH, join *p*, K; H,K cutting T*t* in Z,

Since HP=PK, PZ common to the triangles HPZ, KPZ, and the angle HPZ=angle KPZ by construction,

∴ HZ=KZ, and the angle HZP=angle KZP.

Again, since HZ=KZ, Z*p* common to the triangles HZ*p*, KZ*p*, and angle HZ*p*=angle KZ*p*,

∴ *p*K = *p*H.

But, since any two sides of a triangle are greater than the third side,

$Sp + pK > SK$
$> SP + PK$
$> SP + PH$ ∵ PK=PH by construction·
$> Sp + pH$, by definition 1,
$\therefore pK > pH.$

But we have just proved that *p*K=*p*H, which is absurd, ∴ *p* is not a point in the curve, and in the same manner it may be proved that no point in the straight line T*t* can be in the curve except P.

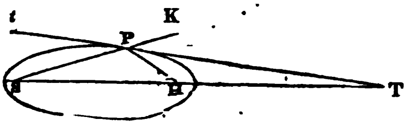
∴ T*t* is a tangent to the curve at P.

*Cor.* 1. Hence, tangents at A and *a*, are perpendicular to the major axis, and tangents at B and *b* are perpendicular to the minor axis.

*Cor.* 2. SP and HP make equal angles with every tangent.

*Cor.* 3. Since HPK, the exterior angle of the triangle SPH, is bisected by the straight line T*t*, cutting the base SH produced in T

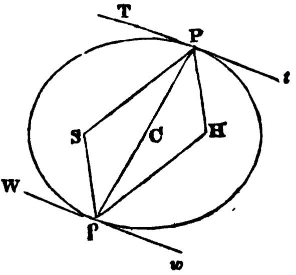
$$\therefore\ \mathrm{ST : HT :: SP : HP}.$$



PROPOSITION V. THEOREM.

*Tangents drawn at the vertices of any diameter are parallel.*

Let T*t*, W*w*, be tangents at P, *p*, the vertices of the diameter PC*p*.



Join S,P ; P,H ; S,*p* ; *p*, H ;

Then, by Prop. II, SH is a parallelogram, and since the opposite angles of parallelograms are equal,

∴ ang. SPH= angle S*p*H

supplement of ang. SPH = supplement of ang. S*p*H

or,

ang. SPT+ang. HP*t*=ang. S*p*W+ang. H*pw*

$$\left.\begin{array}{l}\text{But ang. SPT} = \text{ang. HP}t\\ \text{And ang. S}p\text{W} = \text{ang. H}pw\end{array}\right\}\ \text{by Prop. IV. Cor. 2.}$$

Hence, these four angles are all equal,

∴ ang. SPT=ang. H*pw*.

And since SP is parallel to H*p*,

ang. SP*p* = ang. P*p*H,

∴ whole ang. TP*p* = whole ang. *wp*P, and they are alternate angles,

∴ T*t* is parallel to W*w*.

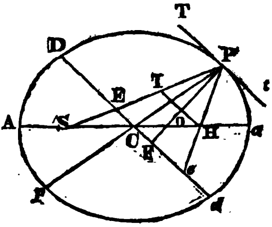
*Cor.* Hence, if tangents be drawn at the vertices of any two diameters, they will form a parallelogram circumscribing the ellipse.

### PROPOSITION VI. THEOREM.

*If straight lines be drawn from the foci to a vertex of any diameter, the distance from the vertex to the insertion of the conjugate diameter, with either focal distance, is equal to the semi axis, major.*

That is, if D$d$ be a diameter conjugate to P$p$, cutting SP in E, and HP in $e$,

$$PE \text{ or } Pe = AC.$$



Draw PF perpendicular to D$d$, and HI parallel to D$d$ or T$t$, cutting PF in O,

Then, since the angles at 0 are right angles, the ang. IPO=ang. HPO, and PO common to the two triangles HPO, IPO,

$$\therefore IP = HP.$$

Also, since SC = HC, and CE is parallel HI, the base of a triangle SHI,

$$\therefore SE = EI.$$

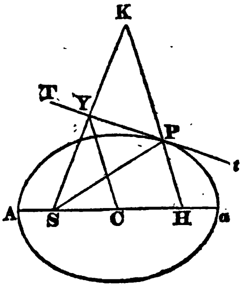
Hence,

$$\begin{aligned} 2\,PE &= 2\,EI + 2\,IP \\ &= SE + EI + IP + HP \\ &= SP + HP \\ &= 2\,AC \\ \therefore PE &= AC. \end{aligned}$$

Also, ang. PE$e$ = ang. P$e$E. $\therefore PE = Pe$, and $Pe = AC$.

### PROPOSITION VII. THEOREM.

*Perpendiculars, from the foci upon the tangent at any point intersect the tangent in the circumference of a circle, whose diameter is the major axis.*



From S let fall SY perpendicular on T$t$ a tangent.

Join S, P; H, P; produce HP to meet SY produced in K.

Join CY;

Then, since angle SPY=angle KPY (Prop. IV.) and the angles at Y are right angles, and PY common to the two triangles, SPY, KPY.

$$\therefore SP=PK$$

And $SY = YK$.

And, since $SY = YK$, and $HC = CS$, CY cuts the sides of the triangle HSK proportionally,

$\therefore$ CY is parallel to HK.

Also, since CY is parallel to HK, $SY = YK$, $HC = CS$,

$$\begin{aligned} \therefore CY &= \tfrac{1}{2} HK \\ &= \tfrac{1}{2} (HP + PK) \\ &= \tfrac{1}{2} (HP + SP) \\ &= \tfrac{1}{2} Aa \\ &= AC \end{aligned}$$

Hence, a circle inscribed with Centre C and radius CA will pass through Y.

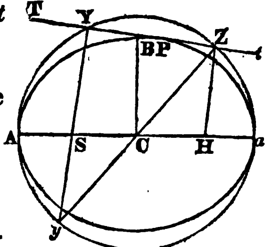
And in like manner, if HZ be drawn perpendicular to T*t*, it may be proved that the same circle will pass through Z also.

### PROPOSITION VIII. THEOREM.

*The rectangle, contained by the perpendiculars, from the foci upon the tangent at any point, is equal to the square of the semi-axis, minor.*

That is, $SY \cdot HZ = BC^2$.

Let T*t* be a tangent at any point P.



On A*a* describe a circle cutting T*t* in Y and Z.

Join S, Y; H, Z;

Then, by the last Prop., SY, HZ are perpendicular to T*t*.

Produce YS to meet the circumference in *y*.

Join C, *y*; C, Z;

Since *y*YZ is a right angle, the segment in which it lies is a semicircle, and Z, *y*, are the extremities of a diameter.

$\therefore$ *y*CZ is a straight line and a diameter.

Hence the triangles CS*y*, HCZ, are in every respect equal.

$$\therefore Sy = HZ$$

$$\therefore SY \cdot HZ = YS \cdot Sy = AS \cdot SA = BC^2 \text{ Prop. III. Cor. I.}$$

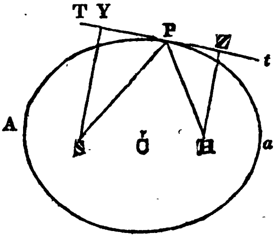
### PROPOSITION IX. THEOREM.

*Perpendiculars let fall from the foci upon the tangent at any point are to each other as the focal distance of the point of contact.*

That is, $SY : HZ :: SP : HP.$

For the triangles SPY, HPZ, are manifestly similar,

$\therefore SY : HZ :: SP : HP.$



*Cor.* Hence,

$$SY = HZ \cdot \frac{SP}{HP}$$

$$SY^2 = SY \cdot HZ \cdot \frac{SP}{HP}$$

$$= BC^2 \cdot \frac{SP}{HP} \text{ last Prop.}$$

$$= BC^2 \cdot \frac{SP}{2\,AC - SP}.$$

So also,

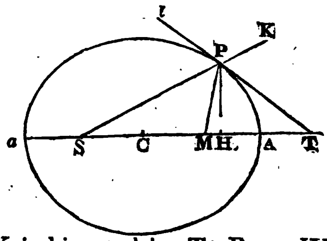
$$HZ^2 = BC^2 \cdot \frac{HP}{SP}$$

$$= BC^2 \cdot \frac{HP}{2\,AC - HP}.$$

### PROPOSITION X. THEOREM.

*If a tangent be applied at any point, and from the same point an ordinate to the axis be drawn, the semi-axis major is a mean proportional between the distance from the centre to the intersection of the ordinate with the axis, and the distance from the centre to the intersection of the tangent with the axis.*

That is, $CT : CA :: CA : CM.$



Since the exterior angle HPK is bisected by T*t*, Prop. IV.

$\therefore$ ST : HT : : SP : HP. (B. IV. Prop. XVI. *El. Geom.*)

$\therefore$ ST+HT : ST−HT : : SP+HP : SP−HP

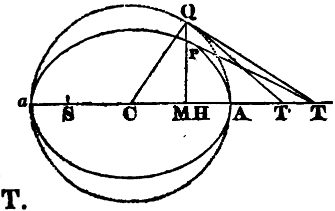
or, 2 CT : SH : : 2 AC : SP−HP
∴ 2 CT : 2 AC : : SH : SP−HP - - (1.)

But since PM is drawn from the vertex of the triangle SPH perpendicular on base SH,
∴ SM+HM : SP−HP : : SP+HP : SM−HM
or, SH : SP−HP : : 2 AC : 2 CM - - - (2.)

Comparing this with the proportion marked (1,) we have,
2 CT : 2 AC : : 2 AC : 2 CM
or, CT : AC : : CA : CM.

### PROPOSITION XI. THEOREM.

*If a circle be described on the major axis of an ellipse, and if any ordinate to this axis be produced to meet the circle, tangents drawn to the ellipse and circle, at points in which they are intersected by the ordinate, will cut the major axis in the same point.*



Let AQ*a*, be a circle described on A*a*.

Take any point P in the ellipse, draw PM perpendicular to A*a*, and produce MP to meet the circle in Q, join C, Q.

Draw PT a tangent to the ellipse at P cutting CA produced in T.

Join TQ.

Then QT is a tangent to the circle at Q.

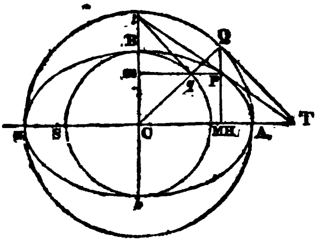
For if TQ be not a tangent, draw QT′ a tangent to Q cutting CA in T′.

Then CQT′ is a right angle.

∴ Since QM is drawn from the right angle CQT′ perpendicular on the hypothenuse.

∴ CT′ : CQ : : CQ : CM. (Prop. XVII, Cor. 2. B. IV. *E. G.*)
or, CT′ : CA : : CA : CM, ∵ CQ=CA.

But, by the last proposition,
CT : CA : : CA : CM,
∴ CT=CT′,
which is absurd, therefore QT′ is not a tangent at Q; in the same manner it may be proved that no line but QT can be a tangent at Q, ∴ &c.



*Cor.* 1. Describe a circle on the minor axis.

Draw P*m* an ordinate to the minor axis cutting the circle in *q*.

Let a tangent at P cut the minor axis produced in *t*.

Then, since P*m* is parallel to AC, and PM to BC,

$\mathrm{C}t : \mathrm{C}m : : \mathrm{CT} : \mathrm{MT}$

$: : \mathrm{CT}^2 : \mathrm{QT}^2$

$: : \mathrm{C}q^2 : \mathrm{C}m^2$ ∴ the triangles CQT, C*mq* are similar.

$: : \mathrm{BC}^2 : \mathrm{C}m^2$, ∴ $\mathrm{C}t : \mathrm{CB} : : \mathrm{CB} : \mathrm{CM}$.

Which is analogous to the property proved in the last proposition for the major axis.

*Cor.* 2. Join *tq*.

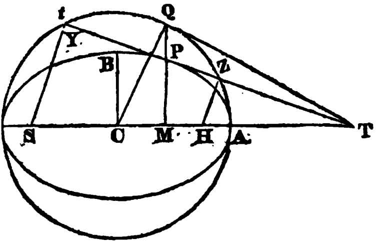
We can prove as above, that *tq* is a tangent to the circle B*qb*.

### PROPOSITION XII. THEOREM.

*The square of any semi ordinate to the axis, is to the rectangle under the abscissæ, as the square of the semi-axis minor is to the square of the semi-axis major.*

That is, if P be any point in the curve,

$$\mathrm{PM}^2 : \mathrm{AM} . \mathrm{M}a : : \mathrm{BC}^2 : \mathrm{AC}^2.$$



Describe a circle on A*a*, and produce MP to meet it in Q.

At the point P and Q draw the tangent PT, QT, which will intersect the axis in the same point T, (Prop. XI.)

Let the tangent to the ellipse intersect the circle in Y, Z.

Join S, Y; H, Z; SY and HZ are perpendicular to T*t*, (Prop. VII.)

Hence the triangles PMT, SYT, HZT, are similar to each other.

$\therefore \mathrm{PM} : \mathrm{SY} : : \mathrm{MT} : \mathrm{TY}$

and $\mathrm{PM} : \mathrm{HZ} : : \mathrm{MT} : \mathrm{TZ}$

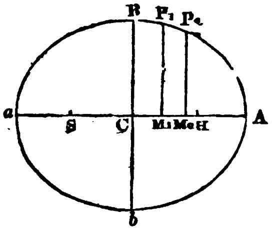
$\therefore \mathrm{PM}^2 : \mathrm{SY} . \mathrm{HZ} : : \mathrm{MT}^2 : \mathrm{TY} . \mathrm{TZ}$

or $\mathrm{PM}^2 : \mathrm{BC}^2 : : \mathrm{MT}^2 : \mathrm{TQ}^2$ (Prop. XI, and Prop. XVI. B. IV. *El. Geom.*)

$: : \mathrm{QM}^2 : \mathrm{CQ}^2$ ∵ MQT, MQC are similar triangles.

$$: : \mathrm{AM} . \mathrm{M}a : \mathrm{AC}^2$$

$$\therefore \mathrm{PM}^2 : \mathrm{AM} . \mathrm{M}a : : \mathrm{BC}^2 : \mathrm{AC}^2.$$



*Cor.* 1

Let $\mathrm{P}_1\mathrm{M}_1$, $\mathrm{P}_2\mathrm{M}_2$, ... be ordinates to the axis from any points $\mathrm{P}_1$, $\mathrm{P}_2$ ... Then by Prop.

$$\mathrm{P}_1\mathrm{M}_1^2 : \mathrm{AM}_1 . \mathrm{M}_1a : : \mathrm{BC}^2 : \mathrm{AC}^2$$

$$\mathrm{P}_2\mathrm{M}_2^2 : \mathrm{AM}_2 . \mathrm{M}_2a : : \mathrm{BC}^2 : \mathrm{AC}^2$$

$$\therefore \mathrm{P}_1\mathrm{M}_1^2 : \mathrm{P}_2\mathrm{M}_2^2 : : \mathrm{AM}_1 . \mathrm{M}_1a : \mathrm{AM}_2 . \mathrm{M}_2a.$$

That is, the square of the ordinates to the axis are to each other as the rectangles of their abscissæ.

*Cor.* 2. By the fifth proportion in Prop.

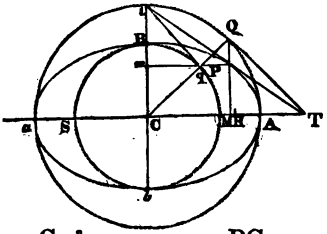
$$\mathrm{PM} : \mathrm{QM} : : \mathrm{BC} : \mathrm{AC}.$$

*Cor.* 3. By Prop. $\mathrm{PM}^2 : \mathrm{AM} . \mathrm{M}a : : \mathrm{BC}^2 : \mathrm{AC}^2$.

But $\mathrm{AM}=\mathrm{AC}+\mathrm{CM}$, $\mathrm{M}a=\mathrm{AC}-\mathrm{CM}$,

$$\therefore \mathrm{PM}^2 : (\mathrm{AC}+\mathrm{CM})\ (\mathrm{AC}-\mathrm{CM}) : : \mathrm{BC}^2 : \mathrm{AC}^2$$

$$\mathrm{PM}^2 : \mathrm{AC}^2-\mathrm{CM}^2 : : \mathrm{BC}^2 : \mathrm{AC}^2.$$



*Cor.* 4. Describe a circle on B*b*, draw P*m*, an ordinate to the minor axis cutting the circle in *q*.

Then, $\mathrm{P}m=\mathrm{CM}$, $\mathrm{PM}=\mathrm{C}m$.

Then by Cor. 3.

$$\mathrm{AC}^2-\mathrm{P}m^2 : \mathrm{AC}^2 : : \mathrm{C}m^2 : \mathrm{BC}^2$$

$$\mathrm{P}m^2 : \mathrm{AC}^2 : : \mathrm{BC}^2-\mathrm{C}m^2 : \mathrm{BC}^2$$

$$: : (\mathrm{BC}^2+\mathrm{C}m)\ (\mathrm{BC}-\mathrm{C}m) : \mathrm{BC}^2$$

$$: : \mathrm{B}m . mb : \mathrm{BC}^2$$

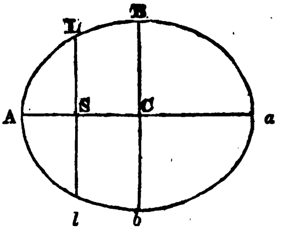
$$\text{or,}\quad \mathrm{P}m^2 : \mathrm{B}m . mb : : \mathrm{AC}^2 : \mathrm{BC}^2$$

Which is analogous to the property proved in the proposition for the major axis.

*Cor.* 5. $\mathrm{P}m : qm : : \mathrm{AC} : \mathrm{BC}.$

### PROPOSITION XIII. THEOREM.

*The latus rectum is a third proportional to the axis major and minor.*



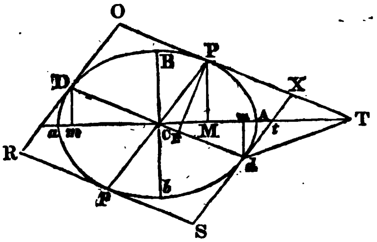
That is, $Aa : Bb :: Bb : Ll$

Since LS is a semi-ordinate to the axis,

$AC^2 : BC^2 :: AS \cdot Sa : LS^2$, Prop. XII.
$:: BC^2 : LS^2$, Prop. III. *Cor.* 1.
$\therefore AC : BC :: BC : LS$
And,
$Aa : Bb :: Bb : Ll$.

### PROPOSITION XIV. THEOREM.

*The area of all the parallelograms, circumscribing an ellipse, formed by drawing tangents at the extremities of two conjugate diameters, is constant, each being equal to the rectangle under the axes.*



Let $Pp$, $Dd$, be any two conjugate diameters, SROX a parallelogram circumscribing the ellipse formed by drawing tangents at P, D, $p$, $d$; then $Pp$, $Dd$, divide the parallelograms SROX into four equal parallelograms.

Draw PM, $dm$, ordinates to the axis; PF perpendicular to $Ad$.

Produce CA to meet PX in T and S$d$ in $t$.

Then, $CT : CA :: CA : CM$
And, $Ct : CA :: CA : Cm$
$\therefore CT : Ct :: Cm : CM$
But, $CT : Ct :: TM : Cm$, by similar triangles.
$\therefore MT : Cm :: Cm : CM$,
$\therefore CM \cdot MT = Cm^2$ . . . . . . . . (1.)
Again, $CM : CA :: CA : CT$
$\therefore CM : CA :: MA : AT$, dividendo.
Or, $CM : Ma :: MA : MT$, componendo.
$\therefore AM \cdot Ma = CM \cdot MT = Cm^2$ . . . . (2.)
But, $AC^2 : BC^2 :: AM \cdot Ma\ (Cm^2) : PM^2$, Prop. XII.
$\therefore AC : BC :: Cm : PM$
Similarly, $AC : BC :: CM : dm$
Or, $BC : dm :: CA : CM$

But, $CT : CA :: CA : CM$

$\therefore \quad CT : CA :: BC : dm$

But, $PF : CT :: dm : Cd$,

for the triangle $CdT = \frac{1}{2}$ the parallelogram $CPXd$,

$\therefore \quad PF : CA :: BC : Cd$.

$\therefore$ rectangle $PF \cdot Cd =$ rectangle $AC \cdot BC$

or, parallelogram $CX =$ rectangle $AC \cdot BC$

$\therefore$ parallelogram $SROX = 4\,AC \cdot BC$

$= Aa \cdot Bb$.

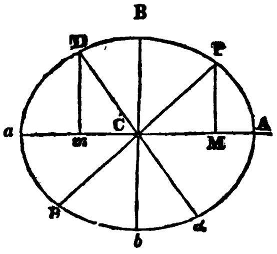
*Cor.* By (2)

$$\begin{aligned} Cm^2 &= AM \cdot Ma \\ &= (CA+CM) \cdot (CA-CM) \\ &= CA^2 - CM^2 \\ \therefore CA^2 &= CM^2 + Cm^2 \end{aligned}$$

And similarly. $CB^2 = PM^2 + dm^2$.

### PROPOSITION XV. THEOREM.

*The sum of the squares of any two conjugate diameters, is equal to the same constant quantity, namely, the sum of the squares of the two axis.*



That is,

If $Pp$, $Dd$, be any two conjugate diameters,

$$Pp^2 + Dd^2 = Aa^2 + Bb^2.$$

Draw $PM$, $Dm$, ordinates the axis.

Then, by Cor. to Prop. XIV,

$AC^2 + BC^2 = CM^2 + Cm^2 + PM^2 + Dm^2$

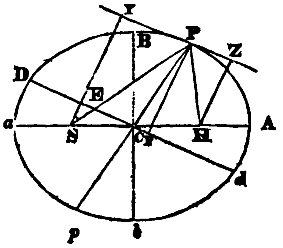
$= CP^2 + CD^2$

$\therefore 4AC^2 + 4BC^2 = 4CP^2 + 4CD^2$

Or, $Aa^2 + Bb^2 = Pp^2 + Dd^2$.

### PROPOSITION XVI. THEOREM.

*The rectangle under the focal distances of any point is equal to the square of the semi-conjugate.*



That is, if $CD$ be conjugate to $CP$,

$$SP \cdot HP = CD^2.$$

Draw $SY$. $HZ$, perpendiculars to the tangent at $P$, $PF$ perpendicular on $CD$.

Then by similar triangles $SPY$, $PEF$.

SP : SY : : PE : PF

Or, SP : SY : : AC : PF ∴ PE=AC, by Prop. VI

Similarly, HP : HZ : : AC : PF,

∴ SP . HP : SY . HZ : : $AC^2$ : $PF^2$,

: : $CD^2$ : $BC^2$, Prop. XIV.

But SY . HZ = $BC^2$, by Prop. VIII.

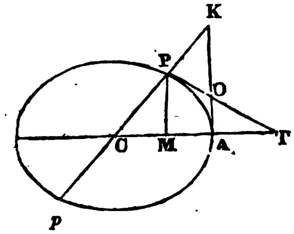
∴ SP . HP = $CD^2$.

### PROPOSITION XVII. THEOREM.

*If two tangents be drawn, one at the principal vertex, the other at the vertex of any other diameter, each meeting the other diameter produced, the two tangential triangles thus formed, will be equal.*

That is,

trian. CPT = trian. CAK.



Draw the ordinate PM, then,

CM : CA : : CP : CK, by similar triangles.

But, CM : CA : : CA : CT, Prop. X.

∴ CA : CT : : CP : CK.

The triangles CPT, CAK, have thus the angle C common, and the sides about the angle reciprocally proportional; these triangles are therefore equal.

*Cor.* 1. From each of the equal triangles CPT, CAK, take the common space CAOP; there remains,

triangle OAT = triangle OKP.

*Cor.* 2. Also from the equal triangles CPT, CAK, take the common triangle CPM; there remains,

triangle MPT = trapez. AKPM.

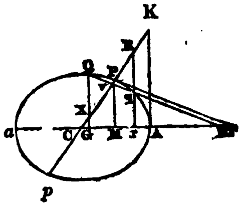
### PROPOSITION XVIII. THEOREM.

*The same being supposed, as in last proposition, then any straight lines,* QG, QE, *drawn parallel to the two tangents shall cut off equal spaces.*

That is,

triangle GQE = trapez. AKXG

triangle *rq*E = trapez. AKR*r*



Draw the ordinate PM.

The three similar triangles CAK, CMP, CGX, are to each other as $CA^2$, $CM^2$, $CG^2$,

$\therefore$ trap. AKPM : trap. : AKXG : : $CA^2 - CM^2$ : $CA^2 - CG^2$, dividendo,

But, $PM^2$ : $QG$ : : $CA^2 - CM^2 : CA^2 - CG^2$,

$\therefore$ trap. AKPM : trap. AKXG : : $PM^2$ : $QG^2$

But trian. MPT : trian. GQE : : $PM^2$ : $QG^2$,

$\therefore$ the triangles are similar.

$\therefore$ trap. AKPM : trian. MPT : : trap. AKXG : trian. GQE.

But, by Prop. XVII. Cor. 2,

trap. AKPM = triangle MPT

$\therefore$ trap. AKXG = triangle GQE

And similarly, trap. AKR$r$ = triangle $rq$E

*Cor.* 1. the three spaces AKXG, TPXG, GQE, are all equal.

*Cor.* 2. From the equals AKXG, EQG, take the equals AKR$r$, E$qr$ ; there remains,

$$RrXG = rqQG.$$

*Cor.* 3. From the equals R$r$XG, $rq$QG, take the common space $rqv$XG there remains,

$$\text{triangle } vQX = \text{triangle } vqR.$$

*Cor.* 4. From the equals EQG, TPXG, take the common space E$v$XG ; there remains,

$$TPvE = \text{triangle } vQX.$$

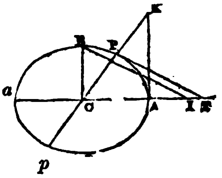
*Cor.* 5. If we take the particular case in which QG coincides with the minor axis,

The triangle EQG becomes the triangle IBC.

The figure AKXG becomes the triangle AKC,

$\therefore$ triangle IBC = triangle AKC

= triangle CPT.



### PROPOSITION XIX. THEOREM.

*Any diameter bisects all its own ordinates.*

That is,

If Q$q$ be any ordinate to a diameter CP,

$$Qv = vq$$

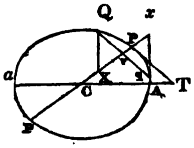
Draw QX, $qx$, at right angles to the major axis ;

Then triangle $v$QX = triangle $vqx$; Prop. XVIII., *Cor.* 3.

But these triangles are also equiangular ;

$$\therefore Qv = vq.$$

*Cor.* Hence, any diameter divides the ellipse into two equal parts.



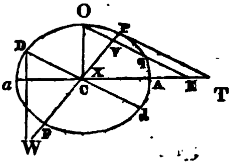
### PROPOSITION XX. THEOREM.

*The square of the semi-ordinate to any diameter, is to the rectangle under the abscissæ, as the square of the semi-conjugate to the square of the semi-diameter.*

That is,

If $Qq$ be an ordinate to any diameter CP,

$$Qv^2 : Pv \cdot vp :: CD^2 : CP^2$$



Produce $Qq$ to meet the major axis in E;

Draw QX, DW, perpendicular to the major axis, and meeting PC in X and W.

Then, since triangles CPT, $CvE$, are similar,

trian. CPT : trian. $CvE$ : : $CP^2 : Cv^2$

or, trian. CPT : trap. $TPvE$ : : $CP^2 : CP^2-Cv^2$

Again, since the triangles CDW, $vQX$, are similar,

triangle CDW : trangle $vQX$ :: $CD^2 : vQ^2$

But triangle CDW = triangle CPT ; Prop. XVIII., Cor. 5.

And triangle $vQX$ = trapez. $TPvE$ : Prop. XVIII., Cor. 3.

$$\therefore\ CP^2 : CD^2 :: CP^2-Cv^2 : vQ^2$$

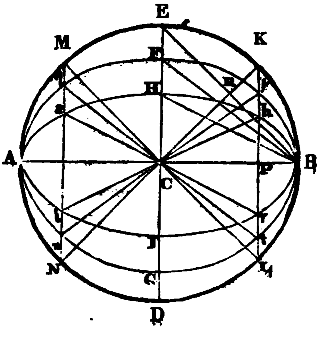
$$\text{Or, } Qv^2 : Pv\ vp :: CD^2 : CP^2$$

*Cor.* 1. The squares of the ordinates to any diameter, are to each other as the rectangles under their respective abscissæ.

*Cor.* 2. The above proposition is merely an extension of the property already proved in Prop. XII, with regard to the relation between ordinates to the axis and their abscissæ.

### PROPOSITION XXI. THEOREM.

*The equal conjugate diameters of all ellipses described on the same axis major, all terminate in the same right lines.*



Let any number of ellipses HJ, FG be described on the same axis major, AB, and the right coordinates to that axis drawn from the extremities of the equal conjugate diameters, will all coincide in the same lines KL, MN, or the vertices of all their equal diameters will all be found in those lines.

Describe on the same axis as a diameter, the circle AEBD, bisect the arcs EB in K and AE in M, from which points through the centre, C, draw the conjugate diameters KN, ML; join KL, MN; draw also the chords BE, BF, BH, also the diameters *fn*, *hi*, and *qt*, *sr*.

The triangles ECB, KPC, being similar, their sides are proportional; and since KC = BC, and KP = BR, the triangle KPC = the triangle BRC = $\frac{1}{2}$ the triangle ECB.

Hence, $CP^2 + PK^2 = \frac{1}{2} CB^2 + \frac{1}{2} CE^2$
and $KC^2 = \frac{1}{2}EB^2 = EC^2 + CB^2$,
or, $KC^2 + CL^2 = CB^2 + CE^2$
Hence $AB^2 + ED^2 = KN^2 + ML^2$

And since the semi-ordinate $KP : fP :: EC : FC$ the triangles FCB, $fPC$, are similar, and $fPC + \frac{1}{2}$ FCB and $fC^2 = \frac{1}{2} FB^2 = \frac{1}{2} FC^2 + \frac{1}{2} CB^2$

Hence, $fC^2 + Ct^2 = FC^2 + BC^2$
Therefore, $fn^2 + qt^2 = FG^2 + AB^2$ agreeably to Prop. XV.

Also in the triangles HCB, $hPC$, being for the same reason as before shown, similar, $hPC = \frac{1}{2}$ HCB, and $hC^2 = \frac{1}{2} HB^2 = \frac{1}{2} HC^2 + \frac{1}{2} cB^2$, and $HC^2 + rC^2 = HC^2 + CB^2$

Therefore, $hi^2 + sr^2 = HI^2 + AB^2$ agreeably to the property of the ellipse.

*Cor.* 1. $CP^2 = \frac{1}{2} CB$
and $Pf^2 = \frac{1}{2} CF^2$
or, $Ph^2 = \frac{1}{2} CH^2$

*Cor.* 2. If AB, instead of being the axis major, should be the minor axis of a series of ellipses, the vertices of their equal conjugate diameters would still all be found in the lines MN, KL, produced.

# HYPERBOLA.

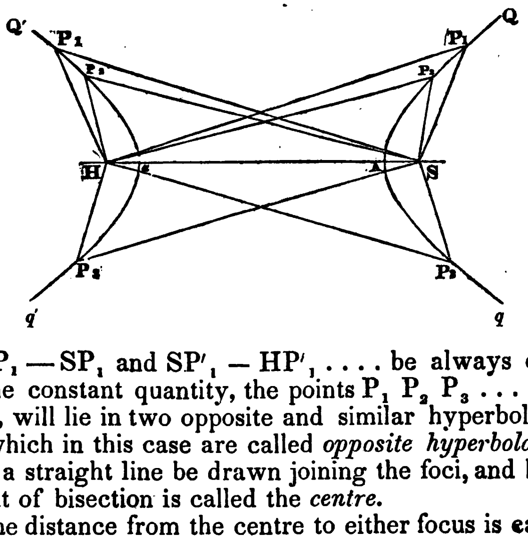
## DEFINITIONS.

1. An Hyperbola is a plane curve, such that, if from any point in the curve two straight lines be drawn to two given fixed points, the excess of the straight line drawn to one of the points above the other will always be the same.

2. The two given fixed points are called the *foci*.

Thus, let $QAq$ be an hyperbola, $S$ and $H$ the foci.
Take any number of points in the curve, $P_1$, $P_2$, $P_3$, . . .
Join $S,P_1$, $H,P_1$, $S,P_2$, $H,P_2$, $S,P_3$, $H,P_3$ ; . . . . then.

$$HP_1 - SP_1 = HP_2 - SP_2 = HP_3 - SP_3 = \ldots.$$



If $HP_1 - SP_1$ and $SP'_1 - HP'_1$ . . . . be always equal to the same constant quantity, the points $P_1$ $P_2$ $P_3$ . . . and $P'_1$, $P'_2$, $P'_3$, will lie in two opposite and similar hyperbolas $QAq$, $Q'aq'$, which in this case are called *opposite hyperbolas*.

3. If a straight line be drawn joining the foci, and bisected, the point of bisection is called the *centre*.

4. The distance from the centre to either focus is called the *eccentricity*.

5. Any straight line drawn through the centre, and terminated by two opposite hyperbolas, is called a *diameter*.

6. The points in which any diameter meets the hyperbolas are called the *vertices* of that diameter.

7. The diameter which passes through the foci is called the

*axis major*, and the points in which it meets the curves the *principal vertices*.

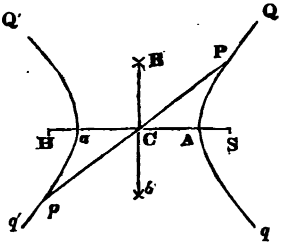
8. If a straight line be drawn through the centre at right angles to the major axis, and with a principal vertex as centre, and radius equal to the eccentricity, a circle be described, cutting the straight line in two points, the distance between these points is called the *axis minor*.

Thus, let Q*q*, Q′*q*′ be two opposite hyperbolas, S and H the foci, join S,H;

Bisect SH in C, and let SH cut the curves in A,*a*.

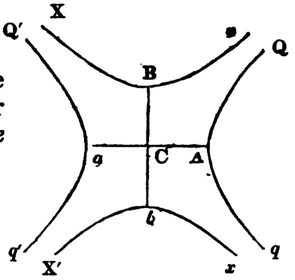
Through C draw any straight line P*p*, terminated by the curves in the points P, *p*.

Through C draw any straight line at right angles to A*a*, and with centre A and radius = CS describe a circle cutting the straight line in the points B, *b*.



Then C is the centre, CS or CH the eccentricity, P*p*, is a diameter, P and *p* its *vertices*, A*a* is the major axis, B*b* is the minor axis.

The hpyerbolas X*x*, X′*x*′, whose major axis is B*b*, and whose minor axis is A*a*, are called the *conjugate hyperbolas* to Q*q*, Q′*q*′.



9. A straight line, which meets the curve in any point, but which, being produced both ways, does not cut it, is called a *tangent* to the curve at that point.

10. A straight line, drawn through the centre, parallel to the tangent, at the vertex of any diameter, is called the *conjugate diameter* to the latter, and the two diameters are called a *pair of conjugate diameters.*

The vertices of the conjugate diameter are its intersections with the conjugate hyperbolas.

11. Any straight line drawn parallel to the tangent at the vertex of any diameter, and terminated both ways by the curve, is called an *ordinate* to that diameter.

12. The segments into which any diameter produced is di-

vided by one of its own ordinates and its vertices, are called the *abscissæ* of the diameter.

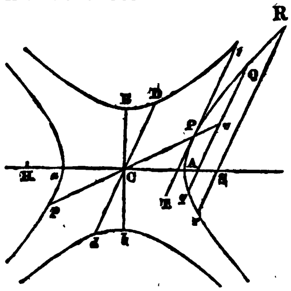
13. The ordinate to any diameter, which passes through the focus, is called the *parameter* of that diameter.

Thus, let P$p$ be any diameter, and T$t$ a tangent at P ;

Draw the diameter D$d$ parallel to T$t$ ;

Take any point Q in the curve, draw Q$q$ parallel to T$t$ and cutting P$p$ produced in $v$ ;

Through S draw R$r$ parallel to T$t$ ;



Then D$d$ is the conjugate diameter to P$p$,

Q$q$ is the ordinate to the diameter P$p$ corresponding to the point Q.

P$v$, $vp$, are the abscissæ of the diameter P$p$ corresponding to the point Q.

R$r$ is the parameter of the diameter P$p$.

14. Any straight line drawn from any point in the curve at right angles to the major axis produced, and terminated both ways by the curve, is called an *ordinate to the axis*.

15. The segments into which the major axis produced is divided by one of its own ordinates and its vertices, are called the *abscissæ of the axis.*

16. The ordinate to the axis which passes through the focus, is called the *principal parameter* or *latus rectum.*

(It will be proved in Prop. IV, that the tangents at the principal vertices are perpendicular to the major axis ; hence definitions 14, 15, 16, are in reality included in the three which immediately precede them.)

17. If a tangent be drawn at the extremity of the latus rectum, and produced to meet the major axis ; and if a straight line be drawn through the point of intersection, at right angles to the major axis ; the tangent is called the *focal tangent*, and the straight line the *directrix.*

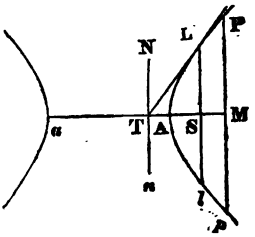
Thus, form P, any point in the curve, draw PM$p$ perpendicular to A$a$, cutting A$a$ in M ;

Through S draw L$l$ perpendicular to A$a$ ;

Let LT, a tangent at L, cut A$a$ in T ;

Through T draw N$n$ perpendicular to A$a$ :

Then, P$p$ is the ordinate to the axis corresponding to the point P.



AM, M$a$, are the abscissæ of the axis corresponding to the point P,

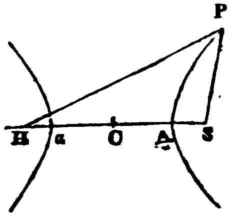
L$l$ is the latus rectum,
LT is the focal tangent,
N$n$ is the directrix.

18. An *asymptote* is a diameter which approaches the curve continually as they are both produced, but which, though ever so far produced, never meets it.

19. If the asymptotes of four opposite hyperbolas cross each other at right angles, the hyperbolas are called *right angled* or *equilaterial hyperbolas.*

### PROPOSITION I. THEOREM.

*The difference of two straight lines drawn from the foci to any point in the curve, is equal to the major axis.*



That is, if P be any point in the curve,

$$\mathrm{HP} - \mathrm{SP} = \mathrm{A}a;$$

For,

$$\left.\begin{aligned}&\mathrm{HP{-}SP}=\mathrm{AH{-}AS}=\mathrm{A}a+a\mathrm{H{-}AS}\\ &\text{And,}\\ &\mathrm{HP{-}SP}=a\mathrm{S}-a\mathrm{H}=\mathrm{A}a-a\mathrm{H}+\mathrm{AS}\end{aligned}\right\}\text{Def. 1.}$$

Or,

$$2(\mathrm{HP} - \mathrm{SP}) = 2\mathrm{A}a$$
$$\mathrm{HP} - \mathrm{SP} = \mathrm{A}a$$

*Cor.* 1. The centre bisects the major axis; for, since

$$\mathrm{AH} - \mathrm{AS} = a\mathrm{S} - a\mathrm{H}$$

Or, $\mathrm{SH} - 2\mathrm{AS} = \mathrm{SH} - 2a\mathrm{H}$

$$\therefore \quad \mathrm{AS} = a\mathrm{H}$$

And $\mathrm{CS} = \mathrm{CH}$, by def. 3.

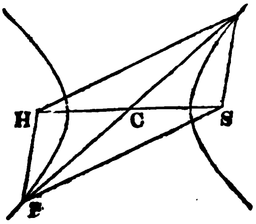
$$\therefore \quad \mathrm{AC} = a\mathrm{C}.$$

*Cor.* 2.

$$\mathrm{HP} - \mathrm{SP} = 2\mathrm{AC}$$
$$\therefore \quad \mathrm{HP} = 2\mathrm{AC} + \mathrm{SP}$$
$$\mathrm{SP} = \mathrm{HP} - 2\mathrm{AC}$$
$$\mathrm{HP} + \mathrm{SP} = 2\mathrm{AC} + 2\mathrm{SP}.$$

### PROPOSITION II. THEOREM.

*The centre bisects all diameters.*



Take any point P in the curve;
Join S, P; H, P; S, H;
Complete the parallelogram SPH$p$;
join C,$p$; C, P;
Then, since the opposite sides of parallelograms are equal,

$$\mathrm{HP} = \mathrm{S}p, \quad \mathrm{SP} = \mathrm{H}p;$$
$$\therefore \mathrm{HP} - \mathrm{SP} = \mathrm{S}p - \mathrm{H}p;$$

∴ $p$ is a point in the opposite hyperbola by definition 2.

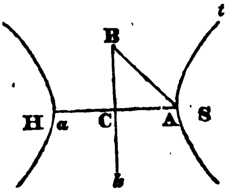
Again, since the diagonals of a parallelogram bisect each other, and since SH is bisected in C, (def. 3,)

∴ P$p$ is a straight line and a diameter, and is bisected in C.

In like manner, it may be proved that any other diameter is bisected in C.

### PROPOSITION III. THEOREM.

*The rectangle under the segments of the major axis produced, made by the focus and its vertices, is equal to the square of the semi-axis, minor.*



That is,

$$AS \cdot Sa = BC^2$$

For,

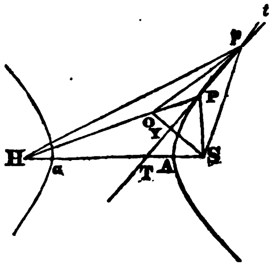
$$\begin{aligned} BC^2 &= AB^2 - AC^2 \\ &= SC^2 - AC^2, \text{ by def. 8,} \\ &= (SC - AC)\,(CS + AC) \\ &= AS \cdot Sa \end{aligned}$$

*Cor.* The square ef the eccentricity is equal to the sum of the squares of the semi-axes.

$$\begin{aligned} \text{For,}\quad SC^2 &= AB^2, \text{ def. 8,} \\ &= AC^2 + BC^2. \end{aligned}$$

### PROPOSITION IV. PROBLEM.

*To draw a tangent to the hyperbola at any point.*



Let P be the given point;

Join S, P; H, P;

Bisect the angle SPH by the straight line T$t$.

T$t$ is a tangent to the curve at P.

For if T$t$ be not a tangent, let T$t$ cut the curve in some other point $p$.

Join S, $p$; H, $p$; draw SYO perpendicular to T$t$, meeting HP in O; join $p$, O.

Since the angles at Y are right angles, and angle SPY = angle OPY by construction, and side YP common to the two triangles SYP, OYP.

$$\therefore \quad SY = OY$$

$$\text{And} \quad SP = OP.$$

Again, since SY = OY, and Y$p$ common to the two triangles SY$p$, OY$p$, and the angles at Y equal;

$$\therefore \quad Sp = Op$$

$$\begin{aligned}\therefore Hp - Op &= Hp - Sp\\ &= HP - SP\\ &= HP - OP\\ &= HO\\ \therefore \quad Hp &= HO + Op\end{aligned}$$

that is, one side of the triangle H$Op$ is equal to the other two, which is absurd;

∴ $p$ is not a point in the curve: and in the same manner, it may be proved that no point in the straight line T$t$ can be in the curve, except P;

∴ T$t$ is a tangent to the curve at P.

*Cor.* 1. Hence tangents at A and $a$, are perpendicular to the major axis.

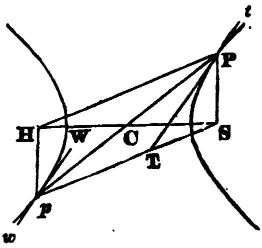
*Cor.* 2. SP and HP make equal angles with every tangent.

*Cor.* 3. Since SPH, the vertical angle of the trian. SPH, is bisected by the straight line PT, which cuts the base in T,

∴ HT : TS : : HP : SP.

### PROPOSITION V. THEOREM.

*Tangents drawn at the vertices of a diameter are parallel.*



Let T$t$, W$w$, be tangents at P, $p$, the vertices of the diameter PCB.

Join S,P ; H, P ; S, $p$ ; H, $p$ :

Then, by Prop. II, SH is a parallelogram, and since the opposite angles of parallelograms are equal,

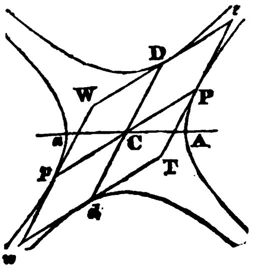
∴ angle SPH = angle S$p$H.

But the tangents T$t$, W$w$; bisect the angles SPH, S$p$H, respectively.

∴ angle W$p$S = angle HPT

= angle PTS, which is the exterior opposite angle to W$p$S.

∴ W$w$ is parallel to T$t$.

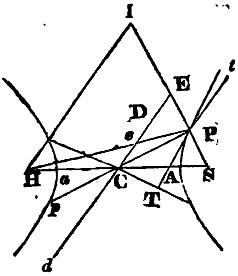


*Cor.* If D$d$ be a diameter conjugate to P$p$, and terminated by the conjugate hyperbolas, tangents drawn at D and $d$ will be parallel.

Hence tangents drawn at the extremities of conjugate diameters form a parallelogram.

### PROPOSITION VI. THEOREM.

*If straight lines be drawn from the foci to a vertex of a diameter, the distance from the vertex to the intersection of the conjugate diameter with either focal distance, is equal to the semi-axis major.*



That is, if D*d* be a diameter to conjugate to P*p*, cutting SP produced in E, and HP in *e*,

PE or P*e* = AC.

Draw HI parallel to D*d*, meeting SP produced in I.

The angle PHI = alternate angle HPT

= angle TPS

= angle HIP

∵ HI is parallel to D*d* or T*t*

∴ IP = HP.

Also, since SC = HC, and CE is parallel to HI, the base of the trian. SHI,

∴ SE = EI. Hence, ∵ PE = PI—EI = HP—SE=HP SP−PE

∴ 2 PE = HP — SP = 2 AC

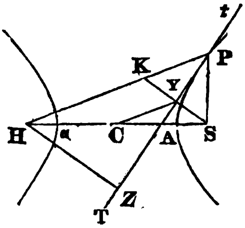
PE = AC.

Also angle PE*e* = angle P*e*E, ∴ P*e* = PE and

P*e* = AC.

### PROPOSITION VII. THEOREM.

*Perpendiculars from the foci upon the tangent at any point, intersect the tangent in the circumference of a circle whose diameter is the major axis.*



From S let fall SY perpendicular on T*t* a tangent at P.

Join S, P ; H, P ; let HP meet SY in K ; join C, Y ;

Then, since angle SPY = angle KPY, and the angles at Y are right angles, and the side PY common to the two triangles SPY, KPY,

∴ SY = KY

and KP = SP.

Again, since SY = YK, and SC = CH, CY cuts the sides of the trian. HSK proportionally,

∴ CY is parallel to HP.

Also, since CY is parallel to HP, $SY = KY$, and $SC = CH$, $\therefore CY = \frac{1}{2} HK = \frac{1}{2} (HP - KP) = \frac{1}{2} (HP - SP) = \frac{1}{2} Aa = AC$.

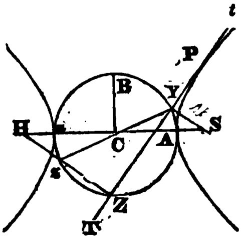
Hence, a circle described with centre C and radius = CA, will pass through Y, and in like manner, if HZ be drawn perpendicular to T$t$, it may be proved that the same circle will pass through Z also.

### PROPOSITION VIII. THEOREM.

*The rectangle contained by perpendiculars from the foci upon the tangent at any point, is equal to the square of the semi-axis, minor.*

That is,

$$SY \cdot HZ = BC^2.$$



Let T$t$ be a tangent at any point P;

On A$a$ describe a circle cutting T$t$ in Y and Z; join S, Y; H, Z.

Then, by last Prop. SY, HZ are perpendicular to T$t$.

Let HZ meet the circumference in $z$;

Join C, $z$; C, Y;

Since $z$ZY is a right angle, the segment in which it lies is a semicircle, and $z$, Y, are the extremities of a diameter;

$\therefore z$CY is a straight line and a diameter.

Hence the triangles CYS, C$z$H are in every respect equal

$$\therefore SY = Hz$$

$$\therefore SY \cdot HZ = Hz \cdot HZ$$

$$= HA \cdot Ha.$$

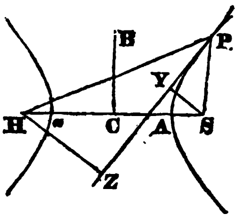
$$= BC^2. \quad \text{Prop. III.}$$

### PROPOSITION IX. THEOREM.

*Perpendiculars let fall from the foci upon the tangent at any point, are to each as the focal distance of the point of contact.*

That is,

$$SY : HZ :: SP : HP.$$



For the triangles SPY, HPZ, are manifestly similar;

$$\therefore SY : HZ :: SP : HP.$$

*Cor.* Hence,

$$SY = HZ \frac{SP}{HP}$$

$$\therefore SY^2 = SY \cdot HZ \cdot \frac{SP}{HP}$$

$$= BC^2 \cdot \frac{SP}{HP} \text{ last Prop.}$$

$$= BC^2 . \quad \frac{SP}{2\,AC + SP}$$

So also,

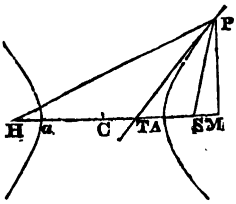
$$HZ^2 = BC^2 \frac{HP}{SP} = BC^2 \frac{HP}{HP - 2\,AC}$$

### PROPOSITION X. THEOREM.

*If a tangent be applied at any point, and from the same point an ordinate to the axis be drawn, the semi-axis major is a mean proportional between the distance from the centre, to the ordinate with the axis, and the distance from the centre to the intersection of the tangent with the axis.*

That is,

CT : CA : : CA : CM.



Since the angle SPH is bisected by PT, which cuts HS, the base of the triangle HPS, in T, ∴

HT : ST : : HP : SP

∴ HT — ST : HT+ST : : HP — SP : HP+SP

or, 2 CT : SH : : 2AC : HP +SP

∴ 2 CT : 2 AC : : SH : HP+SP - - - - (1)

But since PM is drawn from the vertex of triangle HPS perpendicular to HS produced,

HM—SM : HP+SP : : HP—SP : HM+SM

or, SH : HP+SP : : 2 AC : 2 CM - - - - (2)

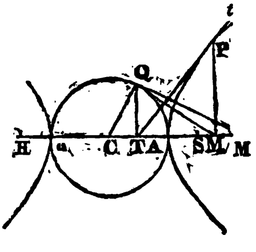
Comparing this with the proportion marked (1), we have

2 CT : 2 AC : : 2 AC : 2 CM

or, CT : CA : : CA : CM.

### PROPOSITION XI. THEOREM.

*Let* AQ*a be a circle described on the major axis, from the point* T, *draw* TQ *perpendicular to* A*a, meeting the circle in* Q, *join* QM.



Then QM is a tangent to the circle at Q

Join C, Q.

For if QM be not a tangent, draw QM′ a tangent at Q, cutting AC in M′

Then CQM′ is a right angle,

∴ Since QT is drawn from the right angle CQM′ perpendicular to the hypothenuse,

$\therefore CM' : CQ :: CQ : CT.$

or, $CM' : CA :: CA : CT, \therefore CQ = CA.$

But by the last Prop..

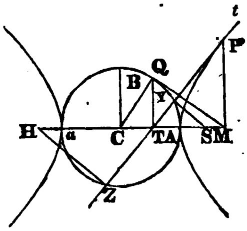
$$CM : CA :: CA : CT$$

$$\therefore CM = CM',$$

which is absurd; $\therefore$ QM is not a tangent at Q; and in the same manner it may be proved that no line but QM′ can be a tangent at Q.

### PROPOSITION XII. THEOREM.

*The square of any semi-ordinate to the axis, is to the rectangle under the abscissæ, as the square of the semi-axis minor, is to the square of the semi-axis major.*



That is, if P be any point in the curve,

$$PM^2 : AM . Ma :: BC^2 : AC^2.$$

Describe a circle on A$a$, and draw PT a tangent to the hyperbola at P, intersecting the circle in the points Y, Z, and the major axis in T.

Draw TQ perpendicular to A$a$, meeting the circle in Q; join QM

Then QM is a tangent to the circle at Q by Prop. II , and $\therefore$ the angle CQM is a right angle.

Join S, Y; , Z; HSY and HZ are perpendicular to T$t$, Prop. VII.

Hence the triangles PMT, SYT, PZT, are similar to each other.

$\therefore PM : SY :: MT : TY$

and, $PM : HZ :: MT : TZ$

$\therefore PM^2 : SY . HZ :: MT^2 : TY . TZ,$

or, $PM^2 : BC :: MT^2 : TQ^2,$

Prop. VII.

$:: QM^2 : CQ^2,$

$\because$ MQT. MCQ are similar triangles,

$:: AM . Ma : AC^2,$

$\because PM^2 : AM . Ma :: BC^2 : AC^2$

*Cor.* 1.

Let $P_1 M_1$, $P_2 M_2$, ...... be ordinates to the axis from any point $P_1$ $P_2$ ......

Then by Prop.

$$P_1 M_1^2 : AM_1 . M_1 a :: BC^2 : AC^2$$

$$P_2 M_2^2 : AM_2 . M_2 a :: BC^2 : AC^2$$

$$\therefore P_1 M_1^2 : P_2 M_2^2 :: AM_1 . M_1 a : AM_2 . M_2 a,$$

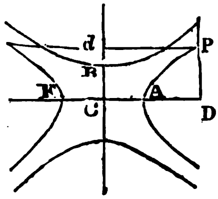
That is, the square of the ordinates to the axis are to each other as the rectangles of their abscisa.

*Cor.* 2. By Prop.

$$PM^2 : AM \, . \, Ma :: BC^2 : AC^2$$

But $AM = CM - CA$, $Ma = CM + CA$,

$$\therefore PM^2 : CM^2 - CA^2 :: BC^2 : AC^2.$$



*Cor.* 3. Since by the proposition
$PM^2 : AM : Ma :: BC^2 : AC^2$,
we have in the conjugate hyberbolas
$CR^2 : CA^2 : CR^2 + Cd^2 : dP^2$,
since $dP^2 = CD^2$.

### PROPOSITION XIII. THEOREM.

*The latus rectum is a third proportional to the axis major and minor.*

That is,

$$Aa : Bb :: Bb : Ll.$$

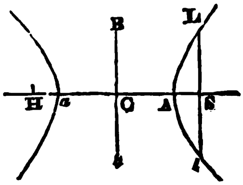
Since LS is a semiordinate to the axis.

$AC^2 : BC^2 :: AS \, . \, Sa : LS^2$, Prop. XII.

$:: \quad BC^2 \quad : LS^2$, Prop. III.

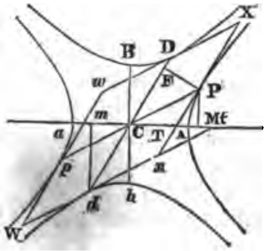
$\therefore AC : BC :: \quad BC \quad : LS$

or, $Aa : Bb :: \quad Bb \quad : Ll.$



### PROPOSITION XIV. THEOREM.

*The area of all parallelograms, formed by drawing tangents at the extremities of two conjugate diameters, is constant, each being equal to the rectangle under the axes.*



Let $Pp$, $Dd$, be any two conjugate diameters, $WwXx$, a parallelogram inscribed between the opposite and conjugate hyperbolas by drawing tangents at $P, p, D, d$; then $Pp$, $Dd$, divide the parallelogram $WxXw$ into four equal parallelograms.

Draw $Pm$. $dm$, ordinates to the axis; $PF$ perpendicular to $Dd$.

Let $CA$ meet $PX$ in $T$ and $Wx$ in $t$;

Then $\quad CT : CA :: CA : CM$

$Ct : CA :: CA : Cm$, Prop XI.

$\therefore CT : Ct :: Cm : CM$

But, $CT : Ct :: MT : Cm$, by similar triangles.

$\therefore MT : Cm :: Cm : CM$

$\therefore CM . MT = Cm^2$ - - - - - - - - (1)

Again, $CM : CA :: CA : CT$

$\therefore CM : CA :: MA : AT$, dividendo:

Or, $CM : Ma :: MA : MT$, componendo:

$\therefore AM . Ma = CM . MT = Cm^2$ - - - - (2)

But, $AC^2 : BC^2 :: AM : Ma : PM^2$

Or, $AC^2 : BC^2 :: Cm^2 : PM^2$

$\therefore AC : BC :: Cm : PM$

Similarly, $AC : BC :: CM : dm$

Or, $BC : dm :: CA : CM$

But, $CT : CA :: CA : CM$

$\therefore CT : CA :: BC : dm$

But, $PF : CT :: dm : Cd$

$\therefore PF : CA :: BC : Cd$

$\therefore$ Rectangle $PF . CD =$ rectangle $AC . BC$

or, Parallelogram $CX =$ rectangle $AC . BC$

$\therefore$ Parallelogram $WwXx = 4\ AC . BC = Aa . Bb$

Cor. By (2),

$$Cm^2 = HM . Ma$$

$$= (CM - CA)\ (CM + CA).$$

$$= CM^2 - CA^2$$

$$\therefore \quad CA^2 = CM^2 - Cm^2$$

And similarly, $CB^2 = dm^2 - PM^2$.

### PROPOSITION XV. THEOREM.

*The difference of the squares of any two conjugate diameters, is equal to the same constant quantity, namely, the difference of the squares of the two axes.*

That is, if $Pp$, $Dd$, be any two conjugate diameters,

$$Pp^2 - Dd^2 = Aa^2 - Bb^2.$$

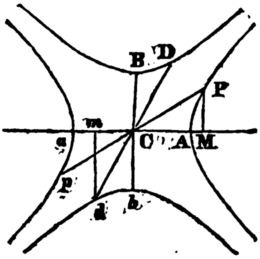
Draw $PM$ $dm$, ordinates to the axis.

Then, by Cor. to last Prop.

$$AC^2 - BC^2 = CM^2 + PM^2 - (Cm^2 + d^2m)$$

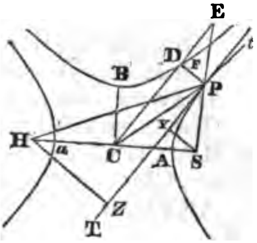
$$= CP^2 - Cd^2$$

$$\therefore Aa^2 - Bb^2 = Pp^2 - Dd^2.$$



### PROPOSITION XVI. THEOREM.

*The rectangle under the focal distance of any point, is equal to the square of the semi-conjugate.*



That is, if CD be conjugate to CP,

$$SP \cdot HP = CD^2.$$

Draw SY, HZ, perpendiculars to the tangent at P, and PF perpendicular to CD;

Then by similar trian. SPY, PEF

$$SP : SY :: PE : PF$$

or, $SP : SY :: DC : PF$

$\because PE = AC$, by Prop. VI.

Similarly, $HP : HZ :: AC : PF$

$\therefore SP \cdot HP : SY \cdot HZ :: AC^2 : PF^2$

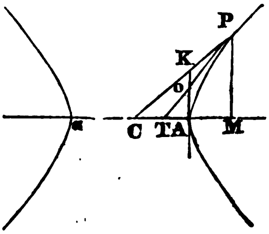
$:: CD^2 : CB^2$, by Prop. XIV.

But $SY \cdot HZ = CB^2$, by Prop. VIII.

$$\therefore SP \cdot HP = CD^2.$$

### PROPOSITION XVII. THEOREM.

*If two tangents be drawn, one at the principal vertex, the other at the vertex of any other diameter, each meeting the other's diameter produced, the two tangential triangles thus formed will be equal*



That is,

triangle CPT = triangle CAK.

Draw the ordinate PM; then

$CM : CA :: CP : CK$, by similar triangles.

But, $CM : CA :: CA : CT$

$\therefore CA : CT :: CP : CK$.

The two triangles CPT, CAK, have thus the angle C common and the sides about that angle reciprocally proportional; these triangles are $\therefore$ equal.

*Cor.* 1. Take each of the equal triangle CPT, CAK, from the common space CAOP; there remains

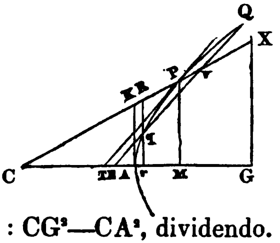
triangle OAT = OKP.

*Cor.* 2. Also take the equal triangles CPT, CAK, from the common triangle CPM; there remains

triangle MPT = trapez. AKPM.

### PROPOSITION XVIII. THEOREM.

*The same being supposed, as in last proposition, then any straight lines* QG, QE, *drawn parallel to the two tangents shall cut off equal spaces.*



That is,
triangle GQE = trapez. AKXG
triangle *rq*E = trapez. AKR*r*
Draw the ordinate PM.

The three similar triangles CAK, CMP, CGX, are to each other as $CA^2$, $CM^2$, $CG^2$,

∴ AKPM : trap. AKXG : : $CM^2 - CA^2 : CG^2 - CA^2$, dividendo.
But, $PM^2 : QG^2 : : CM^2 - CA^2 : CG^2 - CA^2$,
∴ trap. AKPM : trap. AKXG : : $PM^2 : QG^2$
But, trian. MPT : trian. GQF : : $PM^2 : QG^2$,
∵ the triangles are similar.
∴ trap. AKPM : trian. MPT : : trap. AKXG : trian. GQE,
But, by Prop. XVII, Cor. 2.
trap. AKPM = triangle MPT ;
∴ trap. AKXG = triangle GQE.
And similarly, trap. AKR*r* = triangle *rq*E.

*Cor.* 1. The three spaces AKXG, TPXG, GQE, are all equal.

*Cor.* 2. From the equals, AKXG, EQG, take the equals AKR*r*, E*qr* ; there remains,

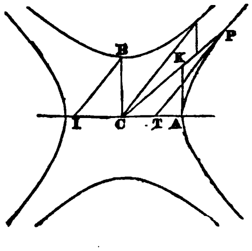
R*r*XG, = *rq*QG.

*Cor.* 3. From the equals R*r*XG, *rq*QG, take the common space *rqv*XG ; there remains,

triangle *v*QX = triangle *vq*R.

*Cor.* 4. From the equals EQG, TPXG, take the common space E*v*XG ; there remains,

TP*v*E = triangle *v*QX.



*Cor.* 5. If we take the particular case in which QG coincides with the minor axis,

The triangle EQG becomes the triangle IBC,

The figure AKXG becomes the triangle AKC,

∴ triangle IBC = triangle AKC
= triangle CPT.

### PROPOSITION XIX. THEOREM.

*Any diameter bisects all its own ordinates.*

That is,

If $Qq$ be any ordinate to a diameter CP,

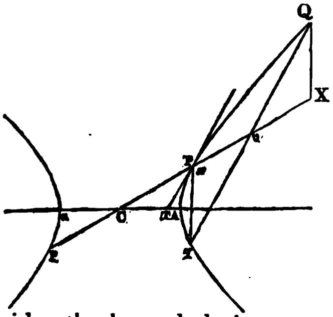
$$Qv = vq$$

Draw QX, $qx$, at right angles to the major axis ;

Then triangle $vQX$ = triangle $vqx$ ; Prop. XVIII., Cor. 3.

But these triangles are also equiangular ;

$$\therefore Qv = vq.$$



*Cor.* Hence, any diameter divides the hyperbola into two equal parts.

### PROPOSITION XX. THEOREM.

*The square of the semi-ordinate to any diameter, is to the rectangle under the abscissæ, as the square of the semi-conjugate to the square of the semi-diameter.*

That is,

If $Qq$ be an ordinate to any diameter CP,

$$Qv^2 : Pv \,.\, vp :: CD^2 : CP^2.$$

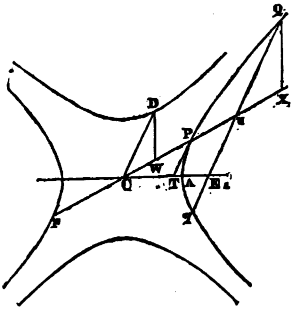
Let $Qq$ meet the major axis in E ;

Draw QX, DW, perpendicular to the major axis, and meeting PC in X and W.

Then, since the triangles CPT, $CvE$, are similar,

trian. CPT : trian. $CvE$ :: $CP^2$ $Cv^2$

or, trian. CPT : trap. $TPvE$ :: $CP^2$ : $Cv^2 - CP^2$



Again, since the triangles CDW, $vQX$, are similar,

triangle CDW : triangle $vQX$ :: $CD^2$ : $vQ^2$ ;

But, triangle CDW = triangle CPT ; Prop. XVIII, Cor. 5,

And triangle $vQX$ = trapez. $TPvE$ ; Prop. XVIII., Cor. 3.

$$\therefore CP^2 : CD^2 :: Cv^2 - CP^2 : vQ^2$$

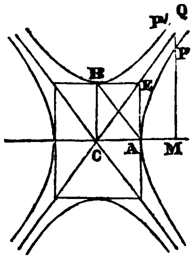
$$\text{Or, } Qv^2 : Pv \,.\, vp :: CD^2 : CP^2.$$

*Cor.* 1. The squares of the ordinates to any diameter, are to each other as the rectangles under their respective abscissæ.

*Cor.* 2. The above proposition is merely an extension of the property already proved in Prop. XII, with regard to the relation between ordinates to the axis and their abscissæ.

### PROPOSITION XXI. THEOREM.

*If tangents be drawn at the vertices of the axes, the diagonals of the rectangle so formed are asymptotes to the four curves.*



Let MP meet CE in Q;
Then, $MQ^2 : CM^2 :: AE^2 : AC^2$
$:: BC^2 : AC^2$
$:: MP^2 : CM^2—CA^2$.

Now, as CM increases, the ratio of $CM^2$ to $CM^2—CA^2$ continually approaches to a ratio of equality; but $CM^2 — CA^2$ can never become actually equal to $CM^2$, however much CM may be increased. Hence, MP is always less than MQ, but approaches continually nearer to an equality with it.

In the same manner it may be proved, that CQ is an asymptote to the conjugate hyperbola BP'.

*Cor.* 1. The two asymptotes make equal angles with the axis major and with the axis minor.

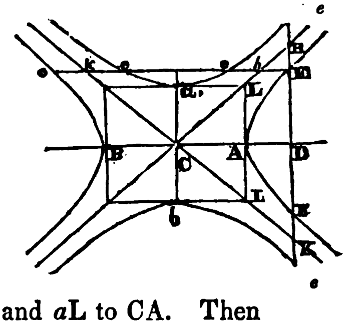
*Cor.* 2. The line AB joining the vertices of the conjugate axes is bisected by one asymptote and is parallel to the other.

*Cor.* 3. All lines perpendicular to either axis and is terminated by the asymptotes are bisected by the axis.

### PROPOSITION XXII. THEOREM.

*If a line be drawn through any point of the curves, parallel to either of the axes, and terminated at the asymptotes, the rectangle of its segments, measured from that point, will be equal to the square of the semi-axis to which it is parallel.*

That is,
the rect. HEK or H*e*K = $CA^2$,
and rect. *h*E*k* or *hek* = $CA^2$.



For, draw AL parallel to C*a*, and *a*L to CA. Then by the parallels, $CA^2 : Ca^2$ or $AL^2 :: CD^2 : DH^2$;
and by Prop. XII, $CA^2 : Ca^2 :: CD^2-CA^2 : DE^2$;
∴ by subtr. $CA^2 : Ca^2 :: CA^2 : DH^2-DE^2$ or HEK.
But the antecedents $CA^2$, $CA^2$ are equal,

Therefore, the consequents $Ca^2$, HEK must also be equal.

In like manner it is again,

by the parallels, $CA^2 : Ca^2$ or $AL^2 :: CD^2 : DH^2$;

$CA^2 : Ca :: CD^2+CA^2 : De^2$;

∴ by subtr. $CA^2 : Ca^2 :: CA^2 : De^2—DH^2$ or $HeK$.

But the antecedents $CA^2$, $CA^2$ are the same,

∴ the conseq. $Ca^2$, $HeK$ must be equal.

In like manner, by changing the axes, is $hEk$ or $hek = CA^2$.

*Cor.* 1. Because the rectangle HEK = the rectangle H*e*K,

∴ $EH : eH :: eK : EK$

And consequently HE : is always greater than H*e*.

*Cor.* 2. The rectangle *h*EK = the rectangle HE*k*.

For, by similar triangles $Eh : EH :: Ek : EK$.

*Scholium.* It is evident that this proposition is general for any line oblique to the axis also, namely, that the rectangle of the segment of any line, cut by the curve, and terminated by the asymptotes, is equal to the square of the semi-diameter to which the line is parallel—since the demonstration is drawn from properties that are common to all diameters.

*Cor.* 3. Hence it is evident that all the rectangles are equal which are made of the segments of any parallel lines, cut by the curves, and limited by the asymptotes ; and therefore, that the rectangle of any two lines drawn from any point in the curve, parallel to two given lines, and limited by the asymptotes, is a constant quantity.

## GENERAL REMARK.

Having thus discussed at length, the properties of the parabola, ellipse and hyperbola, in the relations of their local and peculiar constructions, it may be observed, that there are many properties common to each—especially in the ellipse and hyperbola. These curves have many striking similarities in their determinations, although there is but little similarity in their construction ; for the axes, of the hyperbola are thrown without the curve, while in the ellipse they are within ; hence it should not be surprising, that the same or corresponding quantities, thus differently associated, should propagate by a somewhat similar condition of their mutations, curves so apparently dissimilar in their developments. The ellipse is a curve of limited extent returning into itself as the circle, but the hyperbola is unlimited in its construction, or its determination, for its branches may be extended indefinitely; the same may be observed in relation to the parabola, which may be indefinitely extended, and the branches of the curve become at length parallel to its axis; this, however, is only in their infinite extension. But the hyperbolic curve never approaches toward or even to a parallelism with the axis, but approaches infinitely toward its asymptotes; but without ever touching them, except in their infinite extension. The manner in which these curves are derived from the sections of a cone, as their name indicates, their origin will be shown in another volume, and their quadratures, and some other properties in relation to them, will be there discussed.

# HIGHER GEOMETRY

AND

# MENSURATION:

BEING THE

FOURTH PART OF A SERIES

ON

ELEMENTARY AND HIGHER

GEOMETRY, TRIGONOMETRY, AND MENSURATION,

CONTAINING MANY VALUABLE DISCOVERIES AND IMPROVEMENTS IN MATHEMATICAL SCIENCE, ESPECIALLY IN RELATION TO THE QUADRATURE OF THE CIRCLE, AND SOME OTHER CURVES, AS WELL AS THE CUBATURE OF CERTAIN CURVILINEAR SOLIDS; DESIGNED AS A TEXT-BOOK FOR COLLEGIATE AND ACADEMIC INSTRUCTION, AND AS A PRACTICAL COMPENDIUM OF MENSURATION.

BY NATHAN SCHOLFIELD.

NEW YORK:
PUBLISHED BY COLLINS, BROTHER & CO.
No. 254 Pearl Street.
• • • • •
1845.

pt. 4

Entered according to Act of Congress, in the year 1845, by
NATHAN SCHOLFIELD,
In the Clerk's Office of the District Court of Connecticut.

G. W. WOOD, PRINTER, 29 GOLD ST., NEW YORK.

# PREFACE.

Having, in the former parts of this series, treated of the elements of geometry, trigonometry, conic sections, &c., it now remains for us, in accordance with our original design, to make such application of the former principles, as to elicit such other truths or principles as depend on their various combinations, and to investigate the relations of such subjects as pertain to the higher geometry. And, without attempting to give a full and perfect treatise on the subject, which would require volumes, we shall endeavor to present some portions of this ancient subject in a new dress; hoping, thereby, to render its beauties more plainly visible, and its oracles more intelligible.

Some new solids are introduced into this volume, the most important of which is a class termed *revoloids;* which, from their organization, seem to serve as a connecting link between rectilinear and curvelinear solids. The properties of those solids are discussed, and their surfaces and solidities are determined. Some new curves are also introduced and investigated, among which is the *revoloidal curve,* whose quadrature is determined; and, from its relation to the circle, and also to rectilinear figures, we are enabled to approximate to the circle's quadrature to an indefinite extent. During the investigation of this subject, other important properties of the circle will be developed, by which the area of the segment of a circle whose arc and sine are known, may be computed with as little labor as that of the area of a triangle whose base and perpendicular are given.

We have also introduced into this work a mode of construction for variable quantities, or such magnitudes as depend on variable factors; and have adapted a notation, embracing some of the principles of the calculus, by which variable magnitudes may be algebraically discussed, and their conditions rendered intelligible. By this notation, some of the more difficult geometrical subjects are susceptible of the most elegant solution; and we are also enabled to get a definite algebraic expression for the circle's quadrature, in terms of the diameter. The mensuration of such superficies and solids as depend on the higher geometry, follows at the close of the work.

From the hasty manner with which a considerable portion of this work has been prepared, it can hardly be presumed to be entirely free from errors; but it is believed that if any errors exist, they are such as involve no important principle.

The author, with these remarks, submits the work to the consideration of an intelligent public.

# CONTENTS.

## BOOK I.

PAGE.
On the Species and Quadrature of the Sections of Elementary Solids, embracing the Parabola, Ellipse, and Hyperbola, . . . . . 7
Definitions, . . . . . . . . . . . . . . . 7
The Sections of a Cone, . . . . . . . . . . . . 8
The Sections of a Polyedroid, . . . . . . . . . . . 10
The Sections of a Prism, &c., . . . . . . . . . . . 11
Quadrature of the Parabola, . . . . . . . . . . . 16
On the Ellipse, its Quadrature, &c., . . . . . . . . 20
On the Hyperbola, its Quadrature, &c., . . . . . . . . 23
Equations to the Conic Sections, . . . . . . . . . 31

## BOOK II.

Solid Sections, or Segments of Solids of Revolution, Ungulas, &c., . 34
Definitions, . . . . . . . . . . . . . . . 34
Cylindrical Ungulas and Segments, . . . . . . . . . 36
Conical Segments and Ungulas, . . . . . . . . . 44
Segments and Ungulas of an Elliptical Cylinder, . . . . . 47
Spherical Ungulas and Segments, . . . . . . . . . 49
Parabolic Prisms and Ungulas, . . . . . . . . . 51

## BOOK III.

On Revoloids and Solids formed by the Revolution of the Conic Sections, 54
Definitions, . . . . . . - . . . . . . . . 54
Quadrature of the Surface of a Revoloid determined, . . . 60
Revolvoids and Ungulas equivalent to a Sphere or Spheroid, . . 63
The Cubature of the Revolvoid determined, . . . . . . 68
Segments of Revolvoid, . . . . . . . . . . . 75
Parabolic Revoloid, its Cubature determined, . . . . . 84
Sections of Solids formed by the Revolution of the Conic Sections, . 87
Scholia and Formulæ, in relation to Cylindric and Conical Ungulas, . 91

## BOOK IV.

On the Revolvoidal Curve, the Rectification of the Ellipse, and other Curves, and on the Quadrature of the Circle, . . . . . 98
Definitions, . . . . . . . . . . . . . . . 98
Projection of a Revolvoidal Curve, . . . . . . . . . 100
Equation to the Curve, . . . . . . . . . . . 102
The Ellipse, its Rectification, &c., . . . . . . . . 103
Rectification of the Revoloidal Curve, . . . . . . . 108
On the Circle's Circumference, and its Quadrature, . . . . . 109
Curve of the Circle's Quadrature, . . . . . . . . . 129
On the Quadrature of the Segment of a Circle, . . . . . 131
On Spirals—their Quadrature, &c., . . . . . . . . 139
On the Cycloid, . . . . . . . . . . . . . 143

## BOOK V.

PAGE.
On the Production and Resolution of Geometrical Magnitudes, . . 145
CHAP. I.—Definitions and Principles, . . . . . . . 145
Production of Surfaces and Solids, . . . . . 146
CHAP. II.—On the Construction of Quantities whose Elements are a series either of constant or variable quantities, . . 157
Explanation of Principles, and Notation, . . . 157
Construction of Variables, . . . . . . . . 159
Construction of Curves from their Equations, . . 159
Quadrature of the Circle expressed algebraically, in terms of known functions of the diameter, . . . . 160
Equivalent Constructions for the Equations of Curvelinear Solids, . . . . . . . . . . . . 161
CHAP. III.—The Differential and Integral Calculus, . . . . 166
Differential Calculus, . . . . . . . . 166
Integral Calculus, . . . . . . . . . 169
CHAP. IV.—On the Centre of Surfaces and Solids, the Virtual Centre and Centre of Magnitude, &c., . . . . . . 173
Definitions, . . . . . . . . . . . 173
Virtual Centre of a System of Points, . . . . 174
Virtual Centre of Surfaces, . . . . . . . 175
The Virtual Centre of Solids, . . . . . . 179
The Virtual Centre of a Circular Arc, . . . . 180
The Virtual Centre of the Surface of a Solid, . . 181
CHAP. V.—On the Relations of Lines, Surfaces, and Solids, generated by the motion of the Virtual Centre, . . . 182
General Proposition, . . . . . . . . 182

## MENSURATION OF SUPERFICIES.

Circular Segments and Zones, . . . . . . . . . . 186
To find the Circumference of a Circle, or any Arc, . . . . 192
Mensuration of the Ellipse, . . . . . . . . . 195
Mensuration of the Parabola, . . . . . . . . . 202
Mensuration of the Hyperbola, . . . . . . . . . 207
To find the Area of any Plane Surface by Equi-distant Ordinates, . 210

## MENSURATION OF SOLIDS.

Mensuration of the Sphere and Revoloid, . . . . . . . 212
Segments of a Sphere or Revoloid, . . . . . . . . 214
Segments of a Spheroid, . . . . . . . . . . . 216
Mensuration of the Paraboloid, &c., . . . . . . . . 218
Mensuration of the Hyperboloid, &c., . . . . . . , . 219
Circular Elliptical and Parabolic Spindles, . . . . . . 221
Ungulas, . . . . . . . . . . . . . . . 224
Rings, . . . . . . . . . . . . . . . 227
Gauging of Casks, . . . . . . . . . . . . 228

## SPECIFIC GRAVITY OF BODIES.

Table of Specific Gravities, . . . . . . . . . . 233
Magnitude of Bodies, by their Weight and Specific Gravity, . . 234
QUESTIONS FOR EXERCISE, . . . . . . . . . . . 236
Description of an Instrument for Measuring Distances and Heights by a single observation, . . . . . . . . . . . 242
NOTES, . . . . . . . . . . . . . . . 248

# HIGHER GEOMETRY.

## PART II.—BOOK I.

### SPECIES AND QUADRATURE OF SUPERFICIAL SECTIONS OF ELEMENTARY SOLIDS.

#### DEFINITIONS.

1. *Superficial sections* are surfaces formed when solids are cut by *plane* or *curved surfaces.*

2. If the cutting surface is a *plane,* the section is a *plane section.*

3. Superficial sections of solids take different names, according to the form of the solid in the plane of the section.

4. From the cylinder, we have the *rectangle,* the *circle,* and, as will be shown, (Prop. VIII. *Cor.*) the *ellipse.*

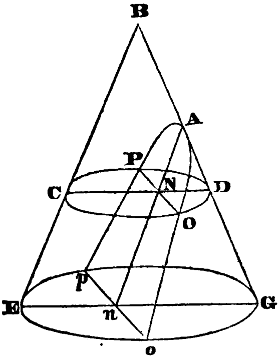
5. From the cone, we have five different figures, viz: a triangle, a circle, and, as will be shown in Propositions I., II., and III., a *parabola,* an *ellipse,* and a *hyperbola.*

6. From the sphere, we have only the circle.

*Scholium.* The *parabola, ellipse,* and *hyperbola,* will be more specially the subjects of this book.

### PROPOSITION I. THEOREM.

*If a right cone* BEG *be cut by a plane* A*p o, which is parallel to a plane touching the cone along the slant side* BE, *the section* A *p o is a parabola.*



Let BEG be that position of the generating triangle which is perpendicular to the cutting plane A*p o*; A *n* their common section, which is parallel to BE. Then, since the plane BEG passes through the axis, it is perpendicular to the base E*o*G, and to every circular section CPD parallel to the base; it is also perpendicular to A*p o*. Hence, the common section PO of the planes A*p o*, CPD, is perpendicular to BEG and therefore to A*n* and CD.

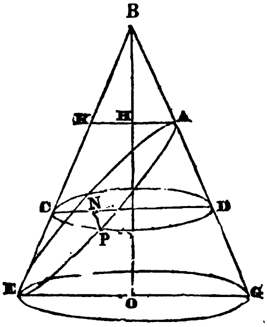
But, AN : ND : : BE : EG, which is a constant ratio; therefore, by the properties of the circle, AN : ND : : CN× ND : NO², since CN is equal and parallel to E*n*, and constant. Hence the curve is a parabola whose axis is A*n*.

*Cor.* If L be the latus rectum of the parabola *p*A*o*, L×AN =NP²=CN×ND.

$$L=CN\times\frac{ND}{AD}$$

### PROPOSITION II. THEOREM.

*If a cone* BEG *be cut by a plane* EAP *through both slant sides the section is an ellipse.*



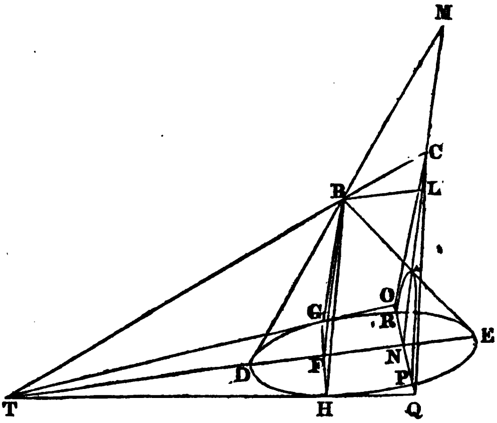
Let BEG be that position of the generating triangle which is perpendicular to the cutting plane: CPD any circular section. Draw AHK parallel to EG, and therefore bisected by the axis BO.

Then EN : CN : : EA : AK
NA : ND : : EA : EG;
∴ EN×NA : CN×ND (NP²)
: : EA² : EG×AK

which is the property of an ellipse, one of whose axes is EA and the other a mean proportional between EG and AK. (Conic Sections, Ellipse, Prop. XII.)

### PROPOSITION III. THEOREM.

*If a right cone* BED *be cut through one side* BE *by a plane* RAP *which being produced backwards, cuts the other side* DB *produced, the section is an hyperbola.*



Let DGEH be any circular section, BGH a triangular section through the vertex B of the cone parallel to the plane RAP.

Then, AN : EN : : BF : EF

NM : ND : : BF : FD

∴ AN×NM : EN×ND ($NP^2$) : : $BF^2$ : EF×FD ($FH^2$)

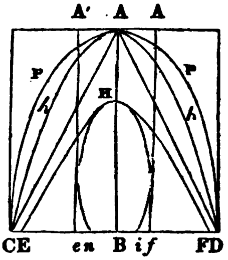
which is the property of an hyperbola, whose axis major is AM, and whose conjugate axis is to AM as FH to BF.

*Cor.* If GT, HT, be tangents to the circle at G, H; and planes passing through GT, HT, respectively, touch the cone along the lines BG, BH; also, if TB, the common section of the planes, meet AM in C, then the common section CO, CQ, of the plane RAP, extended to meet the tangent planes, are the asymptotes of the hyperbola.

Draw BL parallel to DE, meeting AM in L: then the axes of the hyperbola being in the proportion of BF to FH, the angle GBH, or the equal angle OCQ is the angle between the asymptotes.

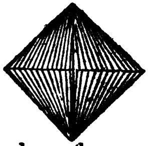
Now, by similar triangles ALB, BFE, and CLB, BFT; AL : CL : : TF : FE, and therefore AC : CL : : TE : FE. In like manner, by similar triangles MLB, BFD, and CLB, BFT; ML : CL : : TF : DF, and therefore CM : CL : : TD : DF. But, by the property of the circle, TE : FE : : TD : DF.

Therefore, CA=CM. Hence C is the centre of the hyperbola, and CO, CQ, are the asymptotes. (Conic Sections, Hyperbola, Proposition XII.)



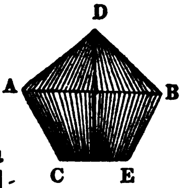
*Scholium* 1. Let EHF represent an hyperbola, and AC, AD the asymptotes; let the two branches HE and HF be brought into the position H*e* and H*f*, so that the asymptotes become A'*e* and A'*f*, or till they become parallel to each other, and the curve becomes a parabola; the parabola then, may be regarded as an hyperbola, whose asymptotes are parallel, and infinitely extended in each direction. If the extremities *e*, *f* of the curve are brought into the positions *n*, *i*, so as to incline toward the axis AB, so that the curve may again return into itself as its axis is extended, it then becomes an ellipse or portion of an ellipse. These different figures are the result of the position of the plane forming the section through the cone.

*Scholium* 2. A section of a polyedroid by a plane oblique to its axis may assume a combination of one, two, or three, of the varieties of surfaces. Thus, a section through a regular vertical quadredroid by a plane, parallel to a plane touching one of its sides, making an angle of 45° with the axis, toward the vertex, will consist of two parabolas on opposite sides of the same base, these will evidently be parabolas of equal type or similar parabolas, when the section passes through the centre of the solid, and as it approaches toward one of its sides, the parabolas become dissimilar.



But if the plane cuts the solid so as to make angles of more than 45° with the axis, the section will consist of the segments of two ellipses on opposite sides of the same ordinate, which ordinate is the line formed by the intersection of the cutting plane, with the plane through the centre of the solid perpendicular to its axis; and if the cutting plane passes through the centre of the solid the two segments will be of similar type, but they will vary as the section recedes from the centre toward either side.

If the plane should be passed through so as to make an angle with the axis less than 45°, then we should have hyperbolas on the same base, these would be similar hyperbolas when the plane should pass through the centre, but would become dissimilar as it recedes from the centre.



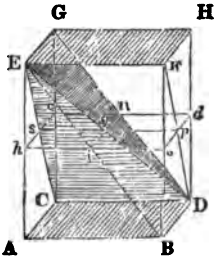
In like manner, if a plane should be passed through the pentadroid AB parallel to one of its sides AD, the section would consist of two species, viz., that part which passes through the portion ABD would be a parabola, that passing through the part ABEC would be a segment of an ellipse. But if the section should be passed so near to BE as to cut the base of the solid, the section would consist of a parabola, and a middle segment of an ellipse, and would have a rectilinear base. If the plane should be parallel to the side EB, then the section through the lower part of the solid would be a parabola, or a segment of a parabola, and the section through the upper part an hyperbola, and in fine, if the plane should make an angle with the axis less than that of the side EB, then the section through both parts of the solid would consist of the dissimilar hyperbolas or segments of hyperbolas.

Let a plane be passed through a polyedroid of a greater number of sides oblique to the axis, and the section may be so made as to consist of segments of all the varieties of the conic sections, and may also, under certain conditions, have one, two, three, and at most four rectilinear sides, but it can have no more than two rectilinear sides, except where the section is parallel to the axis.

### PROPOSITION IV. LEMMA.

*From a rectangular prism there may be taken two pyramids of equal base and altitude with the prism; when there will remain two other pyramids, each equal to half one of the lateral sides as a base, multiplied by one third of the distance of such base to the opposite side.*

For let the rectangular prism AH be divided into two parts, by passing the planes EGDB through the opposite edges, and the portion ABDCGE will consist of the pyramid, whose base is ABDC, and vertex E, plus the triangular pyramid, whose base may be taken as GEC and vertex D, or DCG may be regarded as the base, and E the vertex. And since the other portion of the prism is similar to this, it can be divided in a similar manner. Hence the whole prism consists of two equal pyramids erected on the upper and lower bases of the prism + two other pyramids erected on the lateral sides as



bases, and whose vertices will be in an angle formed by the intersection of its opposite side with one of the bases. Hence as in the proposition.

*Cor.* 1. If a prism have a square base and a section be made through the prism parallel to the base, the sections through the pyramids erected on the upper and lower bases will be squares, and the sections through the pyramids, whose bases are the lateral faces of the prism, will be rectangles whose factors are the sides of the squares composing the sections through the former pyramids.

That the sections through the pyramids, erected on the square bases will be squares, is sufficiently manifest; and since each of the other pyramids coincides with one of these along one of its slant sides, and with the other along another of its sides, it follows that the measure of its section through any parallel portion of the solid will be the rectangle of the edges of the section formed by the same plane through the two former pyramids. And since, if the section is taken in the middle, equidistant between the two bases, the sides of the sections through the pyramids with square bases are=half the sides of their bases, the sections at such place will each be=$\frac{1}{4}$ the section of the whole prism ; the sections of the two quadrangular pyramids will be 2 squares=2 quarters of the whole section; hence the sections through the two triangular pyramids are squares=to the former, and equal to each other, since the four pyramids fill the space, and constitute the whole prism.

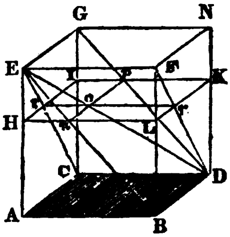
*Cor.* 2. Hence, also any section of a prism with a square base, made by a plane parallel to such base may be expressed by the square of a binomial, whose terms are composed of the sides of the sections through the quadrangular pyramids. Let $hi$ or $hs$, the side of the square forming a section through the pyramid erected on the lower base, be represented by $a$ ; and let $pd$ or $dn$, the side of the square forming a section through the pyramid erected on the upper base, be represented by $b$; then will $a+b$=the line $ho$, the side of the whole section $hodc$, and $a^2$ will represent the section through the pyramid ABDCE, $b^2$ will represent the section through the pyramid EFHGD, and $ab$ will represent a section through each of the other pyramids, hence $2ab$ will represent the sections through both, and $a^2+2ab+b^2$ will represent the whole section in whatever parallel the section is taken, always observing that the values of $a$ and $b$ vary according to the sides of the respective sections, which they represent.

*Cor.* 3. If there be a series of numbers in arithmetical progression, whose first term is 0 and last term $z$, and if the number of terms is infinite between these extremes, then the sum of the squares of the series of numbers will be equal to one third of the square of the last term drawn into the series. For if an infinite number of planes be passed through the pyramid, parallel to its base, and equidistant from each other through the sides of the sections made by those planes, they will be a series of numbers in arithmetical progression, whose first term is 0, and the last term may be called $z$. Now the sum of the sections drawn into their distance will represent the solidity of the pyramid, but the solidity of the pyramid is equal to one third of an equal series of $z$, the last term or base of the pyramid drawn into the distance or ratio ; or= to one third $z$ drawn into the series.

*Cor.* 4. If there be a series of numbers in arithmetical progression increasing from 0 up to $z$, drawn into a similar series, decreasing from $z$ down to 0, then will the sum of their products be equal to half the sum of the squares of one of the series. For the sections through the pyramid HDGE, as we have seen, represent the rectangles of the sides of the corresponding sections through the two pyramids, formed on the two bases, and the sides of these sections are evidently, in each pyramid, a series in arithmetical progression ; one increasing from 0 to $z$, while the other decreases from $z$ to 0. Moreover, the sections through the pyramid CDGE represent the solidity of that body, as the corresponding sections through the pyramid ABDCE, represents the solidity of that body. But the solidity of the pyramid CDGE, is equal to half the pyramid ABDCE, which as we have shown, may be represented by the sum of the squares of a series of arithmeticals, &c.

*Cor.* 5. If in the expression $a^2+2ab+b^2$, $a$ be made to pass successively through all the values from 0 up to $z$ ; and $b$ at the same time pass through all the changes from $z$ to 0 ; then the sum of all the $a^2$ will be equal to the sum of all the $2ab$, = the sum of all the $b^2$.

*Cor.* 6. If $a$ in the expression above, be made to increase from $h$ successively to $z$, and at the same time $b$ be made to pass through all the values from $z$ to $h$, then the series represented by $2ab$, will be a mean proportional between those represented by $a^2$ and $b^2$, since in this condition the two pyramids represented by the series of $a^2$ and $b^2$, would be such as pertain to pyramids inscribed in a frustum of a pyramid, and erected on the two bases, and the portions represented by the series of $2ab$, would be such as pertain to the pyramids erected on the lateral sides as bases, and since the sum of these pyramids is equal to either of the others, when the two bases are equal, and because $ab$ is a mean proportional between $a^2$ and $b^2$, it follows that the series represented by $2ab$ is a mean proportional between those represented by $a^2$ and $b^2$, which agrees with the property of the frustum of a pyramid found in the Elements of Geometry.



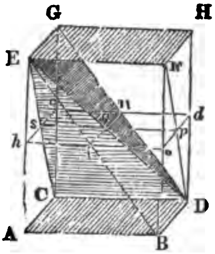
*Cor.* 7. Let any plane KIHL be passed through a prism parallel to its base, cutting the pyramids ABDCE, EFNGD, DCGE, and EFBD in the sections H*not* K*rop*, I*pot*, and L*ron*. Then since I*t* is equal to L*n* or *ro*, and H*n*=*on*, it follows that as the section H*not* : I*pot* : : I*pot* : K*rop* : : H*not* : L*nor*.

And by addition : : (K*not*+I*pot*)=HI*pn* : KL*np*.
Hence H*not* : HI*pn* : : HI*pn* : KIHL.

*Cor.* 8. Hence, of two quantities, the square of the first is to the rectangle of the first and second, as the rectangle of the first and second to the square of the second, and as the sum of the first and second × by the first, is to the sum of the first and second × by the second. Also as the square of the first is to to the sum of the first and second, × by the first, so is the sum of the first and second × by the first, to the first and second × by the first and second. Thus let $a$ and $b$ be two given quantities, then will $a^2 : ab :: ab : b^2, :: a^2+ab : ab+b^2$,
and $a^2 : a^2+ab :: a^2+ab : a^2+2ab+b^2$.

*Scholium.* It has been shown (Prop. XXXIV. B II. *El. Sol. Geom.*) that the solidity of a prismoid is equal to the product of the sum of the areas of the two ends+4 times AD and EH a middle section, equidistant between them $\times \frac{1}{6}$ of the altitude; we may easily infer that this is also true of all prisms and pyramids and pyramidal frusta. Then since the prism AH is equal to the sum of the areas of the two bases+4 times the middle section *hodc* $\times \frac{1}{6}$ of the altitude AE, it follows that whatever parts make up this product are equal to the whole prism.

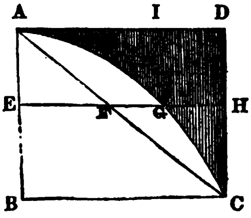
First, then let us take the pyramid ABDCE = (the base ABDC+4 *higs*, *a* middle section)$\times \frac{1}{6}$ AE, and also the pyramid



EFHGD=(the base EFHG$+4gpdn$) $\times \frac{1}{6}$ AD; if from the expression for the whole prism we take the expressions for the two pyramids erected on the bases we have, $4sgnc \times \frac{1}{6}AE + 4iopg \times \frac{1}{6}AE$, that is the two remaining portions, are equal to four times their middle sections, multiplied by $\frac{1}{6}$ of their altitude. Hence we may infer that if any portions of either of the pyramids into which the prism has been conceived to be divided, be cut off by a plane, or planes parallel to the base, the portions so cut off will be equal to the product of the sum of their two bases, + four times a middle section between them, × by $\frac{1}{6}$ of the altitude of such portions. It may be observed that since a regular pyramid has but one base, its solidity is hence equal to the sum of this base + four times a section, midway between the base and vertice$\times\frac{1}{6}$ the altitude; and also in the two pyramids, whose bases are on the lateral sides of the prism, since they have no bases parallel to the middle section, their solidities for that reason are equal to four times their middle sections × $\frac{1}{6}$ of their height. It may be further observed that each portion *sgnc*, GE into which these pyramids are divided by the plane *hodc* is a wedge, whose base is the section forming the division.

### PROPOSITION V. THEOREM.

*If from the extremity of an ordinate to the axis of a parabola and perpendicular thereto, a line be drawn meeting another line, drawn from the vertex perpendicular to the axis, forming with the ordinate and abscissa a rectangle* ABCD, *and if a diagonal be drawn from the vertex* A *to the extremity of the ordinate, forming a right angled triangle* ABC *of the same base* BC *and altitude* AB, *then any line or ordinate drawn from the axis across the triangle, the parabolic area, and the rectangle, parallel to the ordinate, will be cut in continued proportion by the sides of those figures.*



That is, EF : EG : : EG : EH.

Or EF, EG, EH are in continued proportion.

For by (Prop. VII. Cor. of Parabola) AB : AE : : $BC^2 : EG^2$.

And since AB : AE : : BC : EF

hence EF : BC : : $EG^2 : BC^2$.

Or, EF : EH : : $EG^2 : EH^2$,

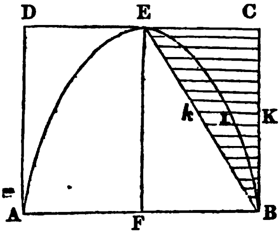
therefore (Prop. XXIV. B. I. *El. Geom.*) EF, EG, EH, are proportionals, or EF : EG : : EG : EH.

*Cor.* Let any number of ordinates GI be drawn across the exterior parabolic space parallel to the axis, and since $EG^2 : BC^2 :: AE : AB$, by equality we have $AI^2 : AD^2 :: IG : DC$, and since this is true from whatever position on the line AD, the line IG may be drawn, it follows that any line IG drawn across the exterior parabolic space is proportional to the square of the distance of such line from the vertice A.

### PROPOSITION VI. THEOREM.

*If a parallelogram be circumscribed about a parabola, the area of the space exterior to the parabola will be equal to one third of the parallelogram, and the interior space will be two thirds of the parallelogram.*

Let ABCD be a parallelogram circumscribed about the parabola AEB, and the space ECB, EDA exterior to the curve will be = $\frac{1}{3}$ the parallelogram ABCD, and the space AEBA within the curve will be = $\frac{2}{3}$ ABCD.



From the vertice E draw EF the axis of the parabola, which will divide the parabola and rectangle into two equal parts; it is to be proved that the exterior space ECB is = $\frac{1}{3}$ EFBC.

Draw the diagonal EB, and let an indefinite number of equidistant ordinates K*k* be drawn across the triangle EBC, and exterior parabolic space EIBC, and those ordinates will represent their respective surfaces in the relation of their magnitudes respectively. Then since the distances of those ordinates on the line BC, estimated from the vertice B, are a series of numbers in arithmetical progression, the ordinates K*k* drawn across the triangle, are also a series in arithmetical progression, for K*k* is proportional to BK in whatever position the ordinate K*k* is drawn; and since it has been shown (Prop. V. Cor.,) that any ordinate IK drawn across the exterior parabolic space is proportional to the square of its distance BK from the vertex B, it follows that the ordinate is proportional also to the square of K*k*, the corresponding ordinate drawn across the triangle. But the ordinate EC or base of the triangle is equal and identical with the base of the parabolic exterior space; hence each of the ordinates IK terminated by the parabolic curve is equal to the square of K*k*, the corresponding ordinate drawn across the triangle.

Hence all the ordinates K*k* may be represented by a series of numbers in arithmetical progression, whose first term beginning at B, is infinitely small or 0, and last term EC, and all the IK will be a similar series of squares of those arithmeticals ; but (Prop. IV. Cor. 3,) the sum of an infinite series of the squares of a series of numbers in arithmetical progression, whose first term is 0, and last term *z*, or EC, is equal to $\frac{1}{3}$ EC×CB, which is $\frac{1}{3}$ of the rectangle EFBC.

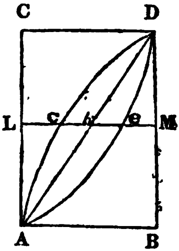
*Cor.* 1. Hence the parabolic segment EIBE, cut off by the diagonal, is equal to $\frac{1}{6}$ of the rectangle, = $\frac{1}{4}$ of the interior parabolic space EIBF, = $\frac{1}{2}$ the exterior parabolic space EIBC.

*Scholium.* It has been shown in the argument above that the ordinates drawn across the exterior parabolic space are severally = the squares of the same ordinates drawn across the triangle EBC, this is true where EC=1, and for all other values of EC, they are in the same proportion.

### PROPOSITION VII. THEOREM.

*The area of the exterior parabolic space included in its circumscribing rectangle is equal to the sum of its base + four times the line or ordinate drawn parallel to the base, and equidistant from the base to the vertex, multiplied by $\frac{1}{6}$ of the altitude.*

*And the area of the interior space of a parabola is equal to the sum of its base, + four times the ordinate equidistant from this base to the vertex, multiplied by one sixth of the altitude.*



Let ABDC be a rectangle circumscribing the semi-parabola A*c*DB, and let L*c* be an ordinate drawn cross the exterior space ACD*c*A equidistant from A to C, and let M be an ordinate drawn across the interior space A*c*DB in a similar manner, then will the area ACD*c*A = (CD+4L*c*) $\frac{1}{6}$ AC. And the area A*c*DB = (AB+4*c*M) $\frac{1}{6}$ BD. For draw the diagonal AD, then, since we have shown in the argument to Prop. VI, that $Lc=Lh^2$, when $CD=CD^2$, and since $Lh=\frac{1}{2}$ CD: then if we call L*h*, *a*, and CD, 2*a*, we shall have $Lc = a^2$, and $CD = 4a^2$ that is $Lc = \frac{1}{4}$ CD or CD=4L*c*.

Hence CD+4L*c*=2CD, then by the proposition 2CD×$\frac{1}{6}$AC = CD× $\frac{1}{3}$ AC = the surface A*c*CDA, as was also found in proposition VI.

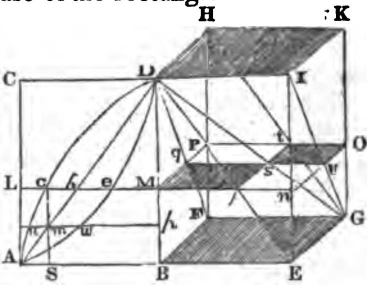
Again since LM = CD or AB, and Lc $=\frac{1}{4}$ CD, cM= $\frac{3}{4}$ AB. Then by the proposition (4cM+AB) × $\frac{1}{6}$ BD=4AB×$\frac{1}{6}$BD= $\frac{2}{3}$ (AB×BD)=the area, as found in the preceding proposition.

*Cor.* 1. Hence the parabolic segment AD*c*A is equal to four times the middle ordinate *ch* × $\frac{1}{6}$ the altitude AB. For since L*h* = $\frac{1}{2}$ LM, *ch*=L*c* = $\frac{1}{4}$ AB, and $\frac{1}{4}$ AB×4=AB, and AB×$\frac{1}{6}$ BD = the surface which agrees with Cor. 1 Prop. VI.

*Cor.* 2. Also if another similar parabolic curve D*e*A be described on the opposite side of the diagonal, the space between the two curves would in like manner be found, = 4 times the ordinate *ce* × $\frac{1}{6}$ of the altitude BD.

*Cor.* 3. The same may be shown in reference to the area of the triangles ACD, and also of the rectangle ABDC, viz., their areas are equal to the sum of their upper and lower bases, + 4 times a middle ordinate, × $\frac{1}{6}$ of their respective altitudes.

*Scholium.* Let there be a rectangle CB, and a prism BK of the same altitude, and let the base of the prism be a square, one of whose sides is = AB the base of the rectangle.



Let the prism be divided into the pyramids BEGFD, DIKHG, DHFG, EGID, as in Prop. IV. Also let the parabolic curves A*c*D A*e*D and the diagonal AD be drawn as in the proposition above. Then if a plane M*n*OP be passed through the prism parallel to its base, and if an ordinate LM be made to pass through the rectangle in the same plane, the plane will cut the pyramidal portions of the prism in the same relation, as the ordinate cuts the parabolic portions of the rectangle; viz., the section M*fsq* through the pyramid BEGCD will be to the section M*n*OP through the prism, as the ordinate *e*M through the exterior parabolic space ABD*e*A, to the ordinate LM through the rectangle ABDC; and the section *sv*O*t* through the pyramid DIKHG is to the whole section through the prism, as the ordinate L*c* through the exterior space ACD*c*A, to the whole ordinate LM through the rectangle, and so for the sections through the other pyramids, and ordinates through the interior parabolic segments, and this is true in whatever parallel position the plane forming the section and ordinate is drawn.

For it has been shown (Prop. VI. Sch.) that the ordinate *e*M is equal to $hM^2$; the section M*fsq* is evidently=the square of M*f*; but M*f* is = *h*M, hence M*fsq* $=hM^2$; therefore M*fsq* will be expressed by the same numbers as $e\mathrm{M}$, and this is true in whatever parallel position the plane is passed.

The same may also be shown in reference to the sections $sv\mathrm{O}t$ through the pyramid DIKHG, and the ordinates $\mathrm{L}c$ and $\mathrm{L}h$, viz., the section $sv\mathrm{O}t=\mathrm{O}t^2=fn^2=\mathrm{L}h^2$ and $\mathrm{L}c=\mathrm{L}h^2$. And since every ordinate LM through the rectangle, has the same relation to the rectangle, as every section through the prism, has to the prism, and each section $\mathrm{M}fsq$, $sv\mathrm{O}t$, has the same relation to the solid as each ordinate $\mathrm{L}c$, $c\mathrm{M}$, has to the surface; it follows that the remaining sections, $fnvs$, $st\mathrm{P}q$, have the same relations to their respective figures, and since the sections $\mathrm{M}fsq$ $sv\mathrm{O}t$ are expressed by the same powers of the same factors as the ordinates $\mathrm{L}c$ and $e\mathrm{M}$ are expressed, $ch$ and $he$ must also be expressed by the same terms as the sections $fnvs$, $st\mathrm{P}q$, viz., $he=\mathrm{L}h\times h\mathrm{M}$, $ch$ also$=\mathrm{L}h\times h\mathrm{M}$, and $ce=2\ (\mathrm{L}h\times h\mathrm{M}.)$ Hence if $\mathrm{L}h=a$ and $h\mathrm{M}=b$ then will $\mathrm{L}c=a^2$, $ce=2ab$ and $c\mathrm{M}=b^2$, and the whole ordinate may also be expressed by $a^2+2ab+b^2$, which is the square of a binomial, the same as has been shown, (Prop. IV. Cor. 2) in reference to the pyramids composing a prism, and the expression is true in whatever parallel position the ordinate is drawn. It follows therefore that the ordinates drawn across any of the parabolic portions of the rectangle, have the same power of determining the area of the exterior or interior parabolic spaces, as the corresponding sections through the pyramidal portions of the prism have, in determining the solidities of those pyramids. But we have shown (Prop. IV. Sch.) that each portion of the prism divided as above, however selected or compounded, is equal to the sum of its two bases+4 times a middle section $\times\ \frac{1}{6}$ of the altitude; hence also the area of any parabolic portion or portions of the rectangle is equal to the sum of its bases + 4 times a middle ordinate $\times\ \frac{1}{6}$ the altitude.

*Cor.* 4. The parabolic area included between two parallel ordinates is=the product of the sum of the two ordinates + 4 times an ordinate equidistant between them $\times\ \frac{1}{6}$ of the altitude of the parabolic segment. For let $\mathrm{ABM}c$ be a segment of the parabola included between the ordinates $c\mathrm{M}$, AB, the axis MB and the curve $\mathrm{A}c$; draw $c\mathrm{S}$ perpendicular to AB, and the parabolic segment $\mathrm{S}c\mathrm{A}$ will be equal to $(\mathrm{AS}+4\ um)\times\frac{1}{6}\ \mathrm{S}c$, and the rectangle $\mathrm{SBM}c$ is evidently$=(\mathrm{S}b+c\mathrm{M}+4mp)\ \frac{1}{6}\ c\mathrm{S}$; hence the whole space $\mathrm{ABM}c=(\mathrm{AS}+\mathrm{SB}+c\mathrm{M}+4um+4mp,)\times\frac{1}{6}\mathrm{S}c=(\mathrm{AB}+c\mathrm{M}+4up)\ \frac{1}{6}\ \mathrm{S}c$.

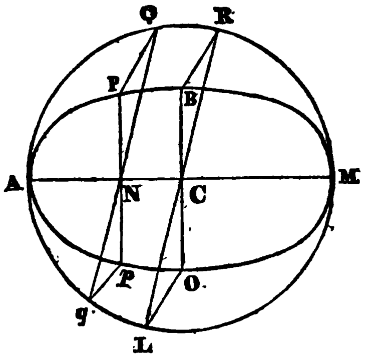
*Cor.* 5. In like manner it may be shown that the space $e\mathrm{MBA}$ is$=(\mathrm{AB}+e\mathrm{M}+4wp)\ \frac{1}{6}\mathrm{MB}$, and hence also that any parabolic portion $\mathrm{A}uchm\mathrm{A}$ or $uchm$ may be determined in the same manner.

*Cor.* 6. Hence generally, if a space be terminated by a parabolic curve on one side, and either a parabolic curve or a right line on the other, the space included between two parallel lines, drawn across the figure cutting those sides, will be = to the product of the sum of those lines + 4 times another line drawn across equidistant between the two multiplied by $\frac{1}{6}$ the perpendicular distance of the two parallel lines.

### PROPOSITION VIII. THEOREM.

*If a circle be cut by a plane through its axis, and perpendiculars be drawn from every point in the circumference to the plane, the orthographic projection of the circle so drawn will be an ellipse.*

Let the circle ARML be inclined to the plane of this paper in such a manner, that the semicircle ARM may be above the paper, and the semicircle ALM below it, and let AM be the common intersection of the two planes. Let the semicircle ARM be projected downwards upon the plane of the paper, by drawing perpendiculars QP, RB, from each point of the circle, and let the semicircle A L M be projected upwards, by drawing the perpendiculars $qp$, LO, &c.; then the curve ABMO, marked out by this projection, will be an ellipse. For draw QN, RC, at right angles to AM, and join PN, BC; then the angles QNP, RCB, will measure the inclination of the planes, and PN, BC will be perpendicular to their common intersection AM. Now QN : : PN : *rad.* : cos. QNP, and RC : BC : : *rad.* : cos. (RCB = QNP); ∴ QN : PN : : RC or AC : BC, which agree with the properties of the ellipse (Prop. XII. Cor. *Conic Sec.*) In a similar manner it may be shown that the semicircle ALM is projected into a semi-ellipse AOM; and thus the whole circle ARML is projected into an ellipse ABMO, whose major axis is AM.



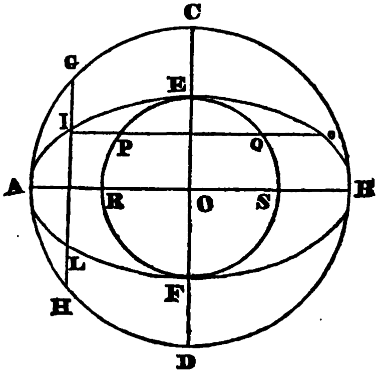
*Scholium.* This proposition is manifestly true, when the plane of the projection does not cut the circle, or cuts it unequally.

*Cor.* Hence any section of a cylinder by a plane not perpendicular or parallel to its axis, is an ellipse, or portion of an ellipse.

### PROPOSITION IX. THEOREM.

*If on the major axis of an ellipse a circle be described, the area of the ellipse will be to that of the circle, as the minor axis of the ellipse to the major axis.*

Let ACBD be a circle described on the major axis AB of the ellipse AEBF, then will the area of the ellipse be to that of the circle as EF to EB.



For it has been shown (Prop. XII. Cor. *Conic Sec.*) that any ordinate IL of the axis AB of the ellipse is to the corresponding ordinate GH of the circle, as EF to CD or AB, as the minor axis to the major axis. And since this is true in whatever position the ordinates to this axis are drawn; and because if we suppose an indefinite number of equidistant ordinates to be drawn across the ellipse and circle, the sum of the ordinates in each will represent those figures in the relation of their areas, the sum of the ordinates in the ellipse will be to the sum of those in the circle, as EF to CD, and hence the area of the ellipse will be to that of the circle in the same ratio.

*Scholium.* As the area of the ellipse bears this given ratio to that of its circumscribing circle, the quadrature of the ellipse must therefore depend on the quadrature of the circle. If $\pi\times(AO\,.\,AO$, or $CO$,) or $\pi AO^2$=the area of the circle, then $\pi\,.\,AO\,.\,EO$=the area of the ellipse. Hence the area of the ellipse is found by multiplying the rectangle under its semi-axis by the constant number $\pi$, the ratio of the diameter to the circumference of a circle, which in the Elements of Geotry we have found, developed to a certain order of decimals to be 3,1415926. But 3,1416 may be regarded as the value of $\pi$ which is a convenient number to use, and is sufficient where extreme accuracy is not required.

From this it also appears, that the area of an ellipse is equal to the area of a circle whose radius is a mean proportional between its semi-axis; for the area of that circle is $= \pi R^2 = \pi \times$ the square of $\sqrt{\overline{AO \times EO}} = \pi . AO . EO$.

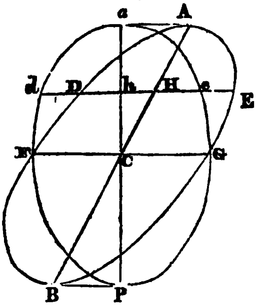
*Cor.* 1. The area of an ellipse has the same ratio to the area of its circumscribed parallogram as the area of a circle has to its circumscribed square. For the area of the parallelogram circumscribing the ellipse is $= 4\ AO \times EO$; hence the area of the ellipse : area of the parallelogram : : $\pi$ . AO . EO : 4AO . EO : : 3,1416 : 4 : : 7854 : 1.

*Cor.* 2. If a circle be described on the minor axis EF of the ellipse, then any ordinate PQ of the circle will be to the corresponding ordinate I*e* of the ellipse as EF to AB. Hence also the area of the ellipse is to that of the circle described on its minor axis, as the major axis of the ellipse to its minor axis.

*Cor.* 3. Hence any segment of the ellipse cut off by ordinates, either of the major or minor axis is to a similar segment of the circle, described on such axis, as its conjugate is to such axis. Thus the segment IAL of the ellipse AEBF is to the segment GAH of the circle described on the axis AB, as the ordinate IL is to the ordinate GH or as the minor axis EF of the ellipse is to the axis CD of the circle, or to AB the transverse axis of the ellipse. And the segment I*e*E is to the segment PQE as the axis AB as conjugate to EF of the ellipse, to the axis RS of the inscribed circle, or EF of the ellipse.

*Cor.* 4. Since the area of a circle BESF is equal to its circumference multiplied by half the radius EO, and since the area of the ellipse AEBF is = to that product increased in the ratio of EO to AO or CO, it follows that the area of the ellipse is equal to the circumference of its inscribed circle multiplied by $\frac{1}{2}$ the radius of its circumscribed circle, and that it is also equal to the circumference of the circumscribed circle multiplied by $\frac{1}{2}$ the radius of the inscribed circle.

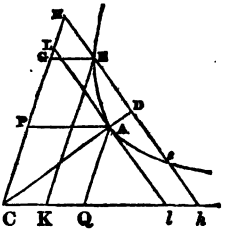
*Cor.* 5. As a circle is to the square of its diameter so is any ellipse to the rectangle of its two axes, or the rectangle of any two conjugate diameters drawn into the sine of their included angle. Any two like segments or zones of the circle and ellipse, are also in the same proportion.



*Cor.* 6. Hence we have the following construction; let ADE be an oblique segment of the ellipse AFBGA, cut off by an ordinate to the diameter AB, FG being the conjugate. Through the centre C draw *a*P perpendicular to FG meeting A*a* and BP, in *a* and P, both parallel to FG; then about the axes *a* P, FG, describe the ellipse *a*FPG*a*, meeting the ordinate produced in *e* and *d*. Then will the right elliptical segment *dae*, be equal to the oblique segment DAE, and so will the whole ellipse *a*FPG*a*=AFBGA; moreover their corresponding ordinates *de* DE parallel to the common diameter FG, are everywhere equal, as are the like parts or zones contained between any two of such ordinates. And the same may be said of all ellipses of the same diameter FG, contained between the parallels *a*A, BP infinitely produced.

### PROPOSITION X. THEOREM.

*All the parallelograms are equal, which are formed between the asymptotes and curve, by lines drawn parallel to the a symptotes of an hyperbola.*



That is, the lines GE, EK, AP, AQ, being parallel to the asymptotes CH, C*h*, then the parallelogram CGEK = parallelogram CPAQ.

For, let A be the vertex of the curve, or extremity of the semi-transverse axis AC, perpendicular to which draw AL or A*l*, which will be equal to the semi-conjugate, by definition XIX. *Hyp*. Also, draw HED*eh* parallel to I*l*.

Then, $CA^2 : AL^2 :: CD^2 - CA^2 : DE^2$,
and by parallels, $CA^2 : AL^2 :: CD^2 : DH^2$;
therefore, by substract. $CA^2 : AL^2 :: CA^2 : DH^2 - DE^2$
or rectangle $HE \cdot Eh$;
consequently, the square $AL^2 =$ the rectangle $HE \cdot Eh$.

But, by similar trian. $PA : AL :: GE : EH$,
and, by the same, $QA : Al :: EK : Eh$;
therefore, by comp. $PA \cdot AQ : AL^2 :: GE \cdot EK : HE \cdot Eh$;
and, because $AL^2 = HE \cdot Eh$, therefore $PA \cdot AQ = GE \cdot EK$.

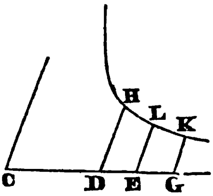
But the parallelograms CGEK, CPAQ, being equiangular, are as the rectangles GE.EK and PA . AQ.

And therefore the parallelogram GK = the parallelogram PQ.

That is, all the inscribed parallelograms are equal to one another.

*Cor.* 1. Because the rectangle GEK or CGE is constant, therefore GE is reciprocally as CG, or CG : CP : : PA : GE. And hence the asymptote continually approaches towards the curve, but never meets it; for GE decreases continually as CG increases; and it is always of *some* magnitude, except when CG is supposed to be infinitely great, for then GE is infinitely small or nothing. So that the asymptote CG may be considered as a tangent to the curve at a point infinitely distant from C.

*Cor.* 2. If the abscissas CD, CE, CG, &c., taken on the one asymptote, be in geometrical progression increasing; then shall the ordinates DH, EI, GK, &c., parallel to the other asymptote, be a decreasing geometrical progression, having the same ratio. For, all the rectangles CDH, CEI, CGK, &c., being equal, the ordinates DH, EI, GK, &c., are reciprocally as the abscissas CD, CE, CG, &c., which are geometricals. And the reciprocals of geometricals are also geometricals, and in the same ratio, but decreasing, or in converse order.



### PROPOSITION XI. THEOREM.

*The three following spaces, between the asymptotes and the curve, are equal; namely, the sector or trilinear space contained by an arc of the curve and two radii, or lines drawn from its extremities to the centre; and each of the two quadrilaterals, contained by the said arc, and two lines drawn from its extremities parallel to one asymptote, and the intercepted part of the other asymptote.*

That is,

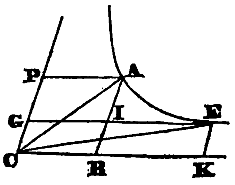
The sector CAE = PAEG = BAEK, all standing on the same arc AE

For, as has been already shown, CPAB = CGEK;

Subtract the common space CGIB,

So shall the parallel PI = the parallel IK;

To each add the trilineal IAE,



Then is the quadril. PAEG=BAEK.

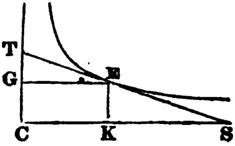
Again, from the quadrilateral CAEK, take the equal triangle CAB, CEK, and there remains the sector CAE=BAEK.

Therefore, CAE=BAEK=PAEG.

### PROPOSITION XII. THEOREM.

*Every inscribed triangle, formed by any tangent and the two intercepted parts of the asymptotes, is equal to a constant quantity ; namely double the inscribed parallelogram.*

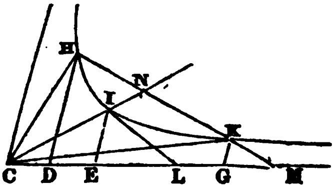
That is, the triangle CTS=2 parallelogram GK.



For, since the tangent TS is bisected by the point of contact E, and EK is parallel to TC, and GE to CK ; therefore CK, KS, GE are all equal, as are also CG, GT, KE. Consequently the triangle GTE = the triangle KES, and each equal to half the constant inscribed parallelogram CK. And therefore the whole triangle CTS, which is composed of the two smaller triangles and the parallelogram, is equal to double the constant inscribed parallogram GK.

### PROPOSITION XIII. THEOREM.

*If from the point of contact of any tangent, and the two intersections of the curve with a line parallel to the tangent, three parallel lines be drawn in any direction, and terminated by either asymptote ; those three lines shall be in continued proportion.*



That is, if HKM and the tangent IL be parallel, then are the parallels DH, EI, GK in continued proportion.

For, by the parallels, EI : IL : : DH : HM ;
and, the same EI : IL : : GK : KM ;
therefore by compos. $EI^2 : IL^2 : : DH . GK : HMK$;
but, the rect. $HMK = IL^2$ ;
and therefore the rectangle $DH . GK = EI^2$,
or DH : EI : : EI : GK.

### PROPOSITION XIV. THEOREM.

*Draw the semi-diameters* CH, CIN, CK ;
(see last diagram.)
*Then shall the sector* CHI = *the sector* CIK,

For, because HK and all its parallels are bisected by CIN,
therefore the triangle CNH=trian. CNK,
and the segment INH=seg. INK ;
consequently the sector, CIH=sec. CIK.

*Cor.* If the geometricals DH, EI, GK be parallel to the other asymptote, the spaces DHIE, EIKG will be equal ; for they are equal to the equal sectors CHI, CIK.

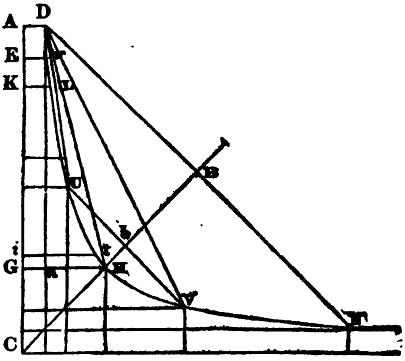
So that by taking any geometricals CD, CE, CG, &c., and drawing DH, EI, GK, &c., parallel to the other asymptote, as also the radii CH, CI, CK ;
then the sectors CHI, CIK, &c.
or the spaces DHIE, EIKG, &c.
will be all equal among themselves.
Or the sectors CHI, CHK, &c.
or the spaces DHIE, DHK, &c.
will be in arithmetical progression.

And therefore these sectors, or spaces, will be analogous to the logarithms of the lines or bases CD, CE, CG, &c. ; namely CHI or DHIE the log. of the ratio of CD to CE, or of CE to CG, &c. ; or of EI to DH, or of CK to EI, &c. ; and CHK or DHKG the log. of the ratio of CD to CG, &c. or of CK to DH, &c.

*Scholium.* If the common logarithms are multiplied by 2,302585093 their products will be the hyperbolic logarithms.

### PROPOSITION XV. THEOREM.

*If any number of ordinates* AD, EF, *&c. to* GH, *between the hyperbola and the asymptote, are taken in geometrical progression increasing, then will their distances* AE, *&c., on the asymptote be a series of geometricals decreasing, and* AG *will represent the sum of the decreasing series.*



That the geometrical ordinates and their distances on the asymptote are reciprocal geometricals is manifest from (Prop. XIV Cor.,) and since the distances AE, EK &c., on the asymptote, intercepted by those ordinates compose the whole series of decreasing geometricals, the sum of that series will be represented by AG, the part of the asymptote taken up by the series.

*Cor.* 1. The last term G*i*, is=AE×EF÷GH. For the area ADFE = the area HG*it*, and if the distances on the asymptote are taken indefinitely small, *ti* may be regarded as =GH, and AD may be regarded as = EF, and if the hyperbola is equilateral, then the space ADFE will be=AE×AD=G*i*×GH converting this equation into a proportion,

we have AE : G*i* : : GH : AD,
or GH : AD : : AE : G*i*.
Hence, G*i*=AE×(AD or EF)÷GH

*Scholium.* It is evident that this proposition is true, however far the series may extend on the asymptote CA.

*Cor.* 2. If the series commence at any point A on the asymptotes, and decrease to infinity, CA will be the sum of the series ; hence the sum of an infinite series of decreasing geometricals is a finite number.

### PROPOSITION XVI. THEOREM.

*If any number of portions* AE, EK, *&c., to* G, (see diagram above) *are the reciprocals of a series of ordinates* AD, EF, KL, *&c., to* GH, *in geometrical progression ; then will the area or space intercepted by* AD *and* GH, *between the curve and asymptote be = the space* ADFE, *multiplied by the number of terms of the series.*

For since the space intercepted by each two consecutive ordinates of the series is equal to ADFE, it follows that the whole space intercepted by all the ordinates, or the space ADHG is = ADFE × by the number of terms in the series.

*Cor.* Hence the area of the interior hyperbolic space HDB may be found, if the exterior space is known. For the interior space is equal the triangle HBD + triangle H$n$D + the rectangle $n$DAG—the exterior hyperbolic space GHDA.

*Scholium.* Let AD=$b$, EF = $e$, GH = $g$, and AE the first term of a geometrical series=$a$ ; the last term = $z$ ; and AG the sum = $s$. Then we shall have $a$ the first term, and $z = \frac{ae}{g}$ the last term, and $s$ the sum of all the terms of a geometrical series to find the ratio and number of terms.

Then by geometrical progression will

$$s = a + ar + ar^2 + ar^3 \quad - \; - \; - \quad + ar^{n-1}$$

multiply by $r$

$$sr = ar + ar^2 + ar^3 \; - \; - \; - \; - \; - \; - \; - \quad + ar^{n-1} + ar^n$$

Subtracting the first from the second,

$$s\,(r-1) = ar^n - a$$

$$\therefore s = \frac{a\,(r^n - 1)}{r-1}$$

And since,

$$z = ar^{n-1}$$

$$s = \frac{rz - a}{r-1}$$

multiplying by $r-1$, $sr - s = rz - a$

By transposing and dividing; $r = \frac{s-a}{s-z}$

and in the equation $z = ar^{n-1}$

multiplying by $r$ and dividing by $a$ we have

$$r^n = \frac{zr}{a} \quad \text{which equation}$$

is irreducible by the ordinary methods when $n$ is the unknown quantity, it being an exponential equation, and can be solved only by logarithms, which see. (Chap. V Trigonometry) ; but if we proceed to raise $r$ to some power, whose value is $\frac{zr}{a}$ the index of that power will be the value of $n$ ; this can always be done to approximate exactness.

Let the series of geometricals commence at the vertex H, making G$i$ the first term = $a$, and let $n$ be a given number, then assuming any value to $s$, $z$, $r$, each of the others may be found by the formula above, and hence the area of the exterior and interior hyperbolic spaces may be found, as well as that of any section or segment.

*Scholium* 2. If the portion of the hyperbolic curve is small, the ordinates AD, EF, &c., and consequently AE, EK, &c., will be very nearly in arithmetical progression as may readily be seen by the rapid manner in which the ordinates DN, DV, DH, DU, &c., approximate toward the curve as the hyperbolic arc is decreased, hence the hyperbolic area may be approximately obtained by first dividing it into portions having known ratios to each other, and finding the value of one when the whole will become known.

### PROPOSITION XVII. PROBLEM.

*Let it be required to find how far from* G *on the asymptote* AC *an ordinate must be drawn parallel to* GH *in order to intercept with* GH *an area* =A

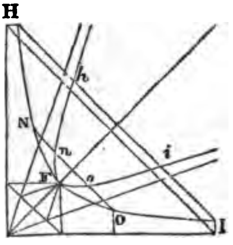
Take any distance G*i* (see fig. to Prop. XV,) and since AG : *it* : : GH : G*i* we have the two parallel sides GH, *it*, and their distance G*i* of the quadrilateral GH*ti* to find its area, which call *f*, then $\frac{A}{f}=n$, let *a* represent the distance, G*i* the first term of the geometric series, and GH ÷ *ti* = *r*, the ratio.

Whence we have $s=a\left(\frac{r^n-1}{r-1}\right)$ = the distance from G on the asymptote, that the ordinate must be drawn to intercept with GH, the area A.

*Scholium.* Since the spaces intercepted by the ordinate GH or AD, with any base IN, are the hyperbolic logarithms of the ratios of AD to GH (P. XIII. Cor.) hence the area may be found by taking the logarithms of the ratio, and multiplying it into the base.

### PROPOSITION XVIII. THEOREM.

*If all the ordinates to the diameter of an equilateral hyperbola be reduced in any ratio r, then the area of the hyperbola will be reduced in the same ratio.*



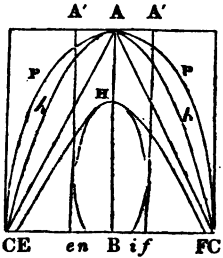
Let all the ordinates HI, NO, &c., of the equilateral hyperbola HFI be reduced to *hi*, *no*, &c., forming a hyperbola *h*F*i*, then will the area of the equilateral hyperbola HFI be to the area of the new hyperbola *h*F*i* as HI to *hi*, or as NO to *no*. For, if an indefinite number of equidistant parallel ordinates be drawn across the two hyperbolas, the areas of each will be as the sum of the lengths of all the ordinates in each, but by hypothesis, each of the ordinates HI, NO, are to their corresponding ordinates *ni*, *no* in the given ratio *r*; hence, their sums in each, must also be in the same ratio.

*Cor.* 1. Since the areas of similar figures, are as the squares of the lines similarly drawn in each; similar segments of similar hyperbolas, are as the squares of their axes, or of their like diameters.

*Cor.* 2. Hyperbolas of the same transverse axis and abscissa are to each other as their conjugate axes; but if their bases, or ordinates, and conjugate axes be the same, they will be as their transverse axes; and generally, hyperbolas having the same abscissa, are as the rectangle of their major and minor axes; and, consequently, having the quadrature of any one hyperbola, we may from it find that of any other. Thus knowing the area answering to any abscissa in any one hyperbola, we can find a similar abscissa in the other; then as the rectangle of the axes of the squared hyperbola, is to the rectangle of the proposed one, so is the area of the former, to that of the latter.

*Scholium.* 1. It may be shown that a parabola is always greater than a triangle of equal base and altitude, and that the hyperbola is always between the two; since from the nature of the section through the cone by which the hyperbola is produced, it may vary from a vertical section through the cone forming a triangle, to the section parallel to one of its sides forming a parabola, each of which extremes it can never reach, but may approach infinitely near.

When the section passes through the vertice and base of a cone, the section is evidently a triangle, whatever angle it may make with the axis. Let the section be removed from the vertice by any quantity however small, and it becomes one of the curves described above; it may be an hyperbola or a parabola, either of which may agree infinitely near with the triangle. Hence, the triangle and parabola are not opposite extremes of the hyperbola as the limits of its dimensions, for the hyperbola and parabola may both terminate in a triangle at the same moment, or the two curves may become assimilated when they are infinitely removed from a triangular form; nevertheless, when the hyperbola *hh*, parabola PP, and the triangle ACC are



described on the same base C C, and of equal altitude B A, the hyperbola may always be described between the triangle and parabola; this arises from the nature of the sections from which they are formed. Let the axis be indefinitely extended as when the section becomes parallel to one of the sides of the cone, in which case the two asymptotes AC, AC of the hyperbola CHC, become parallel to each other, as A′ *e*, A′ *f*, in which case the asymptotes may be conceived to vanish, and the curve, at this point, assumes other properties, as we have seen in our investigations; the axis, after being increased to infinity in the direction, BA, vanishes or becomes changed, and is infinitely extended in the direction B; but the same cause that brings the asympotes to a parallel position, may, by continuing, cause them to assume a greater distance toward A′ A′ than in the opposite direction, and the axis in such case becomes a definite magnitude, in which case the curve will be inclined inward, as at *n i*, when, by extending the axis, it will return into itself and become an ellipse.

*Scholium* 2. As it has been shown (Prop. IX Cor.6) in reference to ellipses, it may be also shown that all hyperbolas having the same centre and equal bases, and described between the same parallels, although infinitely produced, are equal to each other, as are also their corresponding sections parallel to the bases, and likewise any frustum or segments intercepted by the parallels.

The same may also be shown of parabolas of the same base and altitude, or those with the same base, and between the same parallels, that they are equivalent in area, &c.

## EQUATIONS TO THE CURVES FORMED BY SECTIONS OF THE CONE.

### 1. *For the Ellipse.*

Let $d$ denote AC, the semi-axis major or semi-diameter;
$c =$ CM its conjugate ;
$x =$ AK, any abscissa, from the extremity of the diam.
$y =$ DK the correspondent ordinate.
Then, (Prop. XII Ellipse,) $AC^2 : CM^2 :: AK . KB : DK^2$, that is, $d^2 : c^2 :: x\,(d-x) : y^2$, hence $d^2\,y^2 = c^2\,(d-x^2)$ or $dy = c\sqrt{(dx-x^2)}$, the equation of the curve.

And from these equations, any of the four letters or quantities, $d, c, x, y$, may easily be found, by the reduction of equations, when the other three are given.

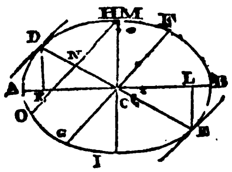
Or, if $p$ denote the parameter, $= c^2 \div d$ by its definition; then, (Prop. XII. Cor. 1) $d : p :: x\,(d-x) : y^2$, or $dy^2 = p\,(dx-x^2)$ which is another form of the equation of the curve.

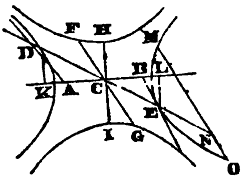
*Otherwise.* Or, if $d$ = AC the semi-axis; $c$= CH the semi-conjugate; $p = c^2 \div d$ the semi-parameter; $x$ = CK the abscissa counted from the centre; and $y$ = DK the ordinate as before.

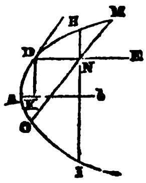
Then is AK $= d - x$, and KB $= d + x$, and AK . KB $= (d-x) \times (d+x) = d^2 - x^2$.

Then, $d^2 : c^2 :: d^2 - x^2 : y^2$, and $d^2y^2 = c^2(d^2-x^2)$, or $dy = c\sqrt{(d^2-x^2)}$, the equation of the curve.

Or, $d : p :: d^2 - x^2 : y^2$, and $dy^2 = p\ (d^2-x^2)$ another form of the equation to the curve; from which any one of the quantities may be found, when the rest are given.







### 2. *For the Hyperbola.*

Because the general property of the opposite hyperbolas, with respect to their abscissæ and ordinates is the same as that of the ellipse, therefore the process here is the same as in the former case for the ellipse; and the equation to the curve must come out the same also, with the exception of the signs which are sometimes changed from + to −, or from − to +, because the abscissæ on the axis, lie beyond or without the curve, whereas they lie within it, in the ellipse. Thus making the same notation for the whole diameter, conjugate, abscissa, and ordinate, as at first in the ellipse; then, the one abscissa AK being $x$, the other BK will be $d + x$, which in the ellipse was $d - x$; so the sign of $x$ must be changed in the general property and equation, by which it becomes $d^2 : c^2 :: x\ (d+x) : y^2$; hence $d^2\ y^2 = c^2\ (dx+x^2)$ and $dy = c\sqrt{(dx+x^2)}$, the equation of the curve.

Or, using $p$ the parameter as before, it is, $d : p :: x\ (d+x) : y^2$, or $dy^2 = p\ (dx+x^2)$, another form of the equation of the curve.

*Otherwise,* by using the same letters $d, c, p$, for the semi-axis, semi-parameter, and parameter, and $x$ for the abscissa CK counted from the centre; then AK$=x-d$, and BK$=x+d$ and the property $d^2 : c^2 :: (x-d) \times (x+d) : y^2$, gives $d^2\ y^2 = c^2\ (x^2-d^2)$ or $dy = c\sqrt{(x^2-d^2)}$, where the signs of $d^2$ and $x^2$ are changed from what they were in the ellipse.

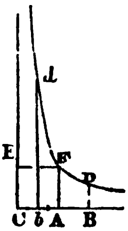
Or, again using the semi-parameter, $d : y : x^2 - d^2 : y^2$, and $dy = p(x^2 - d^2)$ the equation of the curve.

But for the conjugate hyperbola, (Prop. XII, Cor. 3, Hyp.) as the signs of both $x^2$ and $d^2$ will be positive; for the property in that case being $CA^2 : CB^2 :: CD^2 + CA^2 : Dp^2$, it is $d^2 : c^2 :: x^2 + d^2 : y^2 = Dp^2$, or $d^2 y^2 = c^2 (x^2 + d^2)$ and $dy = c\sqrt{(x^2 + d^2)}$, the equation to the conjugate hyperbola.

Or, as $d : p :: x^2 + d^2 : y^2$, and $dy^2 = p(x^2 + d^2)$, also the equation to the same curve.

### *On the Equation to the Hyperbola between the Asymptotes.*

Let CE and CB be the two asymptotes to the hyperbola *d*FD, its vertex being F; and EF, *bd*, AF, BD, ordinates parallel to the asymptotes. Put AF or EF$=a$, CB$=x$, and BD$=y$. Then, (Prop. XII Hyp.,) AF . EF = CB . BD, or $a^2 = xy$, the equation to the hyperbola, when the abscissæ and ordinates are taken parallel to the asymptotes.



### 3. *For the Parabola.*

If $x$ denote any absciss beginning at the vertex, and $y$ its ordinate, also $p$ the parameter. Then AK : KD :: KD : $p$ : or $x : y :: y : p$; hence $px = y^2$ is the equation to the parabola.

### 4. *For the Circle.*

Because the circle is only a species of the ellipse, in which the two axes are equal to each other; therefore making the two diameters $d$ and $c$ equal in the foregoing equations to the ellipse, they become $y^2 = dx - x^2$ $y$ being the mean proportional between $x$ and $d - x$, when the abscissa $x$ begins at the vertex of the diameter : and $y^2 = d^2 - x^2$, when the abscissa begins at the centre.

*Scholium.* In each of these equations, we percieve that they rise to the 2d or quadratic degree, or to two dimensions; which is also the number of points in which every one of these curves may be cut by a right line. Hence it is that these four curves are said to be lines of the 2d order. And these four are all the lines that are of that order, every other curve being of some higher, or having some higher equation, or may be cut in more points by a right line.

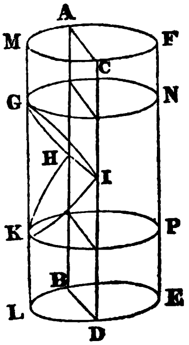
# BOOK II.

## SOLID SECTIONS OR SEGMENTS OF SOLIDS OF REVOLUTION, CYLINDROIDS, AND PARABOLIC PRISMOIDS AND UNGULAS.

### DEFINITIONS.

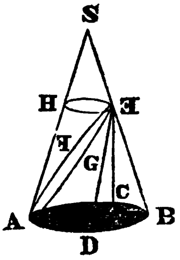
1. *Solid Sections* or segments are the portions of a solid cut off, or out, from another solid by one or more plane or curve surfaces.

2. If any portion of a cylinder or a cone, is cut off by a plane which is not parallel to the base, the solid section, so cut off, is called an *ungula*.



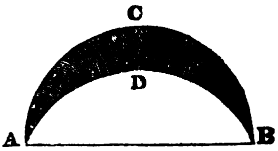
3. Ungulas cut from a cylinder are called *cylindric ungulas*.

Thus the section ACDBEF cut off by the plane ACDB ; as also the several sections ACIHGM, GHIK and HIKLBD, are *cylindric ungulas*.



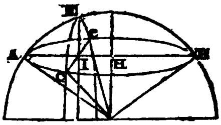
4. Ungulas cut from a cone or conic frustum are called *conical ungulas*.

Thus the sections ECDB and AFEGAB are *conical ungulas*.



5. Portions cut from a sphere by the intersection of two planes, are called *spherical wedges or ungulas*.

Thus the spherical section ACBDAB is a *spherical wedge* or *ungula*.



6. A portion cut from a segment of a sphere by a plane perpendicular to the base of the segment is called a second segment.

The portion C*ce*A is a second segment of a sphere.

7. *Conical ungulas* take particular names according to the figure of the superficial section, viz., *parabolic elliptical* or *hyperbolic*.

8. The portion of a cylinder or cone remaining after an ungula is taken, is called the *complement* of the *úngula*, or ungulical complement, and its altitude is equal to that of the ungula.

Thus GNPKIHG (Fig. at Def. 3) is the complement of the ungula GKIH, and ADCEHA (Def. 4) is the complement of the ungula ECDB.

9. An *ungulical supplement* is what remains of the whole solid after an ungula is taken therefrom.

10. An *elliptical cylinder* is a cylindrical solid, every superficial section of which by planes perpendicular to the axis are equal ellipses.

11. An *elliptical cone* is one, every section of which perpendicular to its axis are similar ellipses, and is the solid included between an elliptical base perpendicular to its axis, and a point as its vertex.

12. A *cylindroid* is a solid included between two bases of equal perimeter, one of which is an ellipse, and the other an ellipse of a different excentricity or a circle.

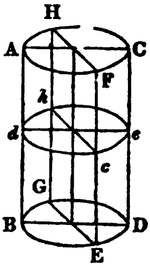
13. If a solid have two parallel bases, consisting of dissimilar ellipses of different perimeters, or one elliptical and one circular base, of different perimeters. The solid may be called a *conoidal frustum.*

14. A parabolic prism is a prismatic solid, whose base is a parabola, and each of whose sections parallel to the base is equal and similar to the base.

15. A parabolic pyramid is a pyramidal solid, whose base is a parabola; and is the solid included between such base, and a point above as its vertex.

### PROPOSITION I. THEOREM.

*If a cylinder be cut by a plane parallel to its axis, the solid section so cut off will be equal to the area of its base multiplied by its altitude, and its curve surface will be equal to the arc of its base multiplied by its altitude.*

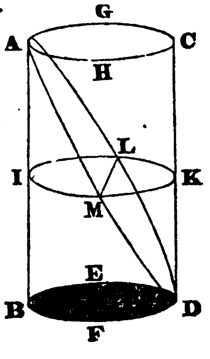


Let HFCGED, be a section of the cylinder, AD, cut off by a plane HFEG, parallel to the axis of the cylinder; then will the solidity of the section be equal to the area GD of the base multiplied by the altitude DC, and its convex surface will be equal to the arc EDG of the base multiplied by the altitude DC.

For, if a plane be passed through a cylinder parallel to the axis, it will divide the two bases proportionally, and every section *dhec* parallel to the base will be divided in the same ratio; hence the curve surface of the cylinder will be divided in the ratio that the circumference of the base is divided. But the convex surface is equal to the circumference of its base multiplied by its altitude, (Prop. I. B. III. *El. S. Geom.*,) and its solidity is equal to the area of its base multiplied by its altitude, (Prop. II. B. III. *El. S. Geom.*) Hence the curve surface of the portion so cut off, proportional to the section of the base, is also equal to the arc of its base multiplied by its altitude; and its solidity, for the same reason, is equal to the area of its base multiplied by its altitude.

### PROPOSITION II. THEOREM.

*If a plane cut a cylinder diagonally, passing through the opposite edges of the two bases, then the cylinder will be divided into two equal ungulas, and the curve surface of each ungula will be equal to the perimeter of its base multiplied by half its altitude; and the solidity of each will be equal to the area of the base multiplied by half the altitude.*

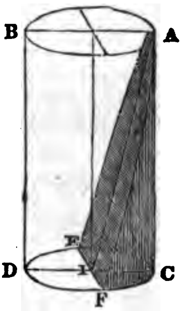


Let ALDMA be a plane passing diagonally through the cylinder BC, cuttingthe opposite edges of the two bases at A and D, and the two ungulas ADB, ADC will be equal in surface and solidity, and the curve surface of each is equal to the circumference of its base multiplied by half its altitude, and their solidities are equal each, to the area of its base multiplied by half its altitude.

For, the two ungulas are symmetrical (Def. 19, B. II. *El. S. Geom.*,) being solids, similarly formed, in opposite sides of the plane ALDMA as a base, and hence are equal each to each. Moreover, let a plane be passed through the cylinder, parallel to, and at equal distances between the two bases, forming the superficial section ILKMI, and the two ungulas will be cut by that plane in the same ratio, since they are similar solids, and since they are cut by a plane parallel to and at equal distances from their bases: hence the solid section, or partial ungula ALMI, is equal to the partial ungula DMLK, both in surface and solidity; and for the same reason, their complemental ungulas ALMKC, ILMDB are equal, each to each, both in surface and solidity. Therefore, the cylindric section KB between the two parallels ILKM and BEDF, is equal to the ungula ABEFDA, both in surface and solidity; but the cylindric surface of KB is equal to the circumference BEDFB of the base multiplied by IB, equal to half the altitude of the ungula ABD; and the solidity of KB is equal to the area BEDF multiplied by IB, equal to half the altitude of the ungula. Hence, &c.

### PROPOSITION III. THEOREM.

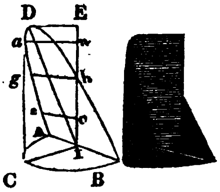
*If a cylinder be cut diagonally by a plane which bissects the base, the ungula cut off by such plane will be equal to its cylindric surface multiplied by one-third of the radius of the circle of the base.*



Let AFEC be an ungula, cut off from the cylinder ABDC by the plane AFE, bisecting the base CFDE in FE, then will the ungula AFEC be equal in solidity to its cylindric surface multiplied by one-third of the radius IC of the base.

For, conceive the ungula to be divided into cylindrical elementary pyramidals (Prop. XIII. *Schol.* 1 and 4, B. III. *El. S. Geom.*) by planes parallel to the axis of the cylinder, and all passing through the centre I of the base, then these will all be perfect pyramidals, since lines drawn from every point in the cylindrical base to the centre I, lie wholly in the solid, or in the plane surfaces of the solid; and since all parts of the solid are included between the cylindrical base of the ungula and centre I, in right lines. And these pyramidals (Prop. XIV. B. III. *El. S. Geom.*) are each equal to its cylindrical base multiplied by one-third of its altitude, viz., one-third of CI. Therefore, the whole ungula, being made up of all the pyramidals, is equal to the sum of all their bases multiplied by one-third of the common altitude CI.

*Cor.* 1. If a line *ec* be passed along the axis of the cylinder, and the edge of the ungula, through its whole extent, being always perpendicular to the axis, in every position *ec* or *gh*, or *an* the solid included within the surface described by the motion of this line, together with the ungula, is equal to the cylindric surface of the ungula multiplied by half the radius of the base.

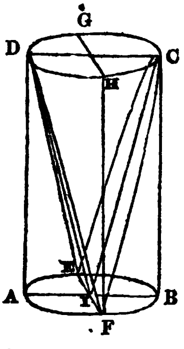


For let the cylindric surface be divided at pleasure, by planes passing through the axis, and the several divisions will all be elementary portions of the cylinder, and (Prop. III, *Cor.* B. III, *El. S. Geom.*) each will be equal to its cylindrical base multiplied by half its altitude, or half the radius of curvature of the cylinder.

*Cor.* 2. Hence the section included between the ungula and the axis of the cylinder is equal to half the ungula; since the ungula, is equal by the proposition to its cylindrical surface, multiplied by one-third of the radius of the base, and the two sections together, are equal to the same surface multiplied by $\frac{1}{3}+\frac{1}{6}=\frac{1}{2}$ the same radius.

### PROPOSITION IV. THEOREM.

*From the complement of two similar ungulas whose bases are together equal to the base of the cylinder, a cone may be taken of equal base and altitude to that of the cylinder, when there will be left a residual portion equal to its cylindric surface multiplied by one-third of the radius of the base.*



Let DEFC be the complement of the two similar ungulas DEFA, and CEFB whose bases AEF and BEF are together equal to the base of the cylinder, and there may be taken a cone DGCHI of equal base and altitude with the cylinder, and the portion of the cylinder remaining will be equal to its cylindric surface multiplied by one-third of the radius IF of the base

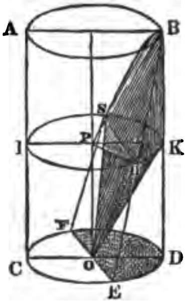
For since each of the ungulas are cut off by planes passing from the centre I, of the lower base of the cylinder, and through opposite edges of the upper base, those planes just pass along the sides, of the cone touching it on the two opposite sides, through its whole length, and be cause the cone terminates in a point I in the centre of the lower base, there exists a portion FIDHC on one side of the cone, and a similar portion EIDGC on the other side, included between the surface of the cone, and that of the cylinder; and these two portions consist of regular pyramidals with cylindric bases and their vertices all centre in I, at the vertex of the cone ; for if lines be drawn from every point in their cylindric surface to the centre I, those lines will pass through every point in the solid portions to which those surfaces belong. And because the cone DGCHI is supposed to be taken from the complement, these pyrimidals are elementary portions of a conesected cylinder, (Prop. XIII. Sch. B. III. *El. S. Geom.* and Def. 2,). Hence (Prop. XIII. Sch. 5, B. III., *El. S. Geom.*) they are equal to their cylindric bases multiplied by one-third of the radius IF.

*Cor.* The opposite ungulas DEFA, CEFB and the complements IFDHC, IEDGC of a conesected cylinder are segments which are in the same proportion to each other in their cylindric surfaces, as in their solidities.

### PROPOSITION V. THEOREM.

*The solidity of an ungula, whose base is less than half the base of the cylinder, is equal to its cylindric surface, multiplied by one-third of the radius of the base of the cylinder, minus the pyramidal segment of a cone, whose base is the base of the ungula, and whose vertex, is the point where the diagonal plane produced forming the ungula, would cut the axis of the cylinder or its axis produced.*

Let BSHK be an ungula whose base SHK is less than half the base of the cylinder, and its solidity will be equal to its cylindric surface, multiplied by one-third of the radius of the base of the cylinder, minus the pyramidal segment KSHO of a cone whose base is KSH, the base of the ungula ; and whose vertex is O, the point where the plane BSH produced to FE cuts the axis of the cylinder.



For the solidity of the ungula BFED, is equal to its cylindric surface multiplied by one-third of the radius of the base of the cylinder (Prop. III,) and the section KSHFED minus the conical section KSHO is a portion of a conesected cylinder, which (Prop. XIII, B. III, *El. S. Geom.*) is equal to the product of its cylindric surface multiplied by one-third of the radius of the base, and because this section of the conesected cylinder is formed by a plane passing through the centre O, cutting it in such manner that every point in its cylindric surface may be connected by right lines with the vertice O, these lines being included in the same solid; this section is also equal to its cylindric surface multiplied by one-third of the distance OD of this vertice from the cylindric surface; hence, the other two portions of the ungula, viz., the conical KSHO, and the ungula BSHK are equal to the remainder of the cylindric surface of the ungula BFED, viz., the cylindric surface of the small ungula BSHK multiplied by one-third of the radius OD of the base of the cylinder. Now, if from this product we take away the conical segment SHKO we shall have the ungula BSHK. Hence, &c.

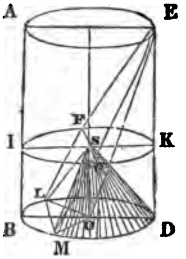
*Scholium.* If on the diagonal plane BSH of the ungula as a base, the pyramidal BSHP be described, P being the vertice of such pyramidal situated in the centre of the circular section of the cylinder, and in the plane KSH of the base of the ungula produced, the sum of this pyramidal and ungula is equal to their cylindric surface multiplied by one-third of the radius of the base of the cylinder; and the pyramidal BSHP is equal to the conical portion, KSHO.

For the pyramidal BSHP together with the ungula BSHK, constitutes the regular elementary pyramidal, BSHKP, with a cylindrical base, which (Prop. XIV. B. III., *El. S. Geom.*) is equal to its cylindrical surface multiplied by one-third of the radius, PK of the base, and because the sum of the ungula, BSHK, and pyramidal section of a cone KSHO, is equal to the same product (Prop. V.) it follows that the pyramidal section BSHP is equal to the conical portion KSHO.

### PROPOSITION VI. THEOREM.

*The solidity of an ungula* ELMD *whose base is greater than half the base of the cylinder, is equal to its cylindrical surface multiplied by one-third of the radius of the base of the cylinder, plus a conical segment, whose base is the base of the ungula, and whose vertice is the point* S, *where the plane* ELM *cuts the axis of the cylinder.*

*Or the solidity of the ungula is equal to the same product plus the pyramidal* ELMO, *whose base is the elliptical sections* ELM *and whose vertice is* O, *the centre of the base of the cylinder.*



For through the point S pass the plane IFKC, and the ungula will be divided into two portions, one of which, EFCK, is equal (Prop. III.) to its cylindrical surface multiplied by one third of the radius of the base of the cylinder; and the portion FCK LMD, from which, if we take a conical section LMDS, we shall have a portion of a conesected cylinder remaining which is equal to its cylindric surface multiplied by one third of its radius.

Hence, the whole ungula ELMD is equal to its cylindric surface multiplied by one third of the radius of the base of the cylinder plus a segment of a cone, whose base is the base of the ungula and vertice, the point where the plane ELM cuts the axis of the cylinder.

Again, if we pass the two surfaces MOE and LOE meeting each other in the right line OE, those surfaces will cut out the pyramidal ELMO, whose base is the elliptical section ELM, and whose vertice is the centre of the base of the cylinder, which if we take from the ungula, will leave a portion which is equal to its cylindrical surface multiplied by one-third of the radius of the base of the cylinder, since it consists of a portion which may be divided indefinitely by planes passing through the vertice O, and each of those portions so divided will be perfect pyramidals, and will remain elementary to the whole section. (Prop. XIII, Schls. B. III, *El. S. Geom.*) Hence, &c.

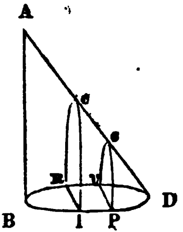
*Cor.* 1. Hence the pyramidal ELMO, is equal to the conical segment LMDS.

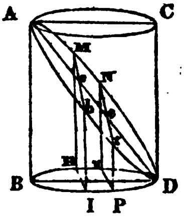
*Cor.* 2. As the plane section ELM is an ellipse, the surface of the pyramidal ELMO contiguous to the other portions of the ungula so divided is a conical surface; for if the several points in the elliptical curve be connected with the vertice O, those lines must include a conical surface, since the section of a cone by a plane passing through both sides is an ellipse.

*Cor.* 3. If the ungula is so cut as to include the whole of the base BLDM, the circumstances will still be the same, and the ellipse of the section ELM becomes perfect, and the pyramidal ELMO becomes a perfect oblique cone, and is equal to the conical body LMDS, which, in such case, becomes a perfect right cone, whose base is the whole base BLDM of the cylinder, and whose altitude is half the altitude of the cylinder.

### PROPOSITION VII. THEOREM.

*If in a cylindric ungula* ADB, *whose base is equal to that of the cylinder, a cone be described on the same base and of an equal altitude with that of the ungula, the cone will be equal to two-thirds of the ungula; and each solid section of the cone made by planes parallel to the axis of the cylinder, and perpendicular to a plane passing through the axis of the cone and cylinder, will be equal to two-thirds of the corresponding section of the ungula cut off by the same plane.*





Let the oblique cone ABD of equal base and altitude to that of the ungula A*b*DNB, be supposed to be inscribed in the ungula, and the solidity of the cone will be equal to two-thirds of the ungula, and each solid section IR*c*D, P*ve*D of the cone made by planes parallel to the axis of the cylinder, and perpendicular to a plane passing through the axis of the cone and cylinder, are equal to $\frac{2}{3}$ their corresponding sections IRM*b*D, P*v*N*t*D of the ungula cut by the same planes.

For, conceive an indefinite number of parallel planes IRM*b*, N*t*P*v*, to be passed through the cone and ungula indefinitely near to each other, and the two bodies will be divided into an indefinite number of strata, the sum of which, in each solid, will be in the relation of the magnitudes of the two bodies; and the corresponding *strata* in each solid will be in the relation of the superficial sections contiguous to such *strata*, in each, since by hypothesis the *strata* are indefinitely thin, so that no appreciable space intervenes between them; hence, if the solidities of any *stratum*, or number of contiguous *strata* in each solid be represented by the superficial sections passing through such *stratum* or associated *strata*, they will exist in each in the relation of the superficial sections in each, made by the same planes.

Now the superficial sections IR*c*, P*ve* &c., made by planes parallel to the side AB of the cone, are all parabolas, (by def.) and all the corresponding sections of the ungula are rectangles circumscribing the several parabolas, since each side of those sections M*b*, N*r* of the ungula are parallel to the opposite sides RI, *v*P, passing through the base. But the area of the parabola (Proposition VI, Book I,) is equal to two-thirds of its circumscribing rectangle or parallelogram, and as each of the several rectangles circumscribe their respective parabolas, throughout the whole extent of the two solids, it follows that not only is the cone equal to two-thirds of the ungula, but also that each solid section, or segment of the cone, cut by the planes RM*b*I, &c., is equal to two-thirds of the corresponding segment of the ungula cut by the same plane.

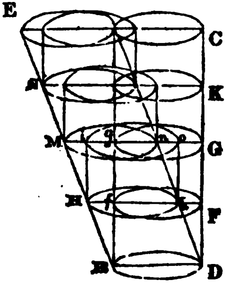
*Cor.* Hence, as in Prop. VIII, Cor. B. III, *El. S. Geom.* a cone is equal to one-third of its circumscribing cylinder of the same base and altitude.

*Scholium.* If the cone and ungula are cut by planes parallel to the plane A*b*DN, the superficial sections of the ungula will be ellipses, or portions of ellipses, and the corresponding sections of the cone will be parabolas.

### PROPOSITION VIII. THEOREM.

*Two cylinders erected on the same base, and between the same parallel planes are equivalent or equal in solidity.*

Let ACDB and AEBD be two cylinders erected on the same base BD, and included between the same parallel planes BD and EAC, and the two cylinders will be equivalent.



For let the cylinder ABDC be divided into sections BDF*f*, F*fg*G by planes parallel to the base, and let each of the sections so divided, be removed from their positions in the cylinder ABDC, so that their several bases shall agree with, and become sections of the oblique cylinder AEBD; viz., let the solid sections *f*FG*g*, be removed to the position H*hoi*, and let the other sections be similarly posited in reference to the two cylinders, so that the bases H*h*, M*n*, &c. of the several solid sections may be included in the oblique cylinder AEBD, and these several sections will still be equal in their altitude to AB, the altitude of the cylinder from whence they are severally derived. Let, now, the number of these solid sections be indefinitely increased, and the altitude of each will be indefinitely small; and hence the right cylinder will become identical with the oblique cylinder AEBD, and will still have the same altitude AB.

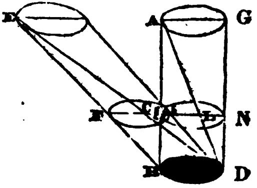
*Cor.* 1. Since the base of the oblique cylinder AEBD is a circle, every section parallel to the base is likewise a circle, and equal to the base.

*Cor.* 2. Hence every section of the oblique cylinder made by the planes perpendicular to its axis, is an ellipse. Therefore, if referred to its axis, it becomes an elliptical cylinder, but when referred to its base, it is circular.

### PROPOSITION IX. THEOREM.

*Cylindrical ungulas of the same base and equal altitude are equivalent.*

Let ABD and EBD, be two ungulas described on the same base BD ; and let their common altitude be AB, and the two ungulas will be equivalent, or equal in solidity.



For each section FC or IL of the two ungulas parallel to the base BD, have the same relation to the respective sections of their cylinders ; viz., the ungulical section FC has the same relation to the cylindric section F*h*, as the ungulical section IL has to the cylindrical section IN ; and the equality of this relation remains through the whole altitude AB of the ungulas, and by Prop. VIII, it appears that two cylinders erected on the same base, and between the same parallels are equal in solidity. Therefore, the ungulas erected on the same base, and of the same altitude, are equivalent.

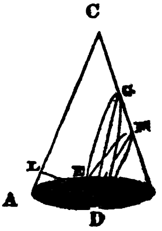
*Scholium.* The above proposition is manifestly true, whether the base of the ungulas is equal to that of the cylinder in which it is erected, or less than that base.

*Cor.* 1. As the same arguments would apply to conical ungulas, it may be inferred that conical ungulas of the same base and equal altitude are equivalent, if the cones from which they are taken are similar.

*Cor.* 2. Since cylinders with equal bases are proportional to their altitudes, cylindric ungulas on the same base are proportional to their altitudes.

### PROPOSITION X. THEOREM.

*If a cone be cut by a plane passing through the centre of the base, the solidity of the ungula formed by such plane will be equal to its curve surface multiplied by one-third of its distance from the centre of the base.*



Let the cone ABC be cut by the plane DEF, cutting off the ungula EFDB ; also by the plane EDG cutting off the ungula GEDB, or the ungula GEDF, and the solidities of the several ungulas will be equal to their curve surfaces multiplied by $\frac{1}{3}$ of the distance IL of the surface of the cone to the centre of the base.

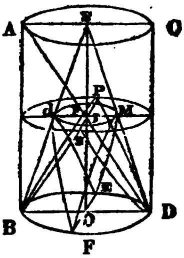
For the sections may each be divided into an indifinite number of pyramidals or elements (Prop. XIII and XIV, B. III, *El. S. Geom.*) by dividing their curve surfaces as bases, and by passing the planes of division through the centre I of the cone's base, at which point the several vertices of the pyramidals all centre ; and because those pyramidals are all perfect; viz.. as they include all the space intercepted between their bases and vertices in right lines, each one is equal to its base, multiplied by one-third of its altitude, and since the altitude is equal in each, and equal to the distance of the curve surface of the cone from the centre of its base, IL is that altitude, and hence IL is the common altitude of each of those pyramidals; hence the sum of the pyramidals constituting any section, is equal to the sum of their bases, constituting the curve surface of such section multiplied by one-third of IL.

*Cor.* Hence, if a cone be cut by a plane passing through the centre of the base, the solidity of the cone, and its convex surface is divided in the same ratio, since the cone (Prop. IX. B. III., *El. S. Geom.*) is equal to its convex surface multiplied by $\frac{1}{3}$ of its distance from the centre of the base.

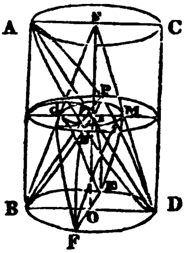
### PROPOSITION XI. THEOREM.

*If a cone be described in a cylinder, and two parabolic ungulas be cut by planes passing through the centre of the cone's base, the ungulas so cut, will be equal to two thirds of the ungulical complement of the cylinder, of equal base and altitude to the parabolical ungulas.*

Let BDS be a cone described in the cylinder ABDC, and let EFB*d*, and EFDM be two equal parabolic ungulas, described on and including the whole base of the cone, and the two ungulas will be equal to two-thirds of the ungulical complement BEDFNP of the cylinder, of equal base and altitude.



For let the cone BDS be brought in the position BDA. and the parabolic ungula IEFD of the cone BDA will be equal to two-thirds of the semi-ungulical complement NPEFD of the cylinder, (Prop. VII ;) and because conical ungulas on the same base and equal altitudes are equivalent (Prop. IX *Cor.*,) the ungula IEFD of the oblique cone BDA is equal to the ungula EFDM of the right cone BDS; hence the parabolic ungula EFMD is equal to two-thirds of the semi-ungulical complement NPEFD of the cylinder; and the two similar ungulas *d*EFB, MEFD are, together, equal,to $\frac{2}{3}$ of the ungulical complement BEDFPN of the cylinder.

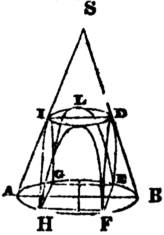


*Cor.* 1. Hence the conical complement EF*d*MS of the parabolic ungulas=the complement PMN*d* EF+the cone PMN*d*S, is equal to two-thirds of the cylindrical ungula APNB. For if the whole cone BDA or BDS is=$\frac{2}{3}$ of the ungula BDA, (Prop. VIII,) and if the two conical ungulas *d*EFB, MEFD, as shown above, is = $\frac{2}{3}$ of the complement PNBD, then must the complement EF*d*MS= $\frac{2}{3}$ the cylindrical ungula APNB

*Cor.* 2. The small cone *d*MS, whose base passes through the vertices of the ungulas, is = to $\frac{1}{8}$ of the cone BDS, since the diameter of its base is necessarily = $\frac{1}{2}$ that of the larger, and since they are similar solids; for similar solids are to each other as the cubes of their like sides, (Prop. XXXV, B. II, *El. S. Geom.*) and cube of 1 is 1, cube of 2 is 8; hence 1 : 8 : : cone *d*MS : cone BDS.

### PROPOSITION XII. THEOREM.

*If a right cone be cut by a plane perpendicular to the plane of its base, the convex surface of the cone, and the plane of the base, will be divided in the same ratio by the cutting plane.*



Let DEF be a plane cutting the cone ABS, perpendicular through the base AEBF, and the convex surface of the cone will be divided by the cutting plane, in the same ratio that the surface of the base is divided.

For since it is shown (Prop. XXII, B. II, *El. S. Geom.,*) that the convex surface of a pyramid and its base is divided in the same ratio, by a plane perpendicular to its base, and because a cone may be considered as a pyramid, whose convex surface consists of an indefinite number of planes, which are indefinitely narrow, it follows that if the convex surface and base of a right cone are cut by a plane perpendicular to the base, those surfaces will be each divided in the same ratio.

*Cor.* 1. Hence if any portion of the base of a right cone is taken, the portion of the curve surface perpendicular above it, will be to the whole curve surface of the cone, as such portion of the base is to the whole base.

*Cor.* 2. Hence, also, if a regular polygon, for instance a square EFHG, be described in the base of the cone, and if on each side of this square, a plane be raised perpendicular to the base, the portion of the conical surface, cut off toward the axis, is to that of the rectilineal polygon EFHG, which corresponds to it perpendicularly below, as the surface of the cone is to the area of its base; or as the slant side AS of the cone is to the radius of the base; and, in fact, whatever figure be inscribed in the base, if we conceive a right prismatic surface raised perpendicular from the perimeter of the figure, it will cut off from the conical surface a portion which will be to it in the same ratio.

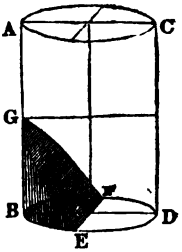
*Scholium.* The solid sections DEFB, LGE, &c., are hyperbolic ungulas, (Def. 7.) And if ungulas DEFB, LGE, &c., are taken from the cone, the remaining portion or compliment will be equal to its curve surface multiplied by one-third of the distance of the curve surface of the cone from the centre of the base, + the surface of the plane hyperbolic sections multiplied by one-third of their respective distance from the same centre of the base.

*Cor.* Hence the portion of the cone included between the centre of the base, and that portion of the convex surface left by the ungulas, since it is equal to its convex surface, multiplied by one-third of its perpendicular distance from the centre, the quadrature of which we have shown to be attainable, becomes known in absolute terms, or its cubature is attained without regard to the circle's quadrature.

### PROPOSITION XIII. THEOREM.

*If an elliptical cylinder, and a circular cylinder, have equivalent bases and equal altitudes, they are equal in solidity; and any ungulas similarly cut from each, with equivalent bases and altitudes, are equivalent.*

For it has been shown (Prop. II. B. III. *El. Geom.*) that the solidity of a cylinder, with circular base, is equal to its base multiplied by its altitude; and because this is true of any prism, whatever be the form of its base, (Prop. XVI. B. II. *El. S. Geom.*) it must be true of a cylinder with an elliptical base. Therefore, an elliptical and a circular cylinder of equivalent bases and equal altitudes, are equivalent.



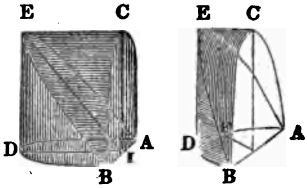
**Again,** let GEFB be an ungula cut from a circular cylinder ABDC; and if the base of this cylinder, including that of the ungula, is drawn out or elongated in the direction from BA toward DC, so as to become elliptical; and if every section of the cylinder parallel to its base should become equal and similar ellipses, then the cylinder becomes an elliptical cylinder.

Now, if planes should be passed through the cylinder parallel to its axis, and in the direction AC or BD of its elongation, in passing from a circular to an elliptical cylinder, this transformation will have been effected by an elongation of these planes proportional to the elliptical elongation of the base, or of the solid; and, as this would be true in every parallel plane, it follows that the elongation of those planes may be regarded as a measure of the ratio of enlargement of the solidity, by the same means. And as every such parallel section becomes enlarged in the same ratio, any specific portion of such section must suffer the same specific enlargement.

And as the increase of any solid sections through which any portions of the plane sections pass, may be measured by the increment of those planes, it follows, that both the ungula and its complement are each increased in the ratio of the increments of the parallel sections passing through each in their enlargement. And the ungula which was cut from a circular cylinder, becomes the ungula of an elliptical cylinder, which ungula has become enlarged in the ratio of the enlargement of its base; and the solidity of the ungula from the elliptical cylinder, is to the solidity of the ungula from the circular cylinder, as the base of the former to the base of the latter. And the same would be true, if instead of an enlargement of the circular cylinder to form the elliptical cylinder, it should be contracted in the direction AC, so as to give it eccentricity in the other direction; but, by hypothesis, the cylinder and an elliptical cylinder have equivalent bases; hence, ungulas similarly cut from each, are equivalent.

*Cor.* 1. Hence, also, if a cone with a circular base, and one with an equivalent elliptical base, have equal altitudes, their solidities will be equivalent; and ungulas with equivalent bases and equal altitudes cut from each, are equivalent.

*Cor.* 2. Ungulas, whose bases are the like parts of circular or elliptical cylinders, are as their altitudes; and it having been shown that they are also as their bases when their altitudes are equal, it follows that they are generally as the rectangle of their bases into their altitudes.



Let ABEC and ABED be two ungulas cut from any cylinders, circular or elliptical, such that AB shall be the same in each, and such that ID, the altitude of the base in the first, shall be equal CE. the altititude of the second, and IC the altitude of the base of the second shall be equal to the altitude ED of the first, and the two ungulas so described will be equivalent.

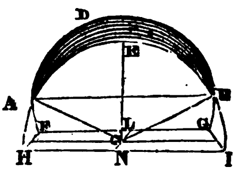
### PROPOSITION XIV. THEOREM.

*The solidity of a cylindroid is equal to the product of the sum of the areas of the two ends, and four times the area of a parallel section, equally distant between the two ends multiplied by $\frac{1}{6}$ of the height.*

Demonstration same as for the prismoid, Prop. XXXV, Cor. B. II, *El. S. Geom.*, which see.

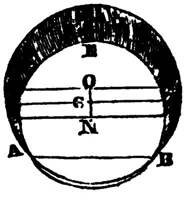
### PROPOSITION XV. THEOREM.

*If an ungula is cut from a sphere by two planes which intersect each other, not in the centre of the sphere, then the solidity of the ungula will be equal to its spherical base multiplied by one-third of the radius of the sphere, plus the products of the superficial sections of the ungula multiplied by one-third of the perpendicular distances of the planes of those sections from the centre of the sphere, estimated from the sides of those planes opposite the ungula.*

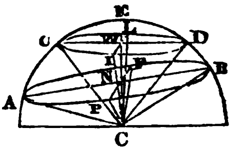


Let ABDE be an ungula cut by the planes HEI and FDG intersecting each other in the line AB, not passing through the centre of the sphere, then will the solidity of the ungula ABDE be equal to the spherical surface ADBEA, multiplied by one-third of the radius of the sphere, — (the plane surface ABE, multiplied by one third of the distance CN) — (the plane surface ABD multiplied by one-third the distance CL) when those planes include the centre on opposite sides of each, from that of the ungula.

For let each point in the perimeter, ABEA, ABDA, be connected by lines to the centre C, and the solid included by such lines will be a spherical pyramid, whose solidity is equal to its spherical base multiplied by $\frac{1}{3}$ the radius of the sphere; from which if we take away those portions included between the planes ABE, ABD and the centre C we shall have the ungula. But the pyramidal ABEC, is equal to the surface ABE × $\frac{1}{3}$ CF, and the pyramidal ABDC is equal the surface ABD × $\frac{1}{3}$ CL. Hence as enunciated above.



*Scholium.* If an ungula is cut by two planes which pass the centre before their intersection, so as to include the centre of the sphere within the ungula, then will its solidity be equal to the spherical surface multiplied by $\frac{1}{3}$ the radius, plus the two pyramidals erected on those plane sections, and whose vertices are in the centre of the sphere, which is agreeable to our proposition; for the sign becomes changed from minus to plus according to the conditions.

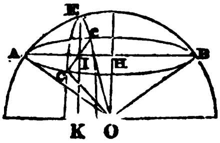


2. Also, if the two intersecting planes pass the centre on one side before their intersection, so as to cut out an oblique ungula ABDG, then since the pyramidal erected on the plane AFBP, and whose vertice is the centre, C, is considered negative, and the pyramid erected on the plane GIDL is considered positive, by the proposition, then will the ungula be equal to its (spherical surface × $\frac{1}{3}$ the radius) — the plane APBF × ($\frac{1}{3}$ CN) + the plane GLDF × $\frac{1}{3}$ CW.

3. And, generally, if the planes do not intersect each other within the sphere, the same proposition will still hold true, even though the planes may be parallel; in which case, the portion cut out will be a segment or zone of the sphere.

### PROPOSITION XVI. THEOREM.

*The solidity of the second segment of a sphere is equal to its spherical surface, multiplied by $\frac{1}{3}$ of the radius of its sphere minus each of the plane surfaces whose planes pass between the segment and centre of the sphere multiplied by $\frac{1}{3}$ of their respective perpendicular distances from the centre, and plus each of the plane surfaces which include the centre of the sphere on the same side with the segment, multiplied by $\frac{1}{3}$ the perpendicular distances of such planes to the centre.*



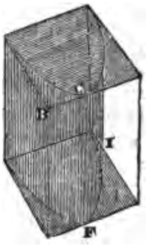
Let ABD be a segment of a sphere cut off by the plane ABCE, and if this segment is cut by the plane CE*e* perpendicular to the former plane, the two portions into which the segment is divided will be second segments. (Def. 6.)

Draw AO, EO, CO, and *e*O, and the spherical pyramidal AE C*e*O will be equal to its spherical base $\times\frac{1}{3}$ AO or EO. Now, from this pyramidal may be taken the pyramidal whose base is the section C*e*E, and whose altitude is KO, and also the pyramidal whose base is the section AC*e* and altitude HO, and there will remain the second segment AE*e*C.

Also the second segment may be shown to consist of the sectoral pyramid, whose base is the spherical surface, plus a pyramidal CE*e*O, and minus the pyramid C*e*BO. Hence the proposition is true, as enunciated.

### PROPOSITION XVII. THEOREM.

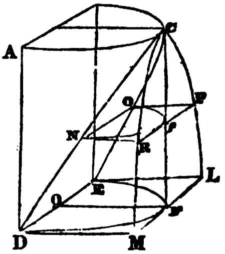
*The solidity of a parabolic prism is equal to two-thirds of its circumscribing quadrangular prism.*



For the solidity of any prism or solid, all of whose sections parallel to the base are equal to the base, is equal to the base multiplied by the altitude: and hence, prisms of the same altitude are as their bases; but the base of the parabolic prism, which is a parabola, is equal (Prop. VI, B. I) $\frac{2}{3}$ its circumscribing rectangle, also the rectangle circumscribing the base of the parabolic prism, is the base of the rectangular prism.

### PROPOSITION XVIII. THEOREM.

*If an ungula be cut from a parabolic prism by a plane passing through the vertical line of the parabolic surface, and whose intersection forms an ordinate to the axis of the parabola of the base; this ungula is equal to two-thirds of the ungulical complement of the prism of the same base and altitude.*

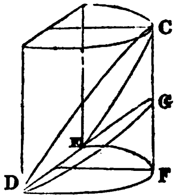


Let DECF be an ungula cut from the parabolic prism DEFCAB, by the plane CDE, from the vertical line CF of the parabolic surface, to the base DEF forming the ordinate ED to the axis FQ of the base, by its intersection with the plane of the base, and the ungula so cut will be equal to $\frac{2}{3}$ of the ungulical complement DECAB.

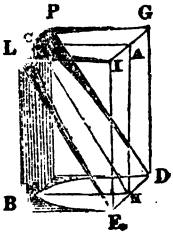
For, let a rectangle DELM be described about the base of the ungula, and let any number of planes

NOf parallel to the base be passed through the ungula; and each of those sections made by those planes are parabolas; and if about each of those parallel sections, rectangles NRPO are described, those parabolic sections will each be equal to two-thirds of their respective circumscribing rectangles, (Prop. VI. B. I.) Now, if these parallel sections through the ungula are infinite in number and equidistant from each other, they will represent the whole ungula, and their sum may be taken as a function of the solidity of the ungula, and the sum of their several circumscribing rectangles may be taken as a similar function of the solid DECLMC; hence the ungula DEFC is equal to two-thirds of the solid CDELMC. Now, if we take the altitude FC of the ungula=QF, the axis of the parabola of the base, the side CLMC of the new solid will be a parabola equal and similar to the base DEF of the ungula, or ABC of the complement, since perpendiculars from every point in the perimeter of the base DEF, trace out the parabolic section DEC; and since perpendiculars from each point in the perimeter DNCOE to the plane CLM, includes and traces out the parabola CLM. And because QF or EL would be equal to EB, the rectangle DELM would be equal to the rectangle DEBA; hence, the two solids CABDEC, CLMDEC being similar figures on opposite sides of the same base, CDE are symmetrical, (Def. 19, B. II. *El. Sol. Geom.*); and hence they are equivalent. But the ungula CDEF has been shown to be equal to two-thirds of the solid CLMDEC, it is therefore equal to two-thirds of the ungulical complement CEDABC.

*Cor.* 1. The above properties are true in whatever part of the base the plane CDE may cut, or whatever be the altitude of the ungula, provided the ungula and its complement have equal bases and altitudes; and because ungulas on the same base are as their altitudes, an ungula CGED is equal to the ungula GEDF, if the altitude CG=the altitude GF, since the two ungulas insist on or above the same base EDF.



*Cor.* 2. Hence the solidity of the parabolic ungula CDEF is equal to $\frac{8}{15}$ of its circumscribing prism, and the complement of a parabolic ungula is equal to $\frac{4}{5}$ of its circumscribing prism. For the parabolic prism (Prop. XVII.) is equal to $\frac{2}{3}$ of its circumscribing rectangular prism; and since the rectangular prism circumscribing the parabolic prism, may be divided into two equal triangular prisms by a diagonal



plane, identical with the plane which divides the ungula from its complement, it follows that the prism circumscribing the ungula DECF is equal to that circumscribing the complement DECI, equal half that circumscribing the parabolic prism. Let U equal the ungula, and C equal the complement, and let P equal the triangular prism circumscribing the ungula, and 2P will equal the quadrangular prism circumscribing the parabolic prism. Then will $U+C=\frac{4}{3}P$ and $U=\frac{2}{3}C$, hence $C=1\frac{1}{2}U$. Substituting the value of C in the first equation, we have

$$2\tfrac{1}{2}U=\tfrac{4}{3}P \text{ or } 15U=8P.$$

Hence $U=\frac{8}{15}P$,

or the ungula is equal to $\frac{8}{15}$ of its circumscribing prism.

And if we substitute the value of U in terms of C in the first equation, we shall have

$$1\tfrac{2}{3}C=\tfrac{4}{3}P \text{ or } 5C=4P.$$

Hence $C=\frac{4}{5}P$, or the complement of the ungula is equal to $\frac{4}{5}$ of its circumscribing triangular prism.

Therefore, the portion CLBFE is equal to $\frac{7}{15}$ of the prism CFNEBL.

And the exterior portion CLEI is equal to $\frac{1}{5}$ of the prism CANELI.

### PROPOSITION XIX. THEOREM.

*A parabolic pyramid or cone is equal to two-thirds of its circumscribing rectangular pyramid.*

Let a pyramid be erected on a base whose figure is a parabola, and if this base is circumscribed by a rectangle, it will $=\frac{2}{3}$ the rectangle; and because every section of each of the solids erected on those figures as bases, and whose sections are in a common point are similar figures and similar to the base, the solids erected on those bases must be in the relation of their bases; hence the parabolic pyramid is $=\frac{2}{3}$ its circumscribing quadrangular pyramid.

# BOOK III.

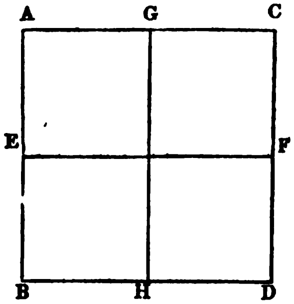
## ON REVOLOIDS AND SOLIDS FORMED BY THE REVOLUTIONS OF THE CONIC SECTIONS.

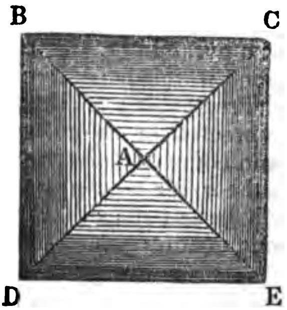
### DEFINITIONS.

1. A REVOLOID is a solid generated by the continued semi-revolution of a polygon on axes parallel to the sides of the polygon respectively, and passing through its centre, which is fixed, and it includes all the space that is not cut off by either side of the plane, in their several semi-revolutions.

2. Every revoloid has as many axes of rotation, as the polygon from which it is conceived to be generated, has independent sides; but the axes of any two parallel sides coincide with each other and are identical.

Thus, if the quadrilaterial ABDC be made to revolve on the two axes, EF and GH, parallel to its sides respectively, the solid generated by the revolutions of those sides will be a revoloid, as represented by BCED.





3. A revoloid is designated by the number of sides contained in the figure from whence it is conceived to be generated.

Thus a *triangular revoloid* is one formed by the continued semi-revolution of the sides of a triangle about their respective axes.

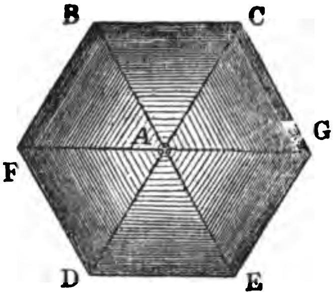
A *quadrangular revoloid*, by the revolutions of the sides of a square about their corresponding axes.

Also a *pentagonal*, a *hexagonal*, or other *polygonal revoloid*, is formed by the continued semi-revolution of the sides of a *pentagon*, a *hexagon* or other polygon, about their respective axes.

AB represents a hexagonal revoloid.

4. The vertices of a revoloid are its two extremities, where its several curve surfaces meet at a point, as at A.

5. The line joining these extremities, is its transverse or vertical axis; and a section through this axis is a vertical section.



6. Any diameter perpendicular to the transverse axis, is a conjugate axis, and any section perpendicular to the transverse axis is a conjugate section.

7. The figure from which a revoloid is supposed to be generated, is called its prime; which is always represented by a conjugate section through the centre of the revoloid.

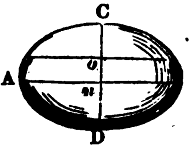
8. A revoloid formed by the revolutions of the several sides of its prime on axes that are fixed or immoveable during the revolutions of each side respectively, may be called right revoloids; and a vertical section through the centre of its opposite sides is a circle.

9. If, during the revolutions of the several sides of the prime of a revoloid, their axes are made to move in the line of the transverse or an opposite conjugate axis, then the revoloid will assume a different character according to the curve described by its sides. It may be *elliptical*, *parabolic*, or of any *regular curve*.

10. An *elliptical revoloid* is one whose transverse and conjugate diameters are unequal, and whose vertical section through the centre of its sides is elliptical.

11. A spheroid or ellipsoid is a solid generated by the revolution of a semi-ellipse about one of its axes, which remains fixed.

If the ellipse revolve round the transverse or major axis AB, the figure is called a prolate or oblong spheroid; if the ellipse revolve round the shorter axis CD, the figure is called an oblate spheroid.



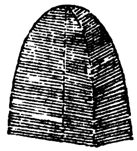
12. A segment of a spheroid or of an elliptical revoloid is

a part cut off by a plane, parallel to the major or minor axis.

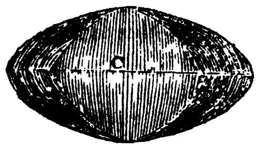
13. A frustum of a spheroid and also of an elliptical revoloid, is a part intercepted between two parallel planes, and is a portion included between two opposite segments.

14. A *parabolic revoloid* is one whose vertical section through the centre of its sides, consists of two parabolas on opposite sides of the same base.

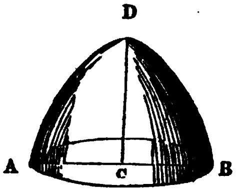
15. A parobolic revoloid is *vertical* when the vertices of the several parobolic sections are identical with the vertices of the revoloid. This may also be called a *parabolic pyramid.*



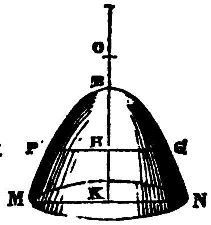
16. It is a *conjugate parabolic revoloid,* when the vertices of the parabolic sections are all in a plane perpendicular to the vertical axis of the revoloid.



17. A **Parabolic conoid** is a solid formed by the revolution of a semi-parabola about its axis; it is also called a paraboloid.



18. A *hyperbolic conoid,* or a hyperboloid, is a solid formed by the revolution of a semi-hyperpola about its greater abscissa, or transverse axis produced. Thus, the hyperbolic conoid MPBGN is formed by the revolution of the semi-hyperbola MPB, about its greater abscissa BK, or the transverse axis BH produced.

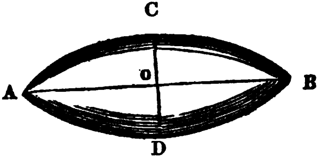


19. A *hyperbolic revoloid* is one whose vertical sections through its sides consists of two hyperbolas whose vertices are in the plane of the conjugate axes.

20. A *hyperbolic pyramid* or *pyramoid,* is one whose base is a polygon, and whose vertical sections through the vertice of the pyramoid, is an hyperbola. This may also be called a vertical hyperbolic semi-revoloid.

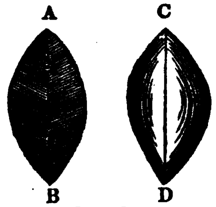
**21.** A *circular spindle* is a solid generated by the revolution of a segment of a circle about its chord, which remains fixed.

**22.** An *elliptical spindle* is a solid generated by the revolution of the segment of an ellipse about its chord.



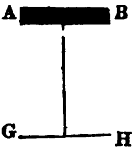
**23.** A *parabolic*, or *hyperbolic spindle*, is a solid formed by the revolution of a segment of a parabola or hyperbola, about its ordinate. Thus, if the segment PBGP of Def. 18 be supposed to revolve about the ordinate PG, which remains fixed, it will describe a spindle.

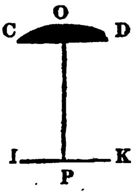
**24.** A *revoloidal spindle* is a revoloid circumscribing a spindle, a vertical section of which through the centre of its opposite sides, is equal and similar to that of the spindle through the same plane. It takes particular names according to the designation of the inscribed spindle.

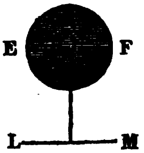


**25.** A ring is a solid formed by the revolution of a plane surface about an axis exterior to itself, which axis is always in the same plane of the revolving surface.

Thus, if the plane surfaces AB, CD, or EF are made to revolve about their several axes GH, IK, LM, they will severally describe solids, which are called rings.

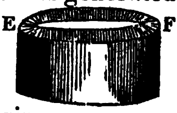






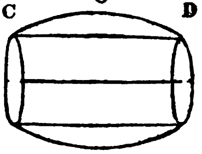
**26.** A ring is designated according to the figure of a conjugate section, or of the plane surface from which it is generated.

Thus from a rectilineal figure AB, is formed a prismatic ring.



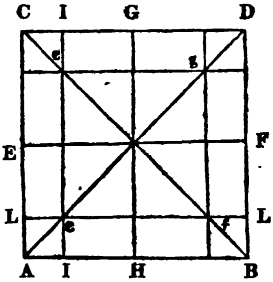
From the circle EF is formed a cylindrical ring.

From the segment CD is formed an ungulical ring, which, if the line OP is equal to the radius of the circular area COD, it may be called a spherical ring, the curve surface forming a portion of the surface of a sphere. Rings formed from other figures or segments of other curves, may be similarly designated.



### PROPOSITION I. THEOREM.

*A right revoloid is composed of cylindrical ungulas, equal in number to the sides of the revoloid; and these ungulas are such as are formed by plane sections, from one side, meeting in the axis of the cylinder, the intersection of which planes forms a diameter to the cylinder.*



For, since the several curve surfaces of a revoloid are conceived to be formed by the revolution of the sides of its prime about axes parallel to those sides respectively; (Def. 1,) the surfaces described by the revolutions of those sides, are cylindrical surfaces, (Def 1. B. III. *El. S. Geom.*) And since the angles formed by the sides of the prime are similar, each to each, in whatever similar part of their revolutions they may be, it follows that they would form similar angles with any plane perpendicular to these sides. Thus, because, when the side AB, of the revoloidal prime ABDC, in its revolution about its axis EF, comes into the position LL; and AC revolving about its axis GH, comes into the position II, &c.; the parts *ce* and *ef* of those sides form the same angle with each other as before; they must also form the same angles with any planes AD or CB perpendicular to the plane ABDC or *cefg*. Hence, the figure *cefg* is similar to the prime ABDC; and because this is true in whatever similar positions, the several sides of the prime may be in the course of their revolutions, it follows, therefore, that sections through the angles meeting in the transverse axis of the revoloid, are plane angles; and as these plane sections cut the cylindric surfaces diagonally through the axis of the revoloid, the segments so cut are cylindric ungulas, (Def. 3. B. II.)

Hence, each side of a right revoloid is a cylindric ungula, such as is formed by plane sections cutting the cylinder diagonally, and meeting in the axis of the cylinder and forming a diameter thereto.

*Cor.* 1. Hence any vertical section through the angles formed by the meeting of the curve surfaces of the revoloid, or any vertical section of the revoloid not at right angles to the sides, is an ellipse.

*Cor.* 2. Hence, also, the elliptical revoloid may be conceived to be made up of similar ungulas cut from an elliptical cylinder; see Prop. XIII. B. II., and a parabolic revoloid may consist of ungulas from a parabolic cylinder or prism.

*Cor.* 3. Since a revoloid is composed of cylindrical ungulas equal in number to the number of the sides, those ungulas are also such as are cut from the cylinder of altitudes or lengths on the cylindric surface equal to the lengths of the sides of the revoloid indicated by the lengths of the sides of its prime.

*Cor.* 4. All sections of a revoloid perpendicular to the vertical or transverse axis, are similar figures, since the sides of its prime retain their parallel position in whatever part of their revolution they are supposed to be taken.

### PROPOSITION II. THEOREM.

*The solidity of every right revoloid, bears the same relation to that of its greatest inscribed sphere, as the area of its prime does to that of its greatest inscribed circle.*

For since the surface of the revoloid is composed of cylindric surfaces, (Prop. I.) and since vertical sections of a right revoloid through the centres of its opposite sides are circles, (Def. 8.) it follows that a sphere may be inscribed in the revoloid so as to touch its cylindric surfaces through the whole circumference of those circular sections, viz., through the centres of the vertical sides, so that vertical sections through the centres of the sides in the revoloid, will be identical with those of the inscribed sphere through the same planes.

Now, if planes be passed through the two solids perpendicular to the vertical or transverse axis, those planes will cut the solids in the relation of their solidities through such sections; and as all such sections (Prop. I. Cor. 4.) are similar figures and similar to its prime, and as all the corresponding sections of the sphere are similar figures, viz., circles; and because the inscribed sphere touches the surface of the revoloid through their whole vertical sections; each of the conjugate sections of the revoloid are polygons, circumscribing the corresponding circular sections of the sphere. And because an indeffinite number of parallel plane sections may be regarded as a function of the solidities of the bodies through which such sections pass, the sum of a series of the revoloidal sections, equidistant and parallel to each other, is to the sum of a similar series of sections of the sphere, as the solidity of the revoloid to the solidity of the sphere; hence the solidity of the revoloid is to the solidity of the sphere as the area of its prime to the area of its inscribed circle.

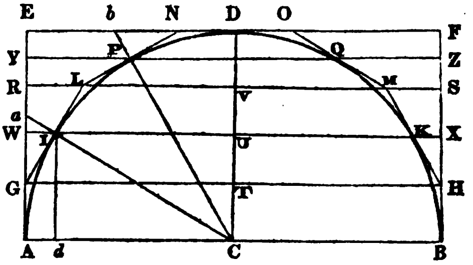
*Cor.* Hence, also, by a parity of reasoning in reference to the perimeters of all the similar polygons formed by the conjugate sections of the revoloid which circumscribe and touch all their corresponding circumferences of the sections of the inscribed sphere; the whole surface of the revoloid is to that of the inscribed sphere as the perimeter of the polygon of the revoloidal section to the circumference of its inscribed circle.

*Scholium.* The proposition and corollary are also manifestly true in reference to any revoloid, and its inscribed solid of revolution, if the surfaces of the two solids correspond through vertical sections through the centres of the revoloidal sides.

### PROPOSITION III. THEOREM.

*The area of each facial side of a right revoloid is equal to that of the corresponding side of its circumscribing prism.*

Let ABDA be a vertical hemisphere of a right quadrangular revoloid, and CD its axis; and let AEFB be a prism circumscribing the hemisphere; then will each facial side of the hemisphere of the revoloid be equal to its corresponding surface of the prism.



For first, let a regular semi-polygon AGLNO, &c. be described about the figure representing the revoloid, and from the angles draw the lines GH, LM, &c., and if this is conceived to be done on every facial side of the revoloid included within such lines as the boundaries of surfaces, it would be a polyedron, whose several faces AGHB, GLMH, LNOM, &c., correspond with, and include the several faces of the revoloid. Now, since by hypothesis the polygon AGLN, &c., circumscribing the figure, is regular the distance across the face of the polyedron, in the plane of a vertical section, is equal in each, viz., TV the width of the face, GLMH is equal VD the width of the face LNOM, and they are each equal to GL or LN; although we cannot so represent those spaces in the diagram in consequence of the curvature in a plane perpendicular to this sheet. But it is evident that the sides GL, IN of the polygon will truly represent the breadth of those faces. And since those several faces are trapezium, they are severally equal to half the sum of their parallel sides multiplied by their altitudes or width. Thus the face MH is equal to $(\frac{1}{2}GH+\frac{1}{2}LM)\times LG$ or VT, and since VT is supposed equal to LG; and since IK is equal $\frac{1}{2}GH+\frac{1}{2}LM$, $IK\times LG$ equal the face GLMH; the face $LNOM=PQ\times LN$, and the face $AGHB=AB\times AG$. Now the areas of the corresponding parallel spaces or sections in the prism being parallelograms, are severally equal to their lengths multiplied by their breadths. Thus the space GRSH corresponding to the face GLMH, is equal $GH\times GR$, and the parallelogram REFS corresponding to the face LNOM is equal $RS\times RE$.

Now let IK be produced each way to W, X and PQ to Y, Z; and through the point I draw the radial lines C*a* C*b*, draw also I*d* perpendicular to AC. Then in the right-angled triangles LRG, *a*IG having an accute angle at G common, they are similar, (Prop. XXII, B. IV. *El. Geom.*), and hence their sides are proportional; and because the right-angled triangle CA*a* has an accute angle at *a* common with the right-angled triangle GI*a*, this triangle is also similar to the former, as also the triangle C*d*I or UIC. Hence $GR : GL :: UI : CI$, and because $IK=2UI$, and AB or $GH=2CI$, we have $GR : GL :: IK : GH$. That is, RG which is a factor of the parallelogram GRSH is to GL, which is the factor of the trapezium GLMH as IK, another factor of GLMH to GH another factor of GRSH; whence, by multiplying extremes and means, we have $GR\times GH=GL\times IK$, viz., the surface of the face GLMH of the polyedron is equal to the corresponding vertical surface GRSH of the prism; and since the same may be shown in reference to any other of the faces with its corresponding portion of the surface of the prism, it follows that the whole sum of the polyedral faces on one side, is equal to the whole cor responding surface of the prism. Let the number of the polyedral faces be indefinitely increased, and the truth of the proposition is still manifest, but when the faces are indefinitely increased, they become assimilated to that of the body about which they are described; therefore, the facial surface of each side of a right quadrangular revoloid is equal to that of the corresponding side of its circumscribing prism.

*Cor.* 1. Let us suppose the revoloid, instead of being quadrangular, to consist of any number of facial sides; then, by hypothesis, its circumscribing prism will consist of an equal number of vertical sides and in the same ratio each to each; and hence the proposition is true for a revoloid of any number of sides. Let, then, the number of sides be indefinitely increased, the reveloid then becomes a sphere, and the circum-

**scribing prism becomes a cylinder. Hence the surface of a sphere is equal to the convex surface of its circumscribing cylinder.**

*Cor.* 2. Hence the surface of a revoloid is equal to the perimeter of its centeral conjugate section multiplied by the vertical axis, for the vertical surfaces of the circumscribing prism are equal to this product; and since in a right quadrangular revoloid its circumscribing prism has six equal sides, four of which are equal to the surface of the revoloid, it follows that the whole surface of the revoloid is to the whole surface of the prism as 2 to 3.

If ↧ represent the perimeter of the central conjugate section, D the vertical axis or diameter, then ↧D will represent the surface, and if H be the altitude of any zone by sections parallel to the conjugate axis, then will H↧=the curve surface of the zone or segment of the revoloid or sphere.

*Cor.* 3. Hence also the surface of a sphere is equal to its circumference multiplied by its diameter or altitude, for the curve surface of its circumscribing cylinder is equal to this product. And since the surface of a great circle of the sphere is measured by the product of its circumference into half the radius, or by $\frac{1}{4}$ the diameter, (Prop. XVI. B. V. *El. Geom.*) Therefore the surface of a sphere is four times the area of its great circle: this is equal to $4\pi R^2$, (Prop. XIII, Sch. 3. *El. S. Geom.*) and because the two bases of the circumscribing cylinder are each equal to one of those circles, it follows that the whole surface of the cylinder is equal to six of those circles, and hence that the whole surface of the sphere is to the whole surface of the cylinder as 2 to 3, as before found in the elements of solid geometry.

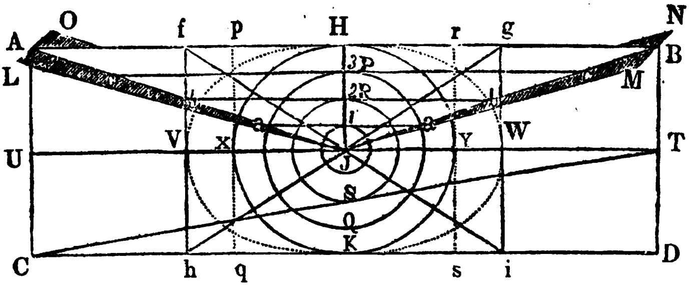
*Cor.* 4. Since we have shown that the surface GLMH of the polyedron is equal to the surface GRSH contained within the same parallel planes, it follows that the surface of any zone or segment LNOM either of a revoloid or sphere, is equal to the perimeter or circumference of a central conjugate section multiplied by the altitude of such zone or segment.

*Cor.* 5. The surface of two zones taken in the same revoloid or sphere, or in equal revoloids or spheres, are to each other as the altitudes; and the surface of any zone, is to the surface of the sphere, as the altitude or diameter of the zone is to that of the sphere. Hence the surfaces of every parallel portion of equal altitude are equal.

### PROPOSITION IV. THEOREM.

*If a cylindrical ungula be cut by two planes from the same side of the cylinder, the intersection of which planes forms a diameter to the cylinder, and if the altitude of the ungula, or the extreme length of the ungula taken in the direction parallel to the axis of the cylinder, is equal to the cylinder's circumference, then the sections or ungulas so cut, will be equal to a sphere described in the cylinder, or to a sphere whose diameter is equal to that of the cylinder.*

Let ABCD be a cylinder, and let HK be a sphere of equal diameter described in the cylinder, and let AJB be an ungula cut from the cylinder by the two planes OLJ, MNJ, from the points A and B, whose distance AB parallel to the axis UT is equal to the circumference of the cylinder, and let the cutting planes meet in J forming a diameter to the cylinder; then will the section AJB be equal to the sphere HK.



For, let planes be passed through the ungula and sphere perpendicular to the diameter formed by the intersection of the planes LOJ and NMJ; and these plane sections, formed by these planes in each solid, will be proportional to the magnitudes of the solid through such sections. Now, any section of the ungula, by a plane perpendicular to the diameter which passes through the pole J of the sphere, is a triangle; and a section through the sphere made by the same plane, is a circle; and the area of a triangle formed by any section, is equal to its base multiplied by half its altitude on such base. Thus the area of the triangle JAB=AB×½HJ; the area of a parallel section J*cc*, is in like manner = the base *cc*×½ the altitude J3; and so for the area of any other parallel section J*bb*, J*aa*, &c.; and because these triangles are equiangular, (Prop. XXII. B. IV. *El. Geom.*) their bases are proportional to their altitudes; thus, HJ : AB : : J3 : *cc* : : J2 : *bb* : : J1 : *aa*, &c. The areas of the several circles formed by the same planes passing through the sphere, are equal to their circumferences multiplied by $\frac{1}{2}$ their several radii; thus the area of the great circle of the sphere HXKY is equal to the circumference HXKY$\times\frac{1}{2}$ the radius HJ, the area of the circle PQ corresponding to the section J*cc*, is equal to the circumference PQ$\times\frac{1}{2}$ radius JP, and the areas of the circles RS, &c., = their several circumferences multiplied by $\frac{1}{2}$ their several radii. And because the circumference of circles are to each other as their radii, (Prop. XV. B. V. *El. Geom.*) as radius JK : circle HK : : radius J3 : circle PQ : : radius J2 : circumference RS, &c.; and, since this is the ratio of the lines AB, *cc*, *bb*, &c., as shown above, the several circumferences are in the same ratio of those lines as bases of their several triangles; but the line AB by hypothesis, is equal to the circumference HK; hence the several circumferences PQ, RS, 1J, &c., are respectively equal to the several bases *cc*, *bb*, *aa*, &c.; hence, also, the areas of the several circles being sections of the sphere, are respectively equal to their several corresponding triangles, being sections of the ungula made by the same planes; and as this is true whatever may be the number of the parallel sections, or in whatever position they are taken, it follows that the solidity of the ungula is equal to that of the sphere.

*Cor.* 1. Since the section AJB may be regarded as composed of the two unglas AJH, BJH, regarding JH as their common base, and because ungulas of the same base are proportional to their altitudes, it follows that if we cut the ungula H*g*J, whose altitude H$g=\frac{1}{2}$ HB, and the ungula HJ*f*, whose altitude H$f=\frac{1}{2}$ HA, then the section *f*J*g* including those ungulas together with an equal opposite section, *h*J*i* are equal to the section AJB, consequently equal to the sphere HK. Or if we take the section CTD, whose base CD=AB, this section will also be equal to the sphere HK.

*Cor.* 2. Since, it may be shown, that all sections of a spheroid or ellipsoid, by planes parallel to its axis of revolution, are similar ellipses; and since ellipses are to their inscribed circles, as the diameter of the circle to the major axis of the ellipse, (Proposition IX of the Ellipse, B. I,) the solidity of an ellipsoid HWKV is to the solidity of the sphere HYKX, as the axis of revolution VW of the ellipsoid is to the axis of revolution XY of the sphere, and hence if an ungula, whose base is equal to half the base of the cylinder be taken, and whose altitude is to that of the ungula CTD as the axis of revolution VW of the ellipsoid to the axis of revolution XY of the sphere, that ungula will be equal to the ellipsoid HWKV made by the revolution of the ellipse on its axis VW.

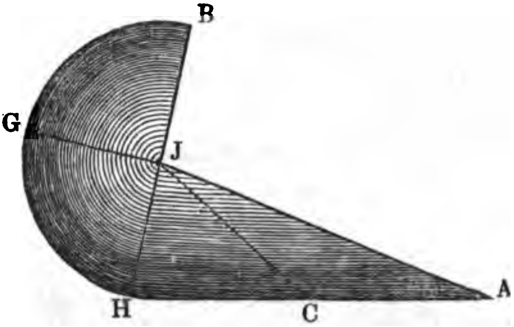
*Cor.* 3. As the ellipsoid HWKV is to the sphere HYKX as the axis VW of the ellipsoid to the axis XY of the sphere, and as the cylinder *fhig* circumscribing the ellipsoid is to the cylinder, *pqrs* circumscribing the sphere, as the same axis VW to the same axis XY, or as the length of those cylinders resrespectively; the ellipsoid HWKV bears the same ratio to its circumscribing cylinder *fhig*, as the sphere HYKX to its circumscribing cylinder *pqrs*. If S=the sphere, and E=the ellipsoid, and if P=the cylinder circumscribing the ellipsoid, and Q=the cylinder circumscribing the sphere,

then, XY : S : : VW : E

and, XY : Q : : VW : P,

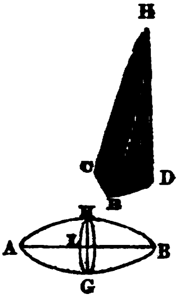
hence, (Prop. XIX. B. I. *El. Geom.*) S : Q : : E : P.

*Scholium.* The segment AJB being equal to the sphere HK, the annexed figure may represent the manner in which they

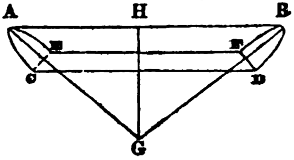


are convertible into each other, as there exists no mathematical reason why the segment AJB may not be changed, as partly represented in the figure.

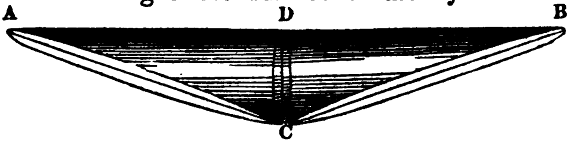
*Cor.* 4. It is also evident that an ungula BCDH, whose base BCD is the segment of a circle, similar to a segment AHB of a vertical section through this circular spindle AHBG; if the altitude DH of the ungula is equal to the circumference of the conjugate section HG of the spindle, the solidity of the ungula will be equal to that of the spindle, so also will its curve surface.

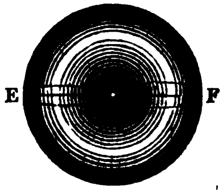


And, if a cylinder segment ABCDH be so cut that the length AB shall be equal to the circumference of a circle of which HG is the radius, and AG and BG lie in the planes ACE, BFD, then may the segment be converted into a ring whose outside diameter is equal 2HG; and every section of the ring will be equal to a section through the segment.

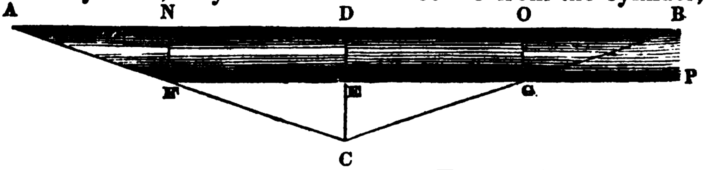


*Cor*, 5. Hence, also, if a cylindric segment ABC be cut by two planes meeting in the surface of the cylinder at C, and terminating at A and B on the opposite side, the distance of which points from each other is equal to the circumference of a circle whose radius is CD; the segment so cut may be changed in the form represented by EF, which is the form of a cylindrical ring, but without an opening through the centre.

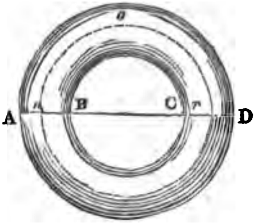




And if instead of the cutting planes meeting in the surface of the cylinder, they meet at a distance EC from the cylinder, and at such an angle that the distance FG shall be equal to the circumference of a circle whose radius is CD, and the section AFGB will form a cylindric ring, whose inner diameter is equal to twice CE.



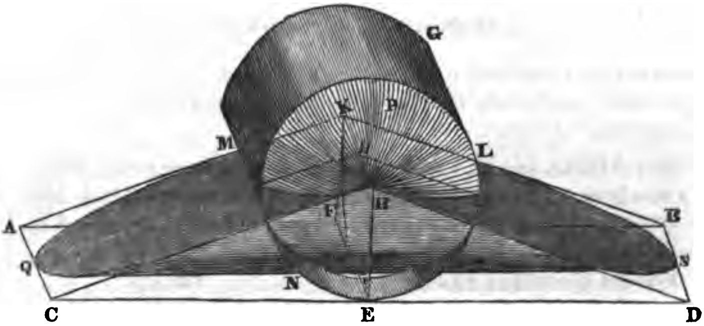
*Cor.* 6. And because the section ANF, (see diagram above,) if cut from its position and placed in the position BGP, completes the cylinder NFPB, which is equal in length to half the sum of sides FG and AB. The solidity of a cylindric ring AB, is equal to that of a cylinder whose base is equal to a radial section of the ring, and whose altitude is equal to half the sum of its inner and outer circles. And hence, cylindric rings whose sections are equal, are proportional to their inner or outer circumferences.



### PROPOSITION V. THEOREM.

*The solidity of a cylinder circumscribing a sphere, is equal to the solidity of a prism circumscribing the cylindrical ungula or ungulas whose solidity is equal to the sphere.*

Let MNGL be a cylinder circumscribing the sphere PK; and let CRDBKA be a prism circumscribing the two similar ungulas QRKV, SRKV described on the base, KRV=half the base of the cylinder from which they are conceived to be taken; and if QS, the sum of their altitudes, is equal in length to the circumference of the cylinder, then (Prop. IV.) will the ungulas equal the sphere of equal diameter to that of the cylinder, and the cylinder equal MNLG will be equal to the prism CRDBKA.



For the altitude AC or KR of the prism, on the base CRD, is equal to the altitude of the cylinder; GL being=the axis, both of the cylinder and the sphere; and the length of the side CD of the prism is equal to the sum of the altitudes VQ, VS, or the length QS of the ungulas taken on the surface of the cylinder; but the length QS is equal to the circumference of

*El. Geom.*) That is, the section of the prism is equal to both the sections of the revoloid and pyramid. And because circles described on the diameters SM, PN, OL, are proportional to the squares described on those diameters, respectively, (Prop. XIV, B. V, *El. Geom.*) it follows that the circle OALB, is equal to the two circles PCND, SEMF; that is, the section of the cylinder is equal to the two sections of the sphere and cone. And because, (Prop. I, Cor. 4,) the sections of a revoloid, of its circumscribing prism, its inscribed pyramid of any number of sides, are similar figures; they are, (Prop. XIV, Cor. 3, B. V, *El. Geom.*) proportional as the squares of the diameters of their inscribed circles, hence the section of a prism of any number of sides circumscribing a revoloid, is equal to the sum of the corresponding sections of the revoloid and pyramid, and as this is always true in any parallel position of the revolving line KL, (Fig. 1.) it follows, that the prism, EB, circumscribing the hemisphere DAB of the revoloid, being composed of all the former sections is equal to the hemisphere DAB of the revoloid and pyramid EIF, composed of all the latter sections; and that the cylinder EB, is, in like manner, equal to the hemisphere DAB and cone IEF. But the pyramid IEF is a third part of the prism DEFB; (Prop. XXVI, Cor. 1, B. II, *El. S. Geom.*) consequently, the hemisphere DAB of the revoloid, is equal to the remaining two-thirds. And the cone IEF, (Prop. VIII, Cor. 1, B. III, *El. S. Geom.*) is equal to one-third of the cylinder, DEFB: hence, the hemisphere DAB, is equal to the remaining two-thirds of the cylinder.

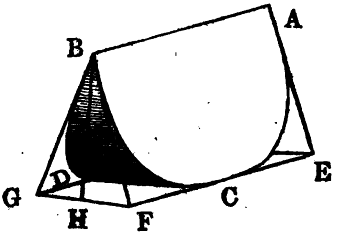
*Cor.* 1. A pyramid, revoloid, and prism, are to each other as the numbers 1, 2, and 3, when the bases of the pyramid and prism are each equal to the prime of the revoloid, and when their altitudes are all equal. Also, a cone, sphere, and cylinder, are, in like manner, proportional as the numbers 1, 2, and 3, if the base of the cone and cylinder are each equal to the great circle of the sphere, and if the altitude of the cone and cylinder are each equal to the diameter of the sphere.

*Cor.* 2. All spheres, and all similar revoloids are to each other as the cubes of their diameters, all being like parts of their circumscribing cylinders or prisms.

*Cor* 3. From the foregoing demonstration, it appears, that the revoloidal frustum DBNP, is equal to the difference between the prism DBLO, and the pyramid SIM, all of the same common height IK. And that the revoloidal segment, PAN, is equal to the difference between the prism, EFLO, and the pyramidal frustum EFMS, all of the same common altitude AK.

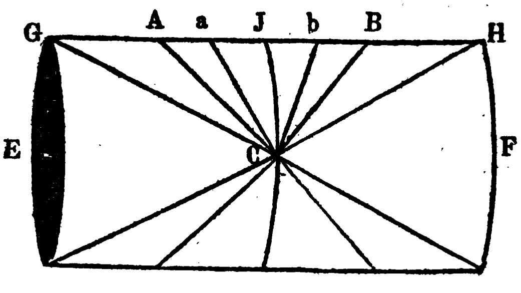
And the same is true of any parallel segment or frustum of the sphere, cylinder and cone.

*Cor.* 4. The sphere may be regarded as a revoloid whose prime has an infinite number of sides, which (Prop. XII, Cor. 4, B. V, *El. Geom.*) is identical with a circle.



*Cor.* 5. From the above demonstration, and from proposition fourth, it may be shown that any ungula, being a portion of the revoloid, is equal to two-thirds of its circumscribing prism or wedge. Thus, if the ungula ABDC be such as forms a portion of a revoloid, it will be equal to two-thirds of its circumscribing prism, BGFEA, since this ungula is the same part of the revoloid as its circumscribing prism, is of the prism, circumscribing the revoloid.

Hence, if from the cylinder EF any ungulas GCH, ACB, GCJ, GC*a*, &c., are taken, whose cutting planes meet in the centre, C, of the cylinder or anywhere in its axis, those ungulas are each equal to two-thirds of their circumscribing prism



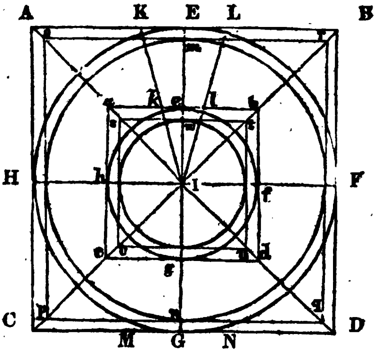
or wedge; these being ungulas such as may compose a right revoloid, and (Prop. IX, Cor. 2, B. II.) they are proportional to their altitudes or lengths taken on the surface of the cylinder; viz: the ungula GH, which may be considered as composed of the two ungulas GCJ, HCJ; their common base being the section through the line CJ, is to the ungula ACB, as the line CH to the line AB, &c. And a prism circumscribing an ungula is evidently proportional to the length of those lines, forming one of the dimensions of one of its sides.

*Cor.* 6. Since all the vertical sections of a right revoloid except those bisecting its sides at right angles, are ellipses, (Prop. I, Cor. 1.) it may be inferred that an elliptical revoloid is also equal to two-thirds of its circumscribing prism; and hence that an ellipsoid is also equal to two-thirds its circumscribing cylinder. For the ellipsoid evidently bears the same relation to the elliptical revoloid as the sphere does to the right revoloid, since the elliptical revoloid may, by multiplying the number of the sides of its prime, be shaded off into the ellipsoid without changing its relation to its circumscribing prism, which in such case becomes a cylinder.

*Scholium.* Lest this latter corollary should not appear sufficiently satisfactory in view of the important principle enunciated; and as it may seem to require a more rigorous demonstration, it will be made the subject of the following proposition.

### PROPOSITION VII. THEOREM.

*An elliptical revoloid is equal to two thirds its circumscribing prism; and an ellipsoid is equal to two-thirds of its circumscribing cylinder.*



Let us imagine the annexed figure ABCD, to be a conjugate section of a right quadrangular revoloid by a plane through its centre, and let *abcd,* be regarded as another conjugate section of the revoloid between the central section and the extremity of the axis, and let them be so projected that the axis I, perpendicular to the plane of projection may be supposed to pass through their centres; and let the circles EFGH, and *efgh* be similar sections of a sphere of the same diameter and similarly posited.

Now let it be conceived that vertical sections or ungulas be taken from each side of the revoloid and sphere; such that KIL, *kIl,* shall be sections of the ungula from the side

AIB, then will the sectoral portions of those triangles be respectively, sections through the spherical ungulas; now let the plane sections of the revoloid and sphere be reduced so as to close the spaces formed by the triangles so removed, but retaining their former figures. Thus let ABCD be reduced to *opqr; abdc* to *stuv*, &c., then, because (Prop. XXIII., B. I., *El. Geom.*) if any number of squares are proportional, the sides of those squares are proportional, it follows that if P be = the length of the side AB of the revoloidal section, and $P'$ = the length of the sides *or*, and if $p$ = the length of the side *ab* and $p'$ = the length of the side *st*; then will $P^2$ = the square ABDC and $P'^2$ = the same square reduced, or *opqr*; $p^2$ will = the square *abdc* and $p'^2$ = the same square reduced, or *stuv*.

And we have shown that $P^2 : P'^2 : : p^2 : p'^2$

Hence also (Prop. XX, B. I, *El. Geom.*) $P : P' : : p : p'$

That is, as the side AB is to the same side reduced, so is the side *ab*, to the same side reduced, and hence the sides, or diameters of those sections are reduced by the removal of the ungulas, in the ratio of those sides, or their diameters respectively, and because the circumferences of circles are as their diameters, and their areas as their squares, the diameters of the circular sections of the sphere, are each reduced in the ratio of these diameters; and the same will hold true with regard to any parallel sections of the revoloid or sphere.

Now the several radii EI, *e*I, &c. of the several sections may be regarded as ordinates to the vertical axis, I, of the revoloid and sphere; and the radii *m*I, *w*I, &c., may be regarded as the corresponding ordinates of the vertical sections of the solids so reduced. Now, since these latter ordinates are severally proportional to their corresponding ordinates of a vertical circular section of the revoloid and sphere, it follows that the same sections made by the same plane, through the solids so reduced are ellipses, for (Prop. XII, *Cor. Ellipse*) this is a property of an ellipse, when compared with a circle; hence the revoloid becomes an elliptical revoloid (Def. 10) and the sphere becomes an ellipsoid.

Now if a prism is supposed to circumscribe the revoloid before being reduced, and if a cylinder is supposed to circumscribe the sphere, they must in order to accommodate themselves to the elliptical revoloid and ellipsoid, be reduced in every conjugate section, equal in amount to the reduction of the central conjugate sections of the revoloid and sphere; and hence the prism will have been reduced in the same ratio as that of the revoloid; and the cylinder will in like manner, have been reduced in the same ratio as that of the sphere, so that if (Prop. VI) a right revoloid is equal to $\frac{2}{3}$ of its circumscribing prism, and a sphere is equal to $\frac{2}{3}$ of its circumscribing cylinder, then also will an elliptical revoloid be equal to $\frac{2}{3}$ of its circumscribing prism, and an ellipsoid will be equal to $\frac{2}{3}$ of its circumscribing cylinder.

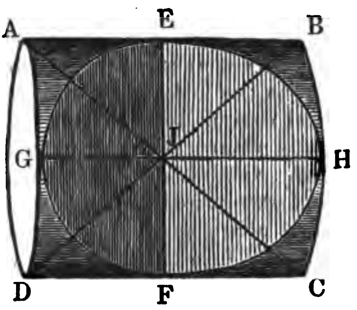
*Cor.* Since an elliptical revoloid is formed of ungulas, cut from an elliptical cylinder (Prop. I, Cor. 2) whose bases are severally the semi-base of the cylinder, it follows that such ungulas of an elliptical cylinder, are equal to $\frac{2}{3}$ of their respective circumscribing prisms.

*Scholium.* Since the solidity of a cylinder may be expressed by $\pi R^2 \times H$ (Prop. II., *Sch.* B. III., *El. S. Geom.*,) that is since its solidity is as the square of its radius or diameter multiplied by its height, it follows that ellipsoids are proportional to each other as the square of their revolving axes multiplied by their fixed axes, and hence the same is true also of the elliptical revoloid, If $R^2\pi H =$ the solidity of a cylinder,
$\frac{2}{3} R^2\pi H =$ the solidity of a spheroid
the solidity of a prism will be $4R^2H$,
and the revoloid will be $\frac{8}{3} R^2H$,
or if $D = 2R$ then the revoloid is $= \frac{2}{3} D^2H$.

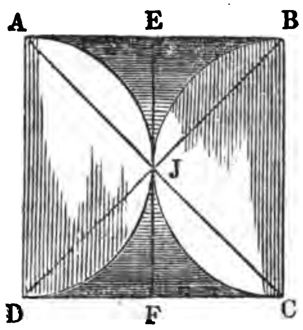
*Cor.* 2. As in Cor. 3, prop. VI, in relation to the segments of a right revoloid, or a sphere, so in relation to the segments of an elliptical revoloid, or spheroid, they are respectively equal to the difference between the corresponding sections of their circumscribing prisms or cylinders, and inscribed pyramids or cones. (See diagrams Prop. VI).

*Cor.* 3. Since a spheroid is equivalent to a sphere drawn out as in the case of a prolate, or contracted, as in the case of an oblate spheroid, and in such manner, as that every line or section through the spheroid, in the direction of the expansion or contraction, is drawn out or contracted, in the ratio of the increase or decrease of the axis in such direction, it follows that any segment of a spheroid, by a plane parallel to its axis of revolution, is to a corresponding segment of its inscribed sphere, if a prolate spheroid, or that of its circumscribing sphere if the spheroid is oblate, as the diameter of that sphere to the axis of the spheroid, or as in the conjugate axis of the segment's base to the transverse of the base, or the axis parallel to the axis of the spheroid.

And any segment by a plane perpendicular to the axis is also proportional to a corresponding segment of the sphere from which it may be conceived to be produced as the axis of the sphere to the axis of the spheroid, or as the height of a similar spherical segment to the height or altitude of the segment of the spheroid.

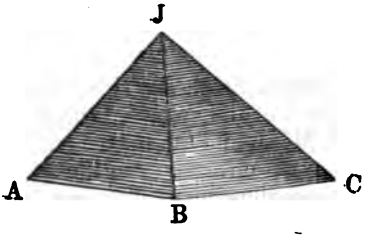


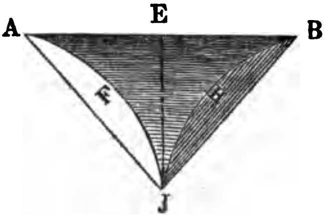
*Scholium.* Let ABCD be the complement of a cylinder from which is taken the ungula GEFH=a quadrant of a revoloid, and also a similar opposite ungula, cut by planes meeting in the diameter EF; and there may be taken two cones whose bases are the two bases of the complement, and whose vertices are in the centre J, and the parts remaining will be equal to the remaining cylindrical surface $\times \frac{1}{3}$ the radius of the base or distance JF. (Prop. IV., B. II.)



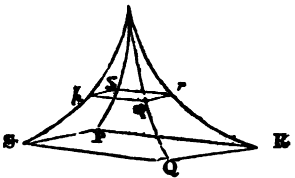
Let the complement be divided in the line EF, and let the segments be inverted so that the bases shall comprehend the line EF, and if the planes ABJ, DCJ cut off the segments AEBJ, DFCJ then there shall be left the pyramid AJ,BJ,CJ,DJ,whose base is equal to a central section of the cylinder along its axis; viz., ABCD, and its vertical height is equal to the radius of the cylinder; and as each side of the cylinder is supposed to be cut alike, we shall have two of those pyramids, which together are equal to one-third of the prism circumscribing the revoloid.

It follows therefore that the two ungulas together with the two pyramids are equal to a full quadrangular revoloid.





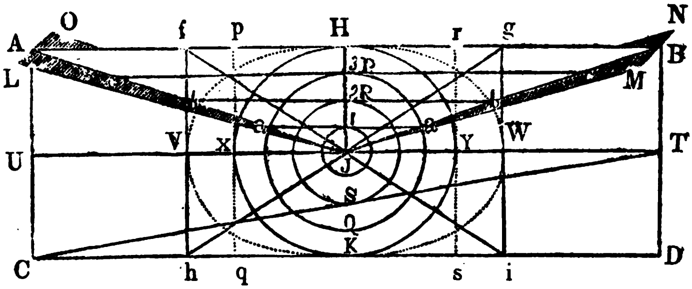
Hence there remains four portions ABJFH to be determined, which when placed together, so that their several vertices J, shall coincide, their cylindric surfaces turned inward, their plane surfaces will be outward, forming a pyramid equal to one of the former pyramids, minus a pyramidal portion PSQRJ, which is required to complete the pyramid. It will be perceived that every section *pqrs* of this latter solid, parallel to the base is a square, and = the square of *pq*, the versed sine of the arc J*p*, therefore this solid is equal to the squares of an infinite series of equidistant versed sines drawn into their distance; or is to its circumscribed prism, erected on the same base PSQR as an infinite series of the squares of equidistant versed sines to a similar series of the squares of radii, as will be more fully discussed in another place.



### PROPOSITION VIII. THEOREM.

*If the solidity of a sphere is equal to one or several cylindrical ungulas of the same cylinder, the surface of the sphere will also be equal to the cylindrical surface of such ungula or ungulas.*

Let HK be a sphere AJB a cylindric ungula equal to the sphere, then will the surface of the sphere be equal to the cylindrical surface of the ungula.



For let an indefinite number of planes be passed through the two solids perpendicular to the axis J of the sphere, formed by the intersection of the planes LOJ, MNJ, and the sphere will be divided into an indefinite number of circles, from the great circle of the sphere down to the smallest about the axis, and the ungula will be divided in like manner into an indefinite number of similar triangles, with bases AB, *cc*, *bb*, &c., which was shown (Prop. IV) to be equal to the circumference of the circles through the corresponding sections of the sphere; and because this is the case throughout, it follows that the surface of the sphere, which may be represented by the sum of the several circumferences of the circles, is equal to the cylindric surface of the ungula, which may likewise be represented by the sum of the several bases of the triangles.

And because cylindric ungulas of the same base are proportional as their altitudes, (Prop. IX, Cor. 2, B. II) and because their cylindrical surfaces are proportional to their altitudes, it follows that if several ungulas, cut from a cylinder of a given diameter are equal to a sphere of the same diameter, the surface of the sphere will be equal to the sum of their cylindrical surfaces.

### PROPOSITION IX. THEOREM.

*The solidity of a sphere as well as a revoloid, is equal to the product of its surface by one-third of its radius.*

For since the revoloid is made upofsections of the cylinder, whose several solidities are equal to their curve surface multiplied by one-third of the radius of the cylinder, whence they are conceived to be taken (Prop. III, B. II) which radius is equal to the vertical height of the several elimentary pyramids of which these sections are formed, and is also the radius of the revoloid, it follows that the solidity of the whole revoloid composed of all the sections, is equal to the whole curve surface of all the sections multiplied by one-third the radius.

Thus, if the revoloid consist of six facial sides $a,b,c,d,e,f$, the solidity of each of which is equal to its surface $\times \frac{1}{3}$ radius or $\frac{r}{3}$, or surface $a \times \frac{1}{3}r =$ solidity $a$, surface $b \times \frac{1}{3}r =$ solidity $b$, &c., their surface $\overline{a+b+c+d+e+f} \times \frac{1}{3}r$ equal to the solidity of the whole revoloid, and as the number of sides of a revoloid may be increased indefinitely without altering the relation of its elimentary pyramids, it follows that the same relation exists between the solidity of the sphere, and its surface, as in the revoloid, viz., the solidity of a sphere, as also of a revoloid, is equal to the product of their respective surfaces by one-third of their respective radii.

*Scholium.* 1. Conceive also a polyedron, all of whose faces touch the sphere; this polyedron may be considerd as formed of pyramids, each having for its vertex the centre of the sphere, and for its base one of the polyedrons faces. Now, it is evident that all these pyramids will have the radius of the sphere for their common altitude; so that each pyramid will be equal to one face of the polyedron multiplied by one-third of the radius; hence the whole polyedron will be equal to its whole surface multiplied by a third of the radius of the inscribed sphere (Prop. XV. Cor. B. III. *El. S. Geom.*) It is therefore manifest that the solidities of polyedrons, as well as revoloids circumscribed about a sphere, are to each other as the surfaces of those polyedrons or revoloids respectively.

Now, also, as with a revoloid, the number of polyedron's faces may be inscribed till the polydron becomes identical with the sphere, and then its solidity is equal to the product of its surface with one-third of its radius; hence the sphere may be conceived to be made up of an indefinite number of indefinitely small pyramidals, whose bases when associated, form the surface of the sphere, and this surface has the same relation to the whole solid, or the sphere, as the base of each individual pyramidal has to the solidity of each.

*Cor.* 1. Hence the solidity of any sector of a sphere or of a revoloid is equal to its spherical or cylindrical surface multiplied by $\frac{1}{3}$ of the radius, for a sector consists of an association of regular pyramidals, the sum of whose bases form the curve surface of the sector.

*Scholium.* 2. Since the axis of the cylinder circumscribing a sphere is equal to its diameter, its solidity is equal to its whole surface, including the two ends multiplied by a third of the radius. For it may be conceived to be made up of two cones, whose bases are two ends of the cylinder, and vertical height = half the axis or length of the cylinder = radius; and the elementary pyramids of the curve surface, whose vertices terminate in the centre of the cylinder with those of the cones, and hence, its solidity bears the same ratio to its surface, that the solidity of a sphere, a right revoloid, or polyedron, circumscribing a sphere do to their respective surfaces.

*Cor.* 2. Hence we have three orders of surfaces, which, taken as bases of pyramids, and multiplied by one-third of the distance of such base to the vertice of the pyramid, will determine the solidity of such pyramid; but, as observed in Schol. to Prop. III, B. III, *El. S. Geom.* in reference to cylindrical surfaces, the vertice of the pyramid with a spherical base, must be in the centre of the spherical curvature.

*Cor.* 3. Hence, as the solidity of a sphere is equal to two-thirds that of its circumscribing cylinder, and as the surface of the sphere is equal to the curve surface of the cylinder; and as the solidities of each of these bodies are equal to the products of their respective surfaces, by one-third of their common radius, the surface of the two ends of the cylinder is equal to half the curve surface, for if $a =$ the surface of the sphere, and $x =$ the two bases of the cylinder, then will $a + x =$ the whole surface of the cylinder, including the ends. Then $a \times \frac{1}{3} r$ or radius = the solidity of the sphere, and $\frac{1}{3} ra + \frac{1}{3} rx =$ the solidily of the cylinder.

But $\frac{1}{3} ra = (\frac{1}{3} ra + \frac{1}{3} rx) \times \frac{2}{3} = \frac{2}{9} ra + \frac{2}{9} rx$.

Transposing and dividing $\frac{1}{3} a = \frac{2}{3} rx$.

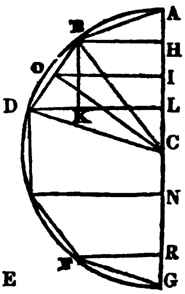
Hence $a = 2x$, therefore the area of the two ends is equal to half the area of the convex surface of the cylinder.

The same may be inferred from the ratio of the inscribed cones, to the remaining portion of the cylinders.

*Scholium.* 3. Since the surface of a sphere whose radius is R, is expressed by $4\pi R^2$ (Prop. III, Cor. 2,) it follows that the surfaces of spheres are to each other as the squares of their radii; and since their solidities are as their surfaces multiplied by their radii, it follows that the solidities of spheres are to each other as the cubes of their radii or diameters, and the same is true also of revoloids. If the diameter is called D, we shall have $R = \frac{1}{2} D$, and $R^3 = \frac{1}{8} D^3$; hence the solidity of the sphere may likewise be expressed by $\frac{4}{3} \pi \times \frac{1}{8} D^3 = \pi D^3$.

### PROPOSITION X. THEOREM.

*Every segment of a sphere is measured by the half sum of its bases multiplied by its altitude, plus the solidity of a sphere whose diameter is this same altitude.*



Let BH, DL, be the radii of the two bases of a segment, HL its altitude, the segment being generated by the revolution of the circular zone DLHB, about the axis AG passing through the centre of curvature C; from C draw CO perpendicular to the chord DB, draw also the radii CD, CB. The solid described by the section BCD is measured by $\frac{2}{3} \pi$, $CB^2$, LH (Prop. XXIII, Sch. 2, B. III., *El. S. Geom.*); but the solid described by the isosceles triangle DCB, has for its measure $\frac{2}{3} \pi$ . CO . LH, (Prop. XVII, Cor. B. III, *El. S. Geom.,*); hence the solid described by the segment BDO = $\frac{2}{3} \pi$ . LH .

$(CB^2—CO^2)$. Now, in the right angled triangle CBO, we have $CB^2 - CO^2 = BO^2 = \frac{1}{4} BD^2$; hence the solid described by the segment BDO, will have for its mearsure $\frac{2}{3} \pi . LH . \frac{1}{4} BD^2$, or $\frac{1}{6} \pi BD^2 LH$.

Again, the solid described by the trapezeium BDLH is $= \frac{1}{3} \pi LH . (BH^2—DL^2+BH . DL,)$ (Prop. X, B. III, *El. S. Geom.*) Hence the segment of the sphere, which is the sum of those two solids, must be equal to $\frac{1}{6} \pi . HL, (2BH+2DL+2 BH . DL+BK^2.)$ But since BK is parallel to HL, we have $DK = DL — BH$, hence $DK^2 = DL^2 — 2DL . BH + BH^2$ (Prop. IX, B. IV., *El. Geom.*); and consequently $BD^2 = BK^2 + DK^2 = HL^2 + DL^2 — 2DL . BH + AH^2$. Substitute this value for $BD^2$ in the expression for the segment, omitting the parts which cancel each other, we shall obtain for the solidity of the segment $\frac{1}{6} \pi HL . (3BH^2 + 3DL^2+HL^2)$, an expression which may be decomposed into two parts; the one $\frac{1}{6} \pi . HL$, $(3BH^2+3DH^2)$, or $HL . \frac{(\pi BH^2+\pi DL^2)}{2}$ being the half sum of the bases × by the altitude; while the other $\frac{1}{6} \pi . HL^3$ represents the sphere of which HL is the diameter. The same may be proved of any other segment DE, EF, &c.; hence the proposition is manifest.

*Cor.* 1. If either of the bases is nothing, the segment in question becomes a spherical segment, with a single base; hence any spherical segment with a single base is equivalent to half the cylinder, having the same base and altitude + the sphere of which this altitude is the diameter.

*Cor.* 2. If IC is perpendicular to AC the solid described by the revolution of the segment about the axis AC, is a ring, when if DC or BC be the radius of curvature of the arc DB of the segment, it is (Def. 26) a spherical ring, and in such case the ring is equivalent to a sphere, whose diameter is the altitude of the segment from which the ring is taken.

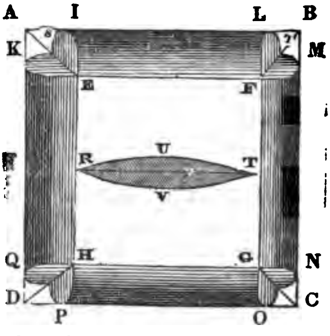
*Cor.* 3. We may hence infer that every segment of a right revoloid included between two parallel planes perpendicular to its transverse axis, is also measured by the half sum of its bases multiplied by its altitude, plus the solidity of a similar revoloid, whose diameter is this same altitude; and that the solidity of the belt ABCD, (see diagram to next proposition,) taken from the middle segment or zone of a revoloid, is equivalent to a right revoloid of similar type, whose diameter is this same altitude.

*Cor.* 4. It may be inferred also that every segment of an elliptical revoloid, or of an ellipsoid, cut by planes perpendicular to their vertical or transverse axis, is equal to half the sum of its bases multiplied by its altitude + the solidity of a similar revoloid or spheroid, whose vertical axis is this same altitude, since the spheroid is the sphere drawn out as in the prolate, and constructed as in the oblate spheroid, and since the same is true of the elliptical revoloid, compared with the right revoloid ; and hence that the solidity of a belt taken from the middle zone or segment of an elliptical revoloid is equivalent to a similar revoloid, whose vertical axis is the altitude of the segment or breadth of the belt, and the same is also true of the ring taken from the middle segment of a spheroid, viz., that it is equivalent to a similar spheroid, whose axis of revolution is the altitude of the segment or breadth of the ring.

### PROPOSITION XI. THEOREM.

*The ungular portions of a revoloidal belt taken from the middle zone of a quadrangular revoloid, are together equal to a revoloidal spindle, whose verticle section through its opposite sides is a double segment of a circle, equal to the segment constituting a conjugate section of the belt.*

Let ABCD be a belt taken from the middle zone of a quadrangular revoloid, and let its conjugate section IES or KES be the segment of a circle ; and the angular portions KAIE, LBMF, NCOG, and PDEQH will be equal to the revoloidal spindle RUTV whose vertical section through the centre of its opposite sides is a similar double segment of the same or an equal circle.



For, let the prismatic cylindric ungulas IESLF, FMNG, OGPH, HQ KES, be removed, and let these several angular spaces be brought in juxta-position by the contact of their corresponding faces, thus let FLM be brought in the position IES, so that LFT shall coincide with IES, let GNO and HQP be brought in contact with FML and EIK, so that GN, HQ, shall coincide with FM, EK. For since the faces of these angular portions are all similar and equal to each other, they would coincide when placed in contact each to each, and because the corresponding faces are parallel to each other in the belt, they would, when placed in contact, form the same angular space as in the belt, and because they form all the angular space about the axis EI when in contact they will form a body, whose conjugate section is a square, and whose vertical section through the centre of its sides is the double segment of a circle, which (Def. 24) is a revoloidal spindle.

*Cor.* 1. Hence, we may infer that the angular portions of any revoloidal belt taken from the middle frustum or zone of a revoloid of any number of sides, is equal to a revoloidal spindle of the same number of sides, and whose vertical section through its opposite sides is equal to a double segment of the circle composing a section of the belt. And since the angular space is the same in a circular ring as in a circle or about a point, (Prop. XXI. B. V. *El. Geom.*) it follows that the angular space of a belt or ring taken from the middle zone of a sphere, is equal to a circular spindle formed by the revolution of a section of the ring, being a circular segment about its chord as an axis.

*Cor.* 2. Therefore, a belt from the middle zone of a revoloid of any number of sides, may be resolved into prismatic ungulas equal in number to the number of the sides of the revoloid, the bases of which are severally equal to segments of the circle forming a conjugate section of the belt, and whose altitudes are each equal to the length of the sides of the polygon forming the inner portion of the belt; and one perfect revoloidal spindle whose vertical section through the centre of its opposite sides, is the double segment of the circle, each equal to the segment formed by a section of the belt.

Hence, the solidity of a revoloidal belt is equal to that of a prismatic cylindrical ungula, whose base is the section of the belt and whose altitude is equal to the perimeter of the inner surface of the belt, plus a revoloidal spindle whose vertical section through its opposite sides is the double segment of the section of the belt.

*Cor.* 3. The solidity of a ring taken from the middle zone of a sphere, is equivalent to that of a prismatic cylindrical ungula, whose base is the section of the ring, and whose altitude is equal to the inner circumference of the ring, plus a circular spindle formed by the revolution of the segment, forming a section of the ring about its chord as an axis.

*Scholium* 1. Let Z = the middle zone of a revoloid, P =

the prism formed by taking a belt from this zone; then will $Z - P$ = the belt.

Let $S$ = the segment formed by a section of the belt, and $p$ = the perimeter of the inner surface of the belt, and $S$ = the solidity of the spindle formed by the angular portions of the belt; then will $s \times p$ or $sp$ = the prismatic cylindrical ungulas, and $Z - P - sp = s$ = the revoloidal spindle.

*Scholium* 2. Let $Z$ = the middle frustum of a sphere, $P$ = the cylinder after taking away the ring whose section is the segment of the circle of which the spherical surface of the segment is formed, and $Z - P$ = the ring.

Let $S$ = the segment formed by a section of the ring and $\pi$ = the inner circumference of the ring, and $s\pi$ will = the prismatic cylindrical ungula which is conceived to form a portion of the ring. Let $s$ = the spindle formed from the angular portions, and $Z - P - s\pi = s$ = the circular spindle.

*Scholium* 3. Let the length or perimeter of a revolidal belt or spherical ring, taken from the middle zone of a revoloid or sphere, be increased while the conjugate section remains the same, and the portion which may be resolved into the prismatic cylindrical ungula is increased in the same ratio: but the angular portion which we resolve into the spindle, remains constant.

Therefore, rings formed from portions, whose sections are a similar segment of a circle, are not proportional to the circumferences of the rings, unless the chord of the segment is perpendicular to the axis of rotation of the segment generating the rings. (See Cor. 6th. Prop. IV.)

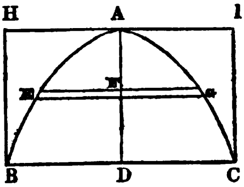
*Cor.* 4. If a belt or ring be taken from a middle frustum or zone of an elliptical revoloid or an ellipsoid, the belt or ring may, in like manner, be resolved into a prismatic ungula of an elliptical cylinder whose base is the section of the belt or ring, and whose altitude is its inner circumference; and an elliptical spindle, such as would be formed by the revolution of the elliptical segment, formed by a section of the belt or ring around its chord as an axis, which spindle is equal to all the angula space contained in the belt or ring.

*Scholium.* 4. The same remarks as were used in reference to segments Prop. X., Cor. 4, are applicable to the belt or ring from a middle zone or frustum of an elliptical revoloid or an ellipsoid, viz: that a belt from a middle zone of an elliptical revoloid is equivalent to a similar revoloid whose axis is equal to the altitude of that of the zone, and that a ring from the middle zone of an ellipsoid is equivalent to a similar ellipsoid whose axis is equal to the altitude of the segment; and this is true whether it is an oblate or prolate revoloid or ellipsoid.

*Scholium.* The middle frustum of a circular or an elliptical spindle, may be resolved into two portions; a cylinder whose base is one of the bases of the frustum, and its altitude equal to the length of the frustum; and a ring remaining after this cylinder is withdrawn. The ring may be computed by finding what portion it is of a spherical ring, and the cylinder may be computed according to its dimensions.

### PROPOSITION XII. THEOREM.

*A vertical parabolic revoloid is equal to half its circumscribing prism; and a vertical paraboloid is equal to half its circumscribing cylinder.*



Let ABC be a vertical section of a segment of a quadrangular parabolic revoloid, or of a paraboloid, and let BCHI be a similar section of the prism circumscribing the revoloid, or cylinder circumscribing the paraboloid through the same plane HBADIC, and if we suppose the axis AD to be divided into an indefinite number of equal parts through which, and through the solid, if planes EFG are passed perpendicular to such axis, the sections of the revoloid made by such planes would be squares, all of which make up the revoloid, and the sections of the paraboloid would be circles, all of which constitute the paraboloid, which, as those squares and circular sections are indefinitely near together, may be represented as a function of those squares or circles.

Now, because the square described on FG is equal to ¼ of the square on EG, $4FG^2 =$ the area of the square described on EG. But by property of parabola, (Prop. VII., parabola,) $p \times AF = FG^2$ where $p$ denotes the parameter of the parabola; consequently $4p \times AF$ will also express the same square section EG, and therefore $4p \times$ the sum of all the AF's will be the sum of all the square sections, or the same function of the whole content of the revoloid; and because circles are as the squares described on their diameters, the sum of all the AF's$\times \pi$ will be a similar function of the whole content of the paraboloid.

But all the AF's form an arithmetical progression, beginning at 0 or nothing, and having the greater term, and the sum of all the terms, each expressed by the whole axis AD.

And since the sum of all the terms of such progression is equal to $\frac{1}{2}$ AD, multiplied by AD, or $\frac{1}{2}$ $AD^2$, half the product of the greatest term, and the number of terms; therefore $\frac{1}{2}$ $AD^2$ is equal to the sum of the AF's, and consequently $4p\times\frac{1}{2}AD^2$, or $2\times p\times AD^2$ is the sum of the revoloid. But by the properties of the parabola $p : DC :: DC : AD$, or $p = \frac{DC^2}{AD}$; consequently $2\times p\times AD^2$, becomes $2\times AD\times DC^2$ for the solid content of the revoloid. But $4\times AD\times DC^2$ is equal to the prism HIBC, consequently the parabolic revoloid is equal to half of its circumscribing prism, and from a parity of reasoning, with regard to the paraboloid and cylinder, the paraboloid is equal to half its circumscribing cylinder.

*Cor.* Hence each of the ungulas of which the parabolic revoloid is composed is equal to half its circumscribing prism, and these ungulas are such as are cut from a parabolic cylinder or prism, by planes meeting in the vertical plane, passing through the vertices of its two parabolic bases.

### PROPOSITION XIII. THEOREM.

*The solidity of a frustum of a polar hemisphere of a parabolic revoloid is equal to a prism of equal altitude, and whose base is half the sum of the two bases of the frustum; and the solidity of a frustum of a polar hemisphere of a paraboloid is equal to that of a cylinder of equal altitude, and whose base is equal to half the sum of the two circular bases of the frustum.*

For in the frustum BEGC last proposition.

$2p\times AD^2 =$ the solid ABC

and $2p\times AF^2 =$ the solid AEG.

Therefore the difference $2p\times(AD^2 - AF^2) =$ the frustum BEGC.

But $AD^2 - AF^2 = DF\times(AD+AF,)$

therefore, $2p\times DF\times(AD+AF) =$ the frustum BEGC.

But, by the parabola $p\times AD = DC^2$, and $p\times AF = FG^2$, therefore $2\times DF\times(DC^2+FG^2) =$ the frustum BEGC, that is frustum BEGC = half the sum DC, FG, of the frustum multiplied by the altitude DF.

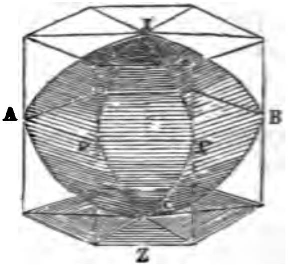
And for the same reasons as adduced in the the last proposition, this demonstration is equally applicable to the frustum of a revoloid or a paraboloid.

*Cor.* Hence any conjugate section of an ungula, being part of a polar hemisphere of a parabolic revoloid, is equal to a prism of equal altitude, and whose base is equal to half the sum of the two bases.

### PROPOSITION XIV. THEOREM.

*A conjugate parabolic revoloid is equal to $\frac{8}{15}$ of its circumscribing prism; and a conjugate parobolic spindle is equal to $\frac{8}{15}$ of its circumscribing cylinder.*

Let AIBC be a parabolic revoloid whose vertical axis is an ordinate to the several parabolic sections, and whose vertices are all in the plane made by the conjugate section ARPB, and the revoloid will be equal to $\frac{8}{15}$ of its circumscribing prism.



For this revoloid is composed of parabolic ungulas, such as are cut from the vertical side of a parabolical prism, which (Prop. XVIII. Cor. 2, B II.) are equal to $\frac{8}{15}$ their circumscribing prisms; hence a number of associated ungulas are equal to $\frac{8}{15}$ their associated prisms, and since the inscribed paraboloid bears the same proportion to its circumscribing cylinder, as the revoloid to its circumscribing prism; hence a conjugate paraboloid or a parabolic spindle is $\frac{8}{15}$ of its circumscribing prism.

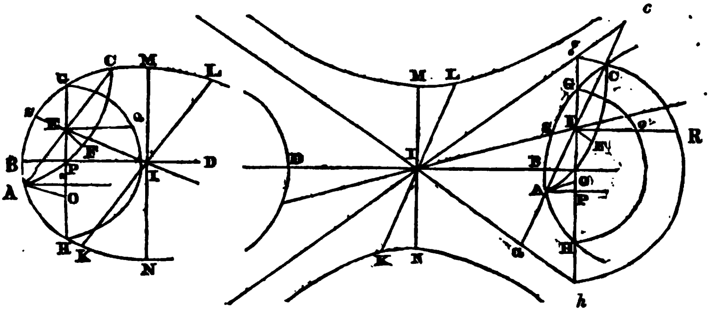
*Scholium.* The frustum of a parabolic spindle may be resolved into three portions—first, a cylinder whose base is one of the bases of the frustum, and whose altitude is the length of the frustum—second, the angular portion of the ring remaining after taking away the cylinder, which is equivalent to a parabolic spindle formed by the revolution of the section of the ring on the chord or double ordinate—third, a parabolic prism, whose base is a section of the ring, and whose altitude is equal to the inner circumference of the ring.

The first is equal to its base multiplied by its altitude. The second is equal to $\frac{8}{15}$ of its circumscribing prism; the third is equal to $\frac{2}{3}$ of its circumscribing rectangular prism.

### PROPOSITION XV. THEOREM.

*If any solid, formed by the rotation of a conic section about its axis, that is a spheroid, paraboloid, or hyperboloid, be cut by a plane in any position; the section will be some conic section, and all the parallel sections will be like and similar figures.*

Let ABC be the generating section, or a section of the given solid through its axis BD, and perpendicular to the proposed section AFC, their common intersection being AC; and GH be any other line meeting the generating section in G and H, and cutting AC in E; and erect EF perpendicular to the plane ABC, and meeting the proposed plane in F.



Then, if AC and GH be conceived to be moved continually parallel to themselves, will the rectangle AE × EC be to the rectangle GE × EH, always in a constant ratio; but if GH be perpendicular to BD, the points G, F, H will be in the circumference of a circle whose diameter is GH, so that GE × EH will be = $EF^2$; therefore AE × EC will be to $EF^2$, always in a constant ratio; consequently AFC is a conic section, and every section parallel to AFC will be of the same kind with it, and similar to it.

*Cor.* 1. The above constant ratio, in which AE × EC is to $EF^2$, is that of $KI^2$ to $IN^2$, the squares of the diameters of the generating section respectively parallel to AC, GH; that is, the ratio of the square of the diameter parallel to the section, to the square of the revolving axis of the generating plane.

This will appear by conceiving AC and GH to be moved into the positions KL, MN, intersecting in I, the centre of the generating section.

*Cor.* 2. And hence it appears, that the axes AC and 2EF of the section, supposing E now to be the middle of AC, will be to each other, as the diameter KL is to the diameter MN of the generating section.

*Cor.* 3. If the section of the solid be made so as to return into itself, it will evidently be an ellipse. Which always happens in the spheroid, except when it is perpendicular to the axis; which position is also to be excepted in the other solids, the section being always then a circle: in the paraboloid the section is always an ellipse, excepting when it is parallel to the axis; and in the hyperboloid the section is always an ellipse, when its axis makes with the axis of the solid, an angle greater than that made by the said axis of the solid and the asymptote of the generating hyperbola; the section being an hyperbola in all other cases, but when those angles are equal, then it is a parabola.

*Cor.* 4. But if the section be parallel to the fixed axis BD, it will be of the same kind with, and similar to, the generating plane ABC; that is, the section parallel to the axis, in a spheroid, is an ellipse similar to the generating ellipse; in the paraboloid, the section is a parabola similar to the generating parabola; and in an hyperboloid, it is an hyperbola similar to the generating hyperbola of the solid.

*Cor.* 5. In the spheroid, the section through the axis is the greatest of the parallel sections; but in the hyperboloid, it is the smallest; and in the paraboloid, all the sections parallel to the axis, are equal to one another.—For, the axis is the greatest parallel chord line in the ellipse, but the least in the opposite hyperbolas, and all the diameters are equal in a parabola.

*Cor.* 6. If the extremities of the diameters KL, MN, be joined by the line KN, and AO be drawn parallel to KN, and meeting GEH in O, E being the midde of AC, or AE the semi-axis, and GH parallel to MN. Then EO will be equal to EF, the other semi-axis of the section.

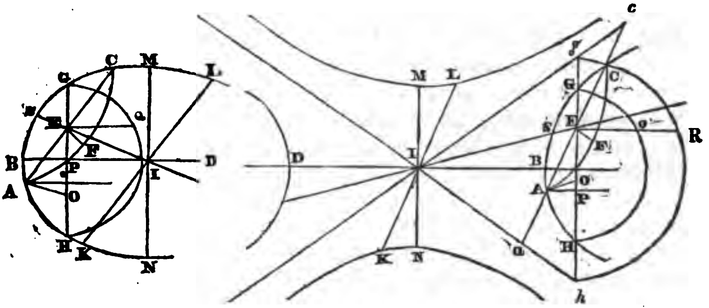
For, by similar triangles, KI : IN : : AE : EO.

Or, upon GH as a diameter, describe a circle meeting EQ, perpendicular to GH, in Q; and it is evident that EQ will be equal to the semi-diameter EF.

*Cor.* 7. Draw AP parallel to the axis BD of the solid, and meeting the perpendicular GH in P. Then it will be evident that, in the spheroid, the semi-axis EF = EO will be greater than EP; but in the hyperboloid, the semi-axis EF = EO, of the elliptic section, will be less than EP; and in the paraboloid, EF = EO is always equal to EP.

*Scholium.* The analogy of the sections of an hyperboloid to those of the cone, are very remarkable, all the three conic sections being formed by cutting an hyperboloid in the same position as the cone is cut.

Thus, let an hyperbola and its asymptote be revolved together about the transverse axis, the former describing an hyperboloid, and the latter a cone circumscribing it ; then let them be supposed to be both cut by a plane in any position and the two sections will be like, similar, and concentric figures ; that is, if the plane cut both sides of each, the sections will be concentric, similar ellipses ; if the cutting plane be parallel to the asymptote, or to the side of the cone, the sections will be parabolas ; and in all other positions, the sections will be similar and concentric hyperbolas.



That the sections are like figures, appears from the foregoing corollaries. That they are concentric, will be evident when we consider that C*c* is = A*a*, producing AC both ways to meet the asymptotes in *a* and *c*. And that they are similar, or have their transverse and conjugate axes proportional to each other, will appear thus : Produce GH both ways to meet the asymptotes in *g* and *h* ; and on the diameters GH, *gh*, describe the semi-circles GQH, *g*R*h*, meeting EQR, drawn perpendicular to GH, in Q and R ; EQ and ER being then evidently the semi-conjugate axes, and EC, E*c*, the semi-transverse axes of the sections. Now if GH and AC be conceived to be moved parallel to themselves, AE × EC or $CE^2$, will be to GE × EH or $EQ^2$, in a constant ratio, or CE to EQ will be a constant ratio ; and since *c*E is as E*g*, and *a*E as E*h* *a*E × E*c* or $cE^2$, will be to *g*E × E*h* or $ER^2$, in a constant ratio, or *c*E to ER will be a constant ratio ; but at an infinite distance from the vertex, C and *c* coincide, or EC = E*c*, as also EG = E*g*, consequently EQ is then = ER, and CE to EQ will be = $c$E to ER ; but as these ratios are constant, if they be equal to each other in one place, they must be always so ; and consequently CE : E$c$ : : QE : ER.

And this analogy of the sections will readily be recognised, when we consider that a cone is a species of the hyperboloid; or a triangle a species of the hyperbola, whose axes are infinitely small.

### PROPOSITION XVI. THEOREM.

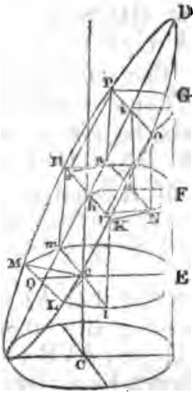
*If* SI *be the semi-diameter belonging to the double ordinate* AEC *of the generating plane,* AEC *being the diameter of the section* AFC, *conceived to be moved continually parallel to itself ; and* x *denote any part of the diameter* SI, *intercepted by* E *the middle of* AC, *and any given fixed point taken in* SI *; then will the section* AFC *be always as* a + bx + cxx ; a, b, c, *being constant quantities ;* b *in some cases affirmative, and in others negative ;* c *being affirmative in the hyperbola, and negative in the ellipse, and nothing in the parabola ; and* a *may always be supposed to denote the distance of the given fixed point from the vertex* s.

In any conic section, $AC^2$ is as $a + bx + cxx$ ; but all the parallel sections are like and similar figures, and similar plane figures, are as the squares of their like dimensions ; therefore the section AFC is as $AC^2$, that is, as $a + bx + cxx$.

*Cor.* If the given fixed point, where $x$ begins, coincide with the vertex $s$, then will $a$ be equal to nothing, and the section will be as $bx \pm cxx$, or as $x \pm dxx$, in the hyperbola and ellipse, and as $bx$, or as $x$, in the parabola.

The subject of hyperbolic revoloids and hyperboloids will be considered in another place ; we will here add a few general scholia and formulæ in relation to cylindric and conical ungulas.

## SCHOLIA AND FORMULÆ IN RELATION TO CYLINDRIC AND CONICAL UNGULAS.



*Scholium.* It has been shown (Prop. VI., Cor. V,) that an ungula pertaining to a right revoloid is equal to $\frac{2}{3}$ its circumscribing prism, and Prop. III, that its curve surface is equal to that of the adjacent side of its circumscribing prism; and because the ungula FHKD is equal to an ungula of a right revoloid, if the intersection HK of the plane FHK, DHK passes through the axis of the cylinder; therefore the convex surface of the ungula FHKG is equal to F$h$ × FD; let GP$o$D be an ungula cut from the former by the plane GP$o$, parallel to the base of the former; and the convex surface of the ungula GP$o$D is = the surface FHKD—the convex surface of the segment HKFGP$o$, and this may be divided into two portions; viz., the curve surface on the segment P$o$N$n$ FG, and the convex surface of the segment $o$N$n$ PHK, the latter of which is equal ($r$K × FD + $n$K × FD) = 2$r$K×FD, and the former is equal the arc PG$o$ ×GF or $o$N.

It will be perceived therefore that, although we have the quadrature of the whole convex surface of a revoloidal ungula, and also of any portion NK$o$ in absolute terms, yet any portion $o$GFN, or $o$GPD is known only in terms of the arc of the circle, and consequently depends on the circle's quadrature, but we are enabled by the principles referred to in this scholium, to determine both the surface and solidity of any portion PBF, with the same degree of accuracy as we have the surface and solidity of the whole cylinder, from which the ungula is derived, and since a revoloidal spindle is conceived to be made up of partial ungulas, the revoloidal spindle is susceptable of the same degree of accuracy in its determination.

Let FD = $h$ the altitude, and $h$G = H$h$ = $r$ the radius of the cylinder from which the ungula is derived; the whole convex surface of the ungula is = $2rh$ - - - - (1)

Let $k$ = the altitude of the ungula GP$o$D, and let $oi$ = half the chord $o$P = $c$, then will the surface ONK = $(r—c) \times k$ = $rh—ch$ - - - - - - - - - - (2)

Let the arc PG$o$ = $p$, and the surface $o$NGFP$n$, will be = $p\times(h—k) = ph—pk$ - - - - - - - (3)

Hence the surface of the ungula PG*o*D $= 2rh - rh + ch - ph + pk = rh + ch - ph + pk$ - - - - - - (4)

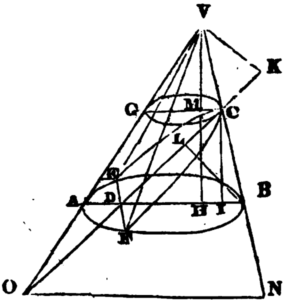
Let an ungula ELMD whose base is greater than half that of the cylinder be considered: if EF = FG, then will the convex surface of this, equal that of the ungula FKHD + the arc (HFK or $\frac{1}{2}\pi r$) × EF + $\frac{1}{2}\pi r$ (KL or N*o*) $- rh + ch - ph + pk$ - - - - - - - - - - (5)

The solidity of the ungula FKHD is $= 2r^2h$ - - (6)

Let $a =$ the base PG*o* of the ungula GP*o*D, and the solidity will be $(rh + ch - ph + pk)\frac{1}{3}r - a \times \frac{1}{3}(h - k)$ (Prop. V, B. II.) $= \frac{1}{3}r^2h + \frac{1}{3}rch - \frac{1}{3}rph + \frac{1}{3}rpk - \frac{1}{3}ah + \frac{1}{2}ak$ (7)

If A equal the base of the ungula ELMD, and if S = its curve surface, its solidity will be $S \times \frac{1}{3}r + A \times (\frac{1}{3}h - \frac{1}{3}k) = \frac{1}{3}Sr + \frac{1}{3}hA - \frac{1}{3}kA$ - - - - - - (8)

*Sch.* 2. Let AEBF be the base of a cone or any other pyramid, right or oblique; AVB a section through the vertex by a plane perpendicular to the base; EVF, ECF two other sections perpendicular to AVB, the former through the vertex, and the latter through the side at C, between V and B. On AB let fall the perpendiculars VH, CI; and on DC the perpendiculars VK, BL, draw CG parallel to AB, meeting AV and VH in G and M.



Then it is evident that EFBV is a pyramid, whose base is EFB, and whose altitude is VH; let the base be called A and the altitude $a$, and the solidity will be $\frac{1}{3}Aa$; and it is evident that EFCV is a pyramid whose base is EFC and altitude VK; let the base be called B and the altitude $b$, and its content will be $\frac{1}{3}Bb$, it is evident also that the ungula EFBC $= \frac{1}{3}Aa - \frac{1}{2}Bb$.

But, by similar triangles, ABV, GVC, it is

AB−CG : CI or HV−VM : : AB : HV, or $a = \frac{AB \times CI}{AB - GC}$;

also AB—GC : CI : : AB : HV : : GC : VM $= \frac{GC \times CI}{AB - GC}$;

and DC : BD : : (by the similar triangles ICD, DBL) CI : BL : : (because of the similar triangles BCI and CVM, VKC and CBL) VM $= \frac{GC \times CI}{AB - GC}$ : VK $(b) = \frac{GC \times CI \times DB}{DC \times (AB - GC)}$

Therefore the ungula EFBC will be

$$= \frac{\frac{1}{3}CI}{AB-GC} \times (A \times AB - B \times \frac{GC \times DB}{DC}); \quad - \quad - \quad (1)$$

which is a general formula for the ungula of any pyramid.

If the base be circular, or the pyramid a cone, and the angle CDB be less than the angle VAD; or which is the same, if CD and VA, produced, intersect in N; the section ECF will be a segment of an ellipse, whose transverse axis is CN, and conjugate $\surd$ (NO $\times$ GC), NO being drawn parallel to AB, and meeting VB produced in O. And then the formula will become $\frac{\frac{1}{3}CI}{AB-GC} \times$ (AB $\times$ circular segment EBF $- \frac{GC \times DB}{DC} \times$ elliptic segment ECF) $= \frac{\frac{1}{3}CI}{AB-GC} \times$ AB $\times$ circular segment EBF $- \frac{GC \times DB \times \surd (NO \times GC)}{DC \times CN} \times$ circular segment, whose diameter is CN, and height CD) =, since sim. seg. are as the squares of their diameters, $\frac{\frac{1}{3}CI}{AB-GC} \times$ (AF $\times$ circular segment EBF$-\frac{GC \times DB \times CN \times \surd NO \times GC}{DC \times AB^2} \times$ segment of the circle AEBF whose height is $\frac{AB \times DC}{CN}$) = the content of the elliptic ungula EFCB - · - - (2)

But, by similar triangles, GC—AD : DC : : GC : CN $= \frac{GC \times CD}{GC-AD}$, and GC—AD : DB : : GC : NO $= \frac{GC \times DB}{GC-AD}$, which values of NO and NC being substituted in the above expression of the elliptic ungula, will give

$$\frac{\frac{1}{3}CI}{AB-GC} \times [AB \times \text{circular segment EBF} - \frac{GC^3}{AB^2} \times \frac{DB}{(GC-AD)}^{\frac{3}{2}} \times \text{segment of the circle AEBF whose height is } \frac{AB \times (GC-AD)}{GC}] = \frac{\frac{1}{3}h}{D-d} \times [D \times \text{circular segment EBF} - \frac{d^3}{D^2} \times (\frac{DB}{BD-D+d})^{\frac{3}{2}} \times \text{segment of the circle AB whose height is } \frac{D \times (DB - D + d)}{d}]. \quad - \quad - \quad - \quad - \quad (3)$$

= the elliptic ungula EFCB; putting *h* for the height of the ungula, D and *d* for the diameters of the base and end, or top, of the frustum, respectively.

If this value be taken from $\frac{1}{3}hn \times \frac{D^3-d^3}{D-d}$ = the whole conic frustum, the remainder will express the complemental elliptical ungula EFCGA $\frac{\frac{1}{3}h}{D-d} \times (D^3-d^3) \times n$ − D circular segment, whose height is $\frac{BD}{D} + \frac{d^2}{D^2} \times \left(\frac{BD}{BD-D+d}\right)^{\frac{3}{2}} \times$ segment whose height is $\frac{BD-D+d}{d}$, or $\frac{\frac{1}{3}h}{D-d} \times -nd^3 + D \times$ segment whose height is $\frac{AD}{D} + \frac{d^2}{D^2} \times \left(\frac{BD}{BD-D+d}\right)^{\frac{3}{2}} \times$ segment whose height is $\frac{BD-D+d}{d}$ - . - - - - - - - (4)

If the points D and A coincide, the section EFC becomes, a whole ellipse, and the formula above, become $\frac{1}{3} Dhn \times \frac{D^2-d\sqrt{Dd}}{D-d}$ = the elliptical ungula ACB - - - - - - (5)

And the complemental ungula ACG = $dhn \times \frac{D\sqrt{Dd-d^2}}{D-d}$ - - - - - - - - - - - - - - - - (6)

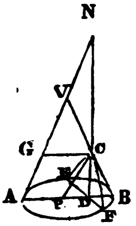
If the angle CDB be equal to the angle VAB, the section will be a parabola, whose axis is CD, and base EF $=2\sqrt{(AD \times DB)} = 2\sqrt{(D-d) \times d}$, and its area, by prop. VI, B. I, $= \frac{2}{3}DC \times EF = \frac{4}{3}DC\sqrt{(D-d)\,d}$; and therefore the expresssion becomes $\frac{\frac{1}{3}h}{D-d} \times (D \times \text{segment EBF} - \frac{Dd-d^2}{DC} \times \frac{4}{3}DC\sqrt{Dd-d^2}$ = the parabolic ungula EFBC - - - (7)

If this be taken from $\frac{1}{3}hn \times \frac{D^3-d^3}{D-d}$, the remainder will express the complemental ungula EFCGA, viz.

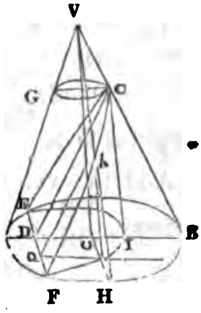
$$\tfrac{1}{3}h \times \left(\tfrac{4}{3}d\sqrt{Dd-d^2} - \frac{nd^3}{D-d} + \frac{1}{D-d} \times \text{segment whose height is } \frac{d}{D}\right)$$

$$\text{or } \frac{\frac{1}{3}h}{D-d} \times \left((D-d)\,\tfrac{4}{3}\sqrt{d\,(D-d)} - nd^3 + D \times \text{segment whose height is } \frac{D}{d}\right) \quad \text{- - - - - - - - - - - - - -} \quad (8)$$

If the angle CDB exceed the angle VAB, the section will be a an hyperbola, whose transverse axis is CN, but the transverse axis $= \frac{CG\times CD}{PD}$ CP being drawn parallel to VA, or $\frac{d\times CD}{D-d-DB}$ and the conjugate also $=GC\sqrt{\frac{DR}{PD}}=d\sqrt{\frac{DB}{D-d-DB}}$ and the area of the hyperbolic section substituted in the general formula (1) will give the solidity of the hyperbolic ungula.



*Scholium* 3. If from the perimeter of the section EFC there be projected, the surface EIF by perpendiculars *h*P,CI, &c., to the base of the cone, the surface EBFIE on the base, will be to that, perpendicular above, viz., ECFB on the surface of the cone as OB to AV; as the radius of the base to the slant height of the cone, (Prop. XII, B. II.,) moreover the area of the section EFC is to the area EIF as DC : DI; hence the convex surface FCEB $= S = \frac{BV}{BO}\times$ EBFIE - - - - - - - - - - - - - - - - - - (1)



And the area FIE $=\frac{DI}{DC}\times$ FCEF

Let ECEF=B and the expression for the surface of the base FIE of the ungula FIEC becomes $\frac{DI}{DC}\times B$ - - - - (2)

Let the base EFB of the ungula EFBC be called A, and the area EBFJE will be $A-\frac{DI}{DC}\times B$ - - - - - - - (3)

And consequently $S=\frac{CB}{IB}\times A-\frac{DI}{DC}\times B$ - - - (4)

If from $(\frac{VB}{OB}\times$ base AFBE of the cone) = the convex surface of the whole cone, there be taken that of the ungula found above; the remainder $\frac{VB}{OB}\times(FAEF+\frac{DI}{DC}\times FCEF)$ - (5) will express the convex surface of the remaining part EFCVA of the cone. And if from the value last found, there be taken $\frac{VB}{OB}\times$ circle CG, $=\frac{VC}{OI}\times$ circle GC the convex surface of the

cone GVC, the remainder $\frac{BV}{OB} \times (TAEF \times \frac{DI}{DC} \times FCEF -$ circle $GC)$ - - - - - - - - - - - - - - - - (6)
will express that of the complement EFAGC.

If the section FCE be an ellipse, the surface of the ungula will be $\frac{VB}{OB} \times$ (circular segment FBE $- \frac{DI}{DC} \times$ elliptical segment FCE) ; and hence $\frac{VB}{OB} \times EBF - \frac{GC^2 \times DI}{AB^2 \times (GC-AD)} \sqrt{\frac{DB}{GC-AD}} \times$ (segment circle AB, whose height is $AB \times \frac{GC-AD}{GC}$,) $= \sqrt{\frac{(4h^2+(D-d)^2)}{D-d}} \times \Big(EBF - \frac{d^2}{D^2} \times \frac{DB-\frac{1}{2}(D-d)}{DB-(D-d)} \times \sqrt{\frac{DB}{DB-(D-d)}} \times$ seg. cir. AB, whose height is $D \times \frac{DB-D-d}{d}$ - - - - - - - - - - - (7)

And the value of the surface EFCVA, (Formula 5,) will become, $\frac{VB}{OB} \times$ (circ. seg. FAE $\times \frac{DI}{DC} \times$ ellip. seg. FCE) (8)

Or it is $= \frac{\surd(4h^2+(D-d)^2)}{D-d} \times (FAE + \frac{d^2}{D^2} \times \frac{\frac{1}{2}D+d-AD}{d-AB} \sqrt{\frac{D-AB}{d-AB}} \times$ seg. circle AB, whose height is $D \times \frac{d-AB}{d}$
- - - - - - - - - (9)

And the value of the surface of the complemental ungula, (Formula 6,) will become $\frac{VB}{OB}$ (cir. seg. FAE $+ \frac{DI}{DC} \times$ ellip. seg. FCE $-$ circle CG) $= \frac{\surd(4h^2+(D-d)^2)}{D-d} \times (-nd^2 + FAE + \frac{d^2}{D^2} \times \frac{\frac{1}{2}(D+d)-AD}{d-AD} \sqrt{\frac{D-AD}{d-AD}} \times$ seg. circle AB, whose height is $D \times \frac{d-AD}{d})$ - - - - - - - - - - - (10)

When D coincides with A the expression will become, $\frac{VB}{BO} \times AB^2 \times n - n \times AB \times \surd(AI \times GC) = \frac{n\surd(4h^2+(D-d)^2)}{D-d} \times (D^2 - \frac{D+d}{2}\surd Dd)$ for the convex surface of the ungula ABC. - - - - - - - - - - - - - - - - (11)

And $\frac{VB}{OB} \times AI\surd(AB \times GC) \times n = 2n \times VB \times AI\sqrt{\frac{GC}{AB}}$

$= \dfrac{n\sqrt{(4h^2 + (D-d)^2)}}{D-d} \times \dfrac{D+d}{2}\sqrt{Dd}$ for the oblique cone ACV. - - - - - - - - - - - - - - - - - (12)

Also $\dfrac{VB \times n}{OB} \times (AI\sqrt{(AB \times GC)} - GC^2)$

$= \dfrac{n\sqrt{(4h^2 + (D-d)^2)}}{D-d} \times (\dfrac{D+d}{2}\sqrt{(Dd - d^2)})$ for the complemental elliptic ungula ACG. - - - - - - (13)

If DC be parallel to AV, or the section a parabola; since its area B is $\frac{4}{3}$ DC × DF $= \frac{4}{3}$ DC $\sqrt{(AD \times DB,)}$ the general formula for the ungula will become

$\dfrac{VB}{OB} \times$ seg. (FBE $- \frac{4}{3}$ DI $\sqrt{(AD \times DB)}) = \dfrac{\sqrt{(4h^2+(D-d)^2)}}{D-d}$ × [seg. FBE to height DB $- \frac{2}{3}(D-d)\sqrt{(d(D-d))}$] for the convex surface of the parabolic ungula FEBC. - - - (14)

And the expression in Formula 5 will become,

$\dfrac{VB}{OB} \times$ (seg. FAE $+ \frac{4}{3}$ DI $\sqrt{(AD \times DB)}) = \dfrac{\sqrt{(4h^2+(D-d)^2)}}{D-d}$ × [(seg. FAE to height AE $+ \frac{2}{3}(D-d)\sqrt{(d(D-d))}$] for that of the part AEFCV.

Also, that in Formula 6, will be $\dfrac{VB}{OB} \times$ seg. FAE $+ \frac{4}{3}$ DI $\sqrt{(AD \times DB)} - AD^2 \times n) = \dfrac{\sqrt{(4h^2+(D-d)^2)}}{D-d}$ × segment FAE to height AD $+ \frac{2}{3}(D-d)\sqrt{(d(D-d))} - nd,)$ for that of the complimental parabolic ungula FAEDG. (15)

If the angle CDB be greater than the angle VAB, or the section be an hyperbola, its area being found, and substituted for B in the general formulæ, will give the surfaces ofthe hyperbolic ungulas.

If the hyperbolic section be perpendicular to the base, DI will vanish, and the expressions will become,

$\dfrac{\sqrt{(4h^2+(D-d)^2)}}{D-d}$ × seg. of cir. AB, whose height is $\dfrac{D-d}{2}$, for the curve surface of the perpendicular ungula CIB. (16)

$\dfrac{\sqrt{(4h^2+(D-d)^2)}}{D-d}$ × seg. of the circle AB, height $\dfrac{D+d}{2}$, for that of the remaining part AICV. - - - - - - - - (17)

$\dfrac{\sqrt{(4h^2+(D-d)\ )}}{D-d}$ × (seg. of the circle AB, whose height is $\dfrac{D+d}{2} - nd^2$,) for that of the complemental perpendicular ungula AIGC. - - - - - - - - - - (18)

# BOOK IV.

---

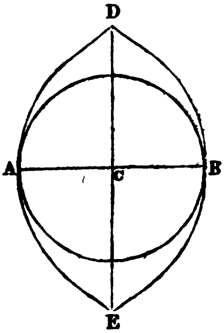
## ON THE REVOLOIDAL CURVE, THE RECTIFICATION OF THE ELLIPSE, AND OTHER CURVES, AND ON THE QUADRITURE OF THE CIRCLE, &C.

### DEFINITIONS.

1. The *revoloidal curve* is the curve forming the contour of one of the facial sides of a revoloid; since this designation may apply to any revoloid, therefore, if the revoloidal curve is mentioned without reference to the species, the curve of a *right revoloid* is understood.

2. If the revoloidal surface is extended on a plane, its contour is called a *plane revoloidal curve;* and the surface is called a *plane revoloidal surface.*

3. The *vertices* of a plane revoloidal surface are the two angular extremities, as D and E.



4. The *vertical* or *transverse* axis of the plane revoloidal surface, is the right line drawn through the vertices, as DE.

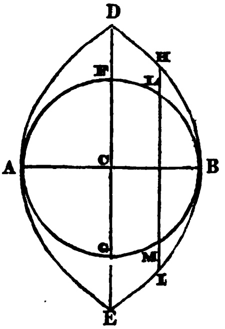
5. Its *conjugate axis* is a line drawn at right angles to its transverse, which it bisects, terminating in the curve, as AB.

6. A *quadrant* of a revoloidal surface is a portion cut off by the two axes, as ACD or BCD.

7. Any area bounded partly by curves and partly by right lines, is sometimes called a *mixtilineal area* or space.

### PROPOSITION I. THEOREM.

*The vertical length of one of the plane surfaces of a quadrangular revoloid, is equal to the semi-circumference of its inscribed circle, and the length of any double ordinate to its conjugate diameter, is equal to the arc of that circle cut off by such ordinate toward the extremity of the conjugate diameter.*



Let ADBE be a plane surface from a quadrangular revoloid, and AFBG its inscribed circle, and the vertical length DE of the revoloidal surface will be equal to the semi-circumference FBG, and the length of the double ordinate HI will be equal to the arc LBM cut off by such ordinate.

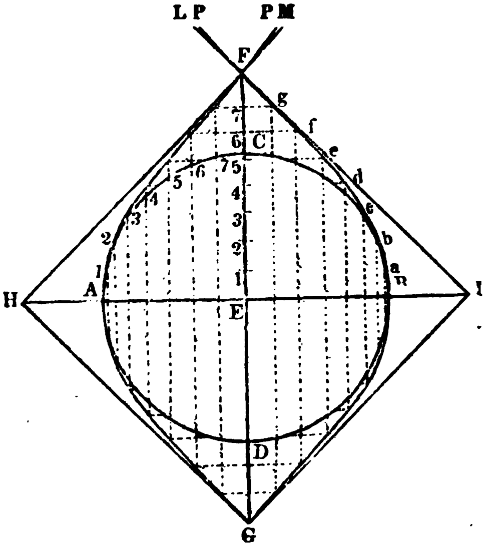
For since (Def. 8, B. III,) the vertical section of a right revoloid through the centre of its opposite sides, is a circle, and since in a quadrangular revoloid this circle is such as may be described on a diameter equal to the conjugate axis of the revoloidal surface, and because each of the facial surfaces of a revoloid extends from one vertice to the other, passing through half the circumference, it follows that its length, DE, is equal to half the length of that circumference which is also equal to the semicircumference FBG.

Again, since any ordinate HI drawn parallel to the transverse axis DE, is the representative of a parallel to a vertical section through the line DE, it follows that the section formed by a plane passing through the ungula of which this face is the surface, is a segment of a circle whose chord terminates in the curve forming the edge of the ungula, and the section of this ungula is similar to the section of its contiguous ungula by a plane perpendicular to this section; which section, through the contiguous ungula, may be represented by a segment LBM; for a quadrangular revoloid has its sides at right angles to each other in a plane perpendicular to the transverse axis. Hence, if HI is equal to the arc containing a segment equal to the segment LBM, it is therefore equal to the length of the arc LBM.

### PROPOSITION II. PROBLEM.

### *To make a plane projection of the facial surface of a right quadrangular revoloid.*

With a radius EA, equal the radius of the circle forming a vertical section of the revoloid, describe a circle ACBD, and draw the diameter AB, and from E perpendicular to AB draw the lines EF, EG, each equal to one-fourth of the circumference of the circle, ACBD, or equal to one-fourth of the circumference of the revoloid. Divide these lines into any number of equal parts, as 1, 2, 3, 4, &c., on the line EF. In like manner divide each quadrant of the circumference into the same number of equal parts, 1. 2, 3, 4, &c.; through the divisions on the circumference draw lines from 1, 2, 3, &c., parallel to DC, and through the divisions on the line EF draw lines both ways parallel to AB, as 7*g*, 6*f*, 5*e*, &c., and where these lines meet the former corresponding lines through the divisions corresponding to the same numbers, will be points in the curve forming the boundary of the surface, through which, if a curved line *a*, *b*, *c*, *d*, *e*, *f*, *g*, &c. is drawn, this line will represent the revoloidal curve, and the space enclosed will represent the plane surface of a right quadrangular revoloid.



*Scholium* 1. The nature of this curve is such, that, as it passes off at the vertices, it reproduces itself again, passing into another curve of the same character but of opposite curvature, for where it passes the vertices F and G, the signs become changed from positive to negative and from negative to positive, so that the curve is reproduced indefinitely as the axes EF and EG are continued.

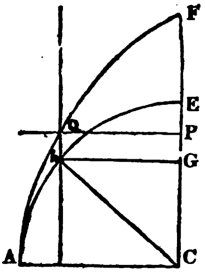
*Scholium* 2. The revoloidal curve passes into and becomes identical with the circle while passing the extremities of the diameter, but its fluxion carries it out of the circle as it leaves these points, and it becomes incorporated with and identical with a right line as it passes off at the extremities of the axis, but its fluction carries it out of the right line as it becomes extended.

For let the diameter AB be produced each way to H and I, so as to be equal to FG, and from the extremities of these lines draw HF, IF, IG, GH, forming a square circumscribing the revoloidal surface; let HF be extended to M, and IF be extended to L, then these lines so produced will cross each other at right angles in F, and form an angle with the axis FG of 45°. Let the revoloidal curves extend to P and P; now if the vertice F be brought to its natural position, on the revoloid, these curve lines evidently cross each other at right angles also, and at the point of contact form an angle of 45° with the axis, which is the same as that formed by the right lines; hence these curve lines agree with the right lines, HM and IL, at that point both in position and inclination, and therefore are identical.

And also, as the revoloidal curve passes into and occupies the space of the circle at the extremities of the diameter, A and B having, in its original position, formed a part of the circle at that point, and the position of that point not having been changed in reference to the axis or diameter AB, it follows, that it is still equal to and identical with the circle at those points, but its fluction, or the law of its propagation, causes it to leave the circle after passing those points.

*Cor.* Each of the ordinates through the quadrant, AEF, parallel to the axis, EF, is equal to the portions A1, A2, A3, &c., of the arc of the quadrant, intercepted by those lines respectively toward the point A.

*Scholium* 3. Hence, this curve is generated by the locus of the intersection of two right lines, AC and EF equal the radius and semi-circumference of a circle moving uniformly from any point E or A, perpendicular to each other, through their respective lengths.



Let A be the origin, and since CF equal the quadrant AE, hence $CF=\frac{1}{4}\pi$, $AC=r$.

Let arc $\theta'$ equal any arc A*h*, measuring the angle AC*h*, then since $Ah=CP$ it is also $\theta'=y$; hence, the equation to the curve is $y=\text{arc }\theta'$

Let F be the origin, and we have $y=QP=hG=\sin.\ \theta'=\sin.\ hCE$.

Then the equation, considering F as the origin, is $y=\sin.\ \theta'$

### PROPOSITION III. THEOREM.

*The contour of a plane revoloidal surface from a right revoloid is equivalent to the perimeter of an ellipse, formed by a vertical section through the angles of the revoloid.*

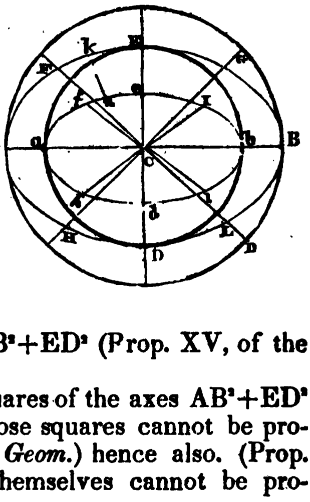
For the section of a revoloid through the angles is an ellipse by definition, and this ellipse terminates the facial surface of the revoloid when in its proper position. Now if the cylindric surface of the revoloid is extended on a plane, its parts are not altered in relation to each other; its vertical length on the plane is equal to its length on the cylindric surface of the revoloid; and its conjugate suffers no change, being a right line parallel to the axis of the cylindric surface while on the revoloid and a right line still when extended; for the surface may be extended in like manner as we would unroll a piece of cloth, or a piece of paper, made to agree with its surface, which suffers no contortion of any of its parts in the change, but the whole surface is the same in reference to its edges after the change as before, and as each facial surface of the revoloid extends through half the circumference of the circle of the revoloid, viz: from one vertice to the other, each side of the surface is terminated by one-half of the ellipse formed by a section through the angles, and as the angles of the revoloid cause generally the ellipses to cross each other at the vertices, forming a vertical angle also on the facial surface, the other side is bounded by one-half of a similar ellipse, so that the whole perimeter of the facial surface of a revoloid is equal to the perimeter of an ellipse by a plane passing through the angles of the revoloid.

*Cor.* Hence, the perimeter of a right quadrangular revoloidal surface is equal to that of an ellipse whose conjugate or *minor* axis is equal to that of the transverse axis of the revoloid, and whose *major* axis is in the ratio to its minor axis as the $\sqrt{2}:1$.

For the plane passing through the angles of the revoloid forming the ellipse, cuts the plane forming a circular section through the centre of the sides at an angle of 45°, therefore, the transverse axis of the ellipse is as the diagonal of a square of which the vertical axis of the revoloid or its diameter forms a side.

### PROPOSITION IV. THEOREM.

*If there be described two ellipses concentric with each other, on axes of which those of the outer one exceed those of the inner one by* N, *then the two ellipses will not be equidistant throughout, but will be nearer to each other at points in the curve between the vertices of the axis, than at the vertices.*



Let AB, ED, and *ab*, *ed*, be the two axes of two concentric ellipses, and let the axis $ab$=Q, and $ed$=R, and let the axis AB=Q+N and ED=R+N, then the distance between the two ellipses, between the vertices E and A will be less than at those vertices.

For, draw the two equal conjugate diameters, HG, FL, and also the two *h*I, *fl*,

Then, $HG^2+FL^2$, or $2HG^2=AB^2+ED^2$ (Prop. XV, of the Ellipse,) and $2hI^2=ab^2+ed^2$

Now, because the sum of the squares of the axes $AB^2+ED^2$ are not greater than $HG^2+FL^2$, those squares cannot be proportional (Prop. XVIII, B. I, *El. Geom.*) hence also. (Prop. XXIII. B. I., *El. Geom.*) the axes themselves cannot be proportional.

Now it is evident, that when the axes AB and ED are nearly equal, then also they will very nearly form the extremes of a proportion of which the two diameters HG, FL are the means; which is the more nearly true the nearer the two axes are to an equality, or the nearer the ellipse approaches to a circle, and hence they are more disproportional, the greater the eccentricity of the ellipse.

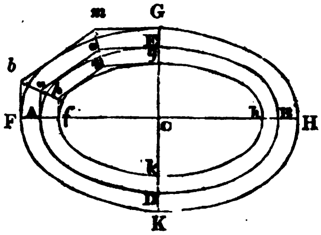
Now it is evident, that the inner ellipse, *a, e, b, d*, is more eccentric than the outer one, AEBD, since the two axes of the inner one are less than those of the outer one by the same constant quantity N, (by hypothesis,) hence the conjugate diameters *h*I, *fl*, are more nearly equal to HG and FL than *ab* to AB, or than *ed* to ED. Hence the elliptical curves are nearer each other at any point F or *k* between the vertices of the major and minor axes, than at those vertices.

*Cor.* 1. Hence, if about an ellipse there be described a curve concentric thereto and equidistant throughout, such curve will not itself be an ellipse, but if it is described very near to the elliptical curve and equidistant, it may be regarded as an ellipse without much error in so considering it, and this will be more nearly correct the more nearly the ellipse approaches to the circle.

*Cor.* 2. If about any curve except the circle, another curve be described, in such manner as to be equi-distant in all its parts from the former, these two curves cannot be similar curves, neither in properties nor in figure, all which may be shown by the same reasoning as in the proposition, viz: a curve described equidistant from a parabola, either within or without, cannot be a parabola, and a curve described equidistant from a hyperbola, cannot itself be a hyperbola; all of which is evident from the properties of those curves, but when they are drawn exceedingly near to those curves they may be regarded as curves of similar character as those near which they are so drawn.

### PROPOSITION V. THEOREM.

*If there be described two curves, one within and the other without an ellipse, of such kind that they shall both be equi-distant from the ellipse, or the ellipse shall be midway between the two, then will the space included between the two curves so described be equal to the circumference of the ellipse multiplied by the distance between the two curves.*



Let FGHK, *fghk*, be two concentric curves described so as to be equi-distant from the elliptical circumference AEBD, one without and the other within, so that AEBD shall be midway between the two, then will the curvilinear space F*f*, G*g*, H*h*, K*k*, F*f*, be equal to the circumference AEBD multiplied by their common distance F*f*, or G*g*.

For, describe about each of those curves polygons F*bm*G, &c., A*no*E, &c., *fpqg*, &c., such that their corresponding sides will be parallel, and draw *bp*, *mq*, &c., and the surface included between the outer and inner polygon will be divided into the trapeziums F*fpb*, *bpqm*, &c., each of which is equal to the half sum of its parallel sides multiplied by their distance F*f*; but any side, A*n* of the polygon described about the middle curve is equal half the sum of its corresponding parallel sides of the trapeziums; hence the trapezium F*fpb* is equal to A*n* × F*f*, and the trapezium *bpqm* is equal to *no* × F*f*; and since this is true for each of the trapeziums, it follows, that the sum of all is equal to the sum of all the sides of the polygon described about the ellipse AEBD, multiplied by the common distance F*f*. And this would be manifestly true, whatever be the number of the sides of the polygon described about the ellipse, but when the number of the sides of the polygon is indefinitely increased, the polygon becomes a curve similar to that about which it is described. (Prop. XII, Cor. 4, B. V, *El. Geom.*) Hence as in the proposition.

*Cor.* It is evident, also, that if there be described curves within and without a parabolic or any other curve, so as to be equi-distant from it, then the space included between the outer and inner one will be equal to the curve situated midway between them, multiplied by the distance between the outer and inner curves, and the same may be affirmed of the revoloidal curves so drawn.

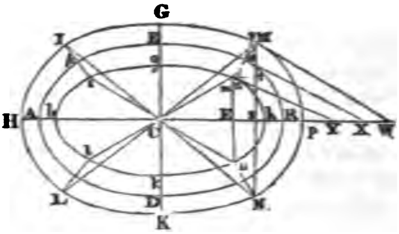
*Scholium* 1. The polygon described about two eccentric curves drawn so as to be equidistant throughout, are not similar polygons, since the figures about which they are described are not similar. (Prop. IV., Cor. 1.)

*Scholium* 2. The last two propositions suggest a method of rectifying the elliptical circumference, and also of finding the lengths of other curves whose quadratures are correctly or approximately known.

For, if about an ellipse AEBD, whose major axis AB=P, and whose minor axis ED=Q, another ellipse FGHK be described, whose major axis FH=P+N, and whose minor axis GH=Q+N, and if another concentric ellipse be also described within the former, whose major axis *fh*=P—N and whose minor axis *gk*=Q — N, we shall have an elliptical ring F*f*, G*g*, H*h*, K*k*, whose area may be found by subtracting the inner ellipse from the outer one, then if this area is divided by the distance of the inner and outer circumference the quotient will be the elliptical circumference AEBD.

### PROPOSITION VI. PROBLEM.

*Let it be required to find the distance* Mm, *between two concentric ellipses, the difference of whose major and minor axes are each equal to a quantiy* N.



Let HGPK, *bghk* be two concentric ellipses, the difference of whose axes, HP—*bh*, is N, also GK—*gk* =N; draw the equal conjugate diameters ML, NI, *ml*, *ni*; draw the right coordinates MN, *mn*, draw *mt* parallel to HP, and join M*m*, which will be the distance between the ellipses at the points M, *m*; we shall have $sC^2=\frac{1}{2}PC^2$ and $sM^2=\frac{1}{2}CG^2$, also $EC^2=\frac{1}{2}hC^2$ and $st^2=\frac{1}{2}Cg^2$. (Prop. XXI. Cor. 1 of Ellipse.) and $sC-EC=sE$ or *mt*, and $sM-st$ or $mE=Mt$. Hence $\sqrt{Mt^2+mt^2}=Mm=$ the distance of the two curves at the points M, *m*.

Let the axis HP=25, and GK=19, and if N=2, then

$bh=23$ and $gk=17$

and $sC=8.83175$ $EC=8.13172$,

hence $tm=.70003$

$sM=6.71751$ and $mE=6.00999$,

hence $Mt=.70752$

and $Mm=.995302$.

*Scholium.* 1. But because the line M*m* is not perpendicular to the curve AFRD at the point of contact, but very nearly perpendicular to MW, it is therefore greater than the true distance of the curves, which we will suppose is *uv*, to find which, we have

$$YC=\frac{Ch^2}{EC}\text{; and } WC=\frac{PC^2}{sC}$$

$$YE=YC-CE.\text{ and } Ws=WC-Cs$$

$$mY=\sqrt{mE^2+EY^2},\ MW=\sqrt{Ms^2+Ws^2}.$$

Therefore, we have the sides of the right-angled triangles *m*EY, M*s*W given to find the angles W and Y, the difference of which, when found, is equal to the angle made by the lines MW and *m*Y with each other produced. Then we have a right-angled triangle M*m*, MW produced, and *m*Y produced right-angled at M, to find the sides MW produced, and *m*Y produced, and also the base *uv* of an isosceles triangle having the same vertical angle, which line *uv* will be the shortest distance required.

*Scholium* 2. Let the equal conjugate diameters IN, LM, *in*, *lm*, be drawn, and the difference of the semi-diameters CI—C*i* will be very nearly equal to the distance I*i* of the curves at the points I, *i*. But $\mathrm{IN}=\sqrt{\frac{1}{4}\mathrm{PH}^2+\frac{1}{4}\mathrm{GK}^2}$, and $in=\sqrt{\frac{1}{4}bh^2+\frac{1}{4}gk^2}$, hence, $\sqrt{\frac{1}{4}\mathrm{PH}^2+\frac{1}{4}\mathrm{GK}^2}-\sqrt{\frac{1}{4}bh^2+\frac{1}{4}gk^2}=$ the distance I*i* very nearly.

Let the axis AR=24, and FD=18; then, if we make PH =25, and GK=19, IN will=22.2036.

Also, we may have *bh*=23, *gk*=17; hence, *in* or *ml*=20.2237.

Therefore, $\frac{22.2036-20.2237}{2}=.9899=$ the difference CI and C*i*, which is nearly equal the true distance of the curves through the point *h*, which will be more accurate as the axes approach equality, and will be approximately true till the eccentricity of the curves becomes very great.

### PROPOSITION VII. PROBLEM.

*To find the length of the elliptical circumference, approximately.*

It has been observed, (Prop. V. Scholium,) that the circumference of an ellipse inscribed between two other concentric ellipses, is equal to the area or space included between the two divided by their distance from each other; but since the distance of the curves is not constant in every part, we must take their average distance.

If G*g*, P*h*, (see diagram to Prop. VI,) be the distance of the two extreme ellipses through the lines of their axes, and *uv* the distance through the point *e*, then the average distance will be very nearly equal $\frac{1}{4}\mathrm{G}g+\frac{1}{2}uv$; hence, if the area of the ring within the exterior and interior circumferences is divided by $\frac{1}{4}\mathrm{G}g+\frac{1}{2}uv$, the quotient will be the length of the whole circumference AFRD; or if a quadrant of the ring is divided in like manner, the quotent will be the length of a quadrant F*e*R of the circumference.

If we assume the axes AR=24, FD=18, as in the last proposition, and the axes of the other ellipses as there assumed, we shall have for the area of the greater ellipses, $\frac{1}{2}\mathrm{PH}\times\frac{1}{2}\mathrm{GH}\times\pi=12.5\times9.5\times\pi=373.06381$; and the area of the smaller ellipse $\frac{1}{2}gk\times\frac{1}{2}bh\times\pi=11.5\times8.5\times\pi=307.09042$, therefore the area of the ring is equal to 65.97339.

Let this be divided by $\frac{995302+1}{2}$, the average distance as found in the last proposition. and we have 66.128 for the circumference of the ellipse AFRD.

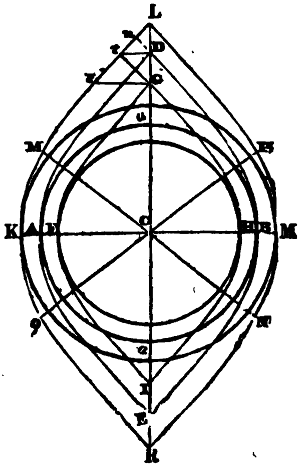
Let the area be divided by the distance as found in (Sch.

2. Prop. V,) and we have $65.97339 \div \frac{9899+1}{2} = 66.3115 =$ the elliptical circumferance, which is true to four places of figures, and very nearly to six; hence, this mode of computing the elliptical circumference is sufficiently accurate for any ordinary calculation.

*Scholium.* The length of the parabolic or hyperbolic arc may be approximately determined in the same manner as is here suggested for the ellipse.

### PROPOSITION VIII. PROBLEM.

*Let it be required to find the length of a revoloidal curve.*



Let ADBE be a revoloidal curve from a right quadrangular revoloid whose length is required; let two other concentric curves KLMR, FGHI, be described on each side of the first and equidistant therefrom, and if these two last described curves are at a small distance only from the former, they will be very nearly revoloidal curves likewise.

Let *aa*, the axis of the revoloid from which the surface ADBE is supposed to be taken, equal P; and since, by hypothesis, ADBE is the surface of a right quadrangular revoloid, the conjugate is also equal to P, and the vertical length DE of the revoloidal surface will be equal $\frac{1}{2}\pi \times P$; since DE is equal to half the circumference A*a*B*a* of the revoloid. Now, let N be the distance AK, or AF, that the concentric curves are proposed to be drawn; and since the angles DL*n* and LD*n* are each$=45°$ in a right quadrangular revoloid, make the semi-circumference of the revoloid from which the surface KLMR is taken, equal $\frac{1}{2}\pi P+2\sqrt{Dn^2+Ln^2}=\frac{1}{2}\pi P+2\sqrt{2N^2}=\frac{1}{2}\pi'$, and make the circumference of the revoloid from which the surface FGHI is supposed to be taken, $=\frac{1}{2}\pi P-2\sqrt{Dn^2+Ln^2}=\frac{1}{2}\pi P-2\sqrt{2N}=\frac{1}{2}\pi'$, then may $P'$=the diameter of the greater revoloid, and $P''$ equal the diameter of the lesser, and $P+2N$ equal the conjugate axis KM of the larger surface, and $P-2N$ equal the conjugate axis of the smaller.

And we have (Prop. III. B. III.) the area of the surface $KLMR=(P+2N)\times P'$, and the area $FGHI=(P-2N)\times P''$, and $(PP'+2P'N)-PP''+2P''N$ equal the space KF, LG, MH, RI between the inner and outer curve. Let this area be divided by the distance KF or N, and the quotient will be the length of the revoloidal curve ADBE very nearly.

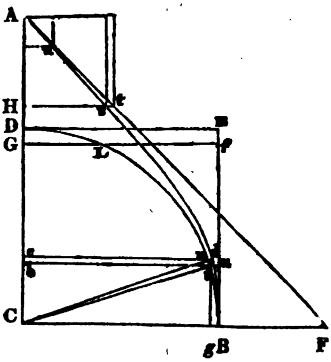
*Scholium* 1. Since the major and minor axes of the outer and inner curves vary in very nearly the same ratio, it follows that the outer and inner curves must be very nearly similar to the central one where the distance KF is small, even if the surface possesses a considerable degree of eccentricity, protracted in the direction KM; but the greater the eccentricity when protracted in the direction LR, the greater is the similarity of the concentric figures, and the greater is the accuracy with which we can rectify the curve.

2. Since the perimeter of a revoloidal curve is also the perimeter of an ellipse, such as is formed by a vertical section through the axis and the angles of a revoloid, this mode of rectifying the revoloidal curve furnishes also a mode of rectifying the elliptical circumference; and reciprocally.

### PROPOSITION IX. PROBLEM.

*To find the vertical length of the revoloidal surface, and consequently the circumference of the circle.*

Let ABC be the quadrant of one of the facial surfaces of a quadrangular revoloid, and BCD a quadrant of the inscribed circle; AC the semi-transverse, and CB the semi-conjugate axis; extend CB to F, making it equal to AC, draw AF, and the triangle ACF will be equal to one-fourth of the square circumscribing the whole revoloidal surface. From the extremities of the radii CD and CB, draw the lines DE and BE, perpendicular to those radii respectively, then will BCD be a square circumscribing the quadrant equal to one-fourth of the square circumscribing the circle.



On the line BE take any distance B*a*, and from the point *a*, draw the line *ab* parallel to the semi-conjugate, BC; then set off from C on the transverse CA, the distance C*c* equal the arc

B*i* cut off by the line *ab*; and draw *cd* also parallel to BC, cutting the curve AB, and from *d* draw *dg* parallel to AC, then will the rectangle CB*ab* equal the portion of the revoloidal surface C*cd*B. (Prop. III.Cor.1, B.III,) and $cd=ib$=cosine of the arc B*i*, and $Cb=ig$=sine of the arc. The portion of the revoloidal surface CB*dc* may consist of the rectangle C*cdg*, and the segment B*dg*; the former of which is equal to the product of the line C*c* or *dg*, into the cosine *ib* or *cd*; and the latter may be divided into the two portions B*ig*, a segment of the circle, and the trilinear space B*id*, the space included between the circle and the revoloidal curve; the former of which is equal to the difference between half the product of the arc B*i*, or its equivalent C*c*, into the radius CB; and half the rectangle of the sine *ig*, into the cosine *ib*; and the latter may be approximately estimated by considering, that the distance between the circle and the curve at any point, and in direction parallel to the base line *id*, is equal to the difference between the arc of the circle included between such point, and the radius CB, and its sine. Thus, the distance *id*=arc B*i* or *dg*, its equivalent, —*gi*, its sine. Hence, it will be perceived that its value converges rapidly as we approach the semi-transverse AC. Though the ratio of this space is constantly changing with regard to its linear dimensions, as the vertical length of a conjugate section is varied; yet, when the distance B*a* is taken very small, its area may be regarded as equal to one-half the product of the base *id*, into its vertical height perpendicular to that base, viz., into *g*B.

Let $x$=the arc B*i*=C*c*,
and $s$=sine of the arc B*i*,
and $c$=cosine.
Then will $cx$=the rectangle C*cdg*,
and $\frac{1}{2}rx-\frac{1}{2}cs$=the segment B*ig*,
and $\frac{rx}{2}-\frac{rs}{2}-\frac{cx}{2}+\frac{cs}{2}$=the space B*id*,
and . . . . $rs$=the rectangle BC*ba*.

Hence, $cx+\frac{1}{2}rx-\frac{1}{2}cs+\frac{1}{2}rx-\frac{1}{2}rs-\frac{1}{2}cx+\frac{1}{2}cs=rs$.

By transposing and condensing $\frac{1}{2}cx+rx=\frac{3}{2}rs$

Therefore, . . . . . . . . $x=\frac{\frac{3}{2}rx}{r+\frac{1}{2}c}=\frac{3is}{2r+c}$ - - (1)

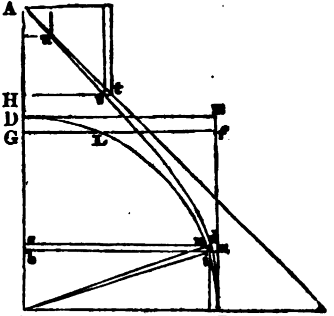
Or if we make the curvilinear space *id*B=*ai*B, making the mixtilineal space C*cdb* equal to the arithmetical mean between the mixtilineal space *di*BC*c* and *dia*BC*c*, we shall have

$xc+\frac{1}{2}rx-\frac{1}{2}sc$=the space *di*BC*c*
and $xc+rs-cs$=the space *dia*BC*c*
half the sum of which make=the rectangle
C*ba*B=$rs$.

Thus, $\frac{2cx+\frac{1}{2}rx+rs-\frac{3}{2}cs}{2}=rs$

and $2cx+\frac{1}{2}rx=\frac{3}{2}cs+rs.$

Hence, $x=\frac{\frac{3}{2}cs+rs}{2c+\frac{1}{2}r}$ . . . . . . . (2.)



Again, draw *f*G parallel to CB or DE, cutting off any arc DL, and set off from A the distance, AH, equal the length of the arc DL, and from the point H, draw the line H*t*, perpendicular to the axis AC, to cut the curve in *v*, and the trilinear space AH*v* will be equal to the rectilineal DE*f*G, (Prop. III, and Cors. B. III.)

Now, if we compute the area of this trilinear space AH*v* in terms of its sides, those sides may be rectified by its known quadrature. The length of the side AH equal the arc DL, and the area AH*v* is less than half the rectangle of AH into H*v*, and greater than $\frac{1}{2}v\mathrm{H}^2$; but will be very nearly an arithmetical mean between the two.

Let AH=$x$=arc DL,
H*v*=$s$=sine of $x$,
DG=$v$=versed sine of $x$.

Then $\frac{\frac{1}{2}x^2+\frac{1}{2}xs}{2}$=area AH*v*,

and $vr$=area DG*f*E;

but it has been shown that the area AH*v*=area DG*f*E.

Hence, $vr=\frac{\frac{1}{2}x^2+\frac{1}{2}sx}{2}$

Or, $4vr=x^2+xs.$

Completing the square $x^2+sx+\frac{s^2}{4}=4vr+\frac{s^2}{4}$,

Hence, $x=\sqrt{4vr+\frac{s^2}{4}}-\frac{s}{2}$ - - (3.)

The smaller the arc is taken, the greater the accuracy; for when the arc is taken very small, the sine and arc are very nearly identical; when no appreciable error would occur in so considering them, and the area AH*v* would still be found between half the square of AH, and half the rectangle of AH into H*i*; and if the difference of $x$ and $s$ becomes $=o$, then the expression for the surface becomes barely $\frac{x^2}{2}$

and $x=\sqrt{2rv}$ - - - - - - - - (4.)

Let the two arcs, DL, *i*B, be taken such that the diagonal A*t* is nearly equal the arc *i*B, and if we make AH*t* equal the rectangle DG*f*E, and *n*BC*c* equal the rectangle *a*BC*c*, the result will be a close determination of the length of the arc of the circumference; for the area AH*t*, in such case, is very nearly as much in excess, above the area AH*v*, as the area *n*BC*c* is in defect of the area *d*BC*c*, and if these arcs are taken very small, any error may be rendered evanescent.

For this purpose, let $x$=the arc *i*B

and $x'$=the arc DL.

Then let $cx-\frac{1}{2}cs+\frac{1}{2}rx=rs$

which reduced gives $x=\frac{\frac{1}{2}cs+rs}{c+\frac{1}{2}r}$ - - - - - - - - (5.)

And $\frac{1}{2}x'^2=rv$

Then $x'=\sqrt{2vr}$ - - - - - - - - (6.)

Whence, if we take an arithmetical mean between the results of these two equations, we shall arrive at a very approximate determination of the circumference, if the arc is taken very small.

For an example, let us take an arc$=\frac{1}{24576}$ part of the circumference. We have, by trigonometry, the sine of that arc =.00025566346=$s$, and its cosine=.99999996732=$c$. Hence, by formula 1, we have

$$x=\frac{\frac{3}{2}rs}{r+\frac{1}{2}c}$$

Therefore. 00025566346$\times\frac{3}{2}=\frac{3}{2}rs$=.00038349819

and 1+9999996732÷2$=r+\frac{1}{2}c$=1.4999998366

And $\frac{3rs}{r+\frac{1}{2}c}$=.00038349819÷1.4999998366=.00025566346278

hence $x$=.00025566346.278, which multiplied by 24576, gives the whole circumference=628318526133 when the diameter is=2, or 3.1415926306=$\pi$, when the diameter is 1, which result is true to eight places of figures, but the 9th should be 5 instead of 3.

If instead of this arc, we take that of $\frac{1}{40000}$ of the circumference, the sine of which is=.000157079632033, and the cosine =.999999987462994

$\frac{3}{2}rs$=.000235619448050

$r+\frac{1}{2}c$=1.499999993831497

$\frac{\frac{3}{2}rs}{r+\frac{1}{2}c}$=000157079632676

whence $\pi$=.000157079632676$\times\frac{40000}{2}$=3.1415926535.2, which is true to eleven places, but the last figure should be 8 instead of 2.

Hence, it will be perceived, that, by taking a very small arc, we can rectify the circumference to any degree of exactness.

*Cor.* 1. Since in formula (6) $\frac{1}{2}x^2=vr$; if $r=1$, then $v=\frac{1}{2}x^2$ (1.)

Hence, the versed sine may be taken as half the square of the arc, which is approximately true when the arc is small, and if very small, may be taken as an accurate determination, but as the arc is increased, if we proceed as in formula (3,) where $rv=\frac{1}{4}x^2+\frac{1}{4}sx$, we have the value of $v$ or the versed sine, equal to $\frac{1}{4}$ of the square of the arc$+\frac{1}{4}$ the rectangle of the arc and sine; which is approximately true for any arc of the quadrant, and may be taken for an accurate determination when the arc is small.

Hence this general formula $v=\frac{1}{4}x^2+\frac{1}{4}s3x$ - - - - (2.)

*Cor.* 2. Hence, also, from formula (1,) $x=\frac{\frac{3}{2}rs}{r+\frac{1}{2}c}$ we may deduce expressions for the sines, of small arcs, in terms of the cosine and arc; and also for the cosine in terms of the sine and arc. For from this equation we obtain

$$s=\tfrac{2}{3}x+\tfrac{1}{3}cx \quad - - - - - - - - \quad (1.)$$

and

$$c=\frac{3s}{x}-2 \quad - - - - - - - - \quad (2.)$$

*Scholium.* 1. If $s$ is taken equal to the sine of 30°, we shall avoid in some measure the inconvenience of using imperfect decimal terms; for the sine of 30° is equal to half the radius, and we then have only one decimal term entering into the expression, viz., the cosine which is, .866025, &c.; but we have a more difficult determination of the area of that portion of the revoloidal curve existing without the circle, viz., the space *Bid*; which, if determined, would lead to the true determination of the arc of the circumference.

2. If the ratio between the quadrant of the circle and the portion of the revoloid without the quadrant, viz., if the portion ADB is determined, then the ratio of the circumference may also be determined.

Thus, if $\frac{1}{2}rx=$the area of the quadrant CDB, and $z=$the area of the space included between the quadrant and the curve, viz., ABD, and if the ratio between these terms are known,

that is, if $\frac{1}{2}rx : z :: m : n$

then $\frac{1}{2}rx+z=r$ (Prop. III, B. III.)

by uniting extremes and means,

$$\tfrac{1}{2}rnx = zm$$
$$z = r^2 - \tfrac{1}{2}rx$$

By substituting this value of $z$ in the former equation $\tfrac{1}{2}rnx = mr^2 - \tfrac{1}{2}rmx$; by dividing and transposing $\tfrac{1}{2}nx + \tfrac{1}{2}mx = mr$;

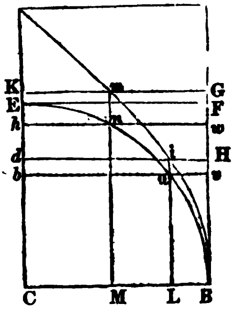
hence $$x = \frac{mr}{\tfrac{1}{2}n + \tfrac{1}{2}m} = \tfrac{1}{4} \text{ of the circumference.}$$

3. Or if the ratio between the quadrant of the revoloidal surface and its circumscribing parallelogram is determined, the circumference may be computed also.

For the revoloidal quadrant being equal to the square of the radius, (Prop. III. B. III,) it is in the same ratio to its circumscribing parallelogram, as the radius to one-fourth of the circumference, seeing the parallelogram is equal to the product of the radius by one-fourth of the circumference.

### PROPOSITION X. THEOREM.

*If a quadrant* ABC *of a plane revoloidal surface from a right quadrangular revoloid be described, and a rectangle* ACBD *circumscribe the quadrant, then if the axis* AC *be divided into any number of equal parts, and if ordinates be drawn from the points of division across the rectangle, then the parts of those ordinates included between the axis* AC, *and the revoloidal curve* BA, *will be equal to the sum of a series of sines of arcs of the quadrant* EB, *of a circle in arithmetical progression, equal in number to the number of the ordinates, and the parts of those ordinates intercepted by the curve* AB, *and line* CD; *will be equal to the sum of a similar series of versed sines, and the sum of the whole ordinates will be equal to an equal series of radii of the quadrant* EB.



Draw any ordinate as KG, and from $m$, where it cuts the curve, draw $m$M parallel to AC, and let it cut the circumference of the circle described on the ordinate CB, in $n$ draw $nh$ parallel to $m$K, and the line $m$K = the line $nh$, is the sine of the arc $n$E, and because (Prop. II,) the line AC = the quadrant EB of the circumference, and the line $m$M = KC is also = the arc $n$B, the line KA equal the arc $n$E; take K$d$ = KA, and from $d$, draw the ordinate $d$H parallel to the former ordinate, and from the point $i$, where it cuts the curve, draw $i$L, and from the point *a* where the latter line cuts the quadrant, draw *ab*, which will be the sine of the arc *a*E=to the part *id* of the ordinate *d*H; and because *i*L is equal to *a*B (Prop. I,) and *m*M =*n*B, and AC=EB, and because AC—*m*M=*m*M—*i*L, E*n* =*na*; and since the same may be shown in reference to any other ordinate, drawn from a point on the axis AC, whose distance from *d* toward C is equal to K*d* or AK, it follows that the parts of equidistant ordinates drawn across the rectangle intercepted between the curves AB, and the axis AC, are equal to the sum of a similar series of sines of arcs of the quadrant EB taken in arithmetical progression; which is the first branch of the proposition.

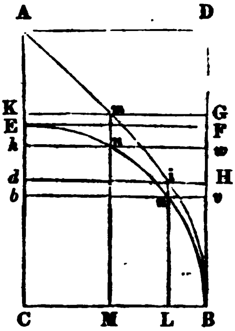
And since *av*=*i*H, is the versed sine of the arc *a*B corresponding to the sine *a*L with its complement *ab*, and *nw*=*m*G versed sine of the arc *n*B, its complement being *m*K; and since they have been shown to be at distances from each other proportional to the arcs E*n*, *na*, &c.; and since the same may be shown in reference to the portion of any ordinate intercepted by the curve AB, and line BD, wherever drawn, it follows that the sum of the portions of the equidistant ordinates intercepted by the curve AB and line CD, is equal to the sum of a similar series of versed sines of arcs taken in arithmetical progression; which is the second branch of the proposition. And since the lines CA and BD are parallel by hypothesis, the ordinates are all equal in length, and equal to the radius CB, hence the whole series of ordinates is a similar series of radii.

*Cor.* 1. From the preceding demonstration, it appears that any line or ordinate drawn from any point on the axis AC, parallel to the conjugate CB, and terminating in the curve AB, is equal to the sine of the arc on the quadrant BE, cut off by a line drawn from the point of termination of such ordinate in the curve parallel to the axis AC.

*Cor.* 2. Hence, also, the sum of the series of sines is to the sum of a similar series of radii as the square ECBF, described on the radius, to the rectangle CB, AC, of the radius and arc of the quadrant. Since the square ECBF is equal (Prop. III. B. III.) to the surface ABC.

### PROPOSITION XI. THEOREM.

*The area of a plane revoloidal surface is to that of its circumscribed parallelogram, as the sum of an indefinite series of sines in the circle, to the sum of an equal series of radii.*



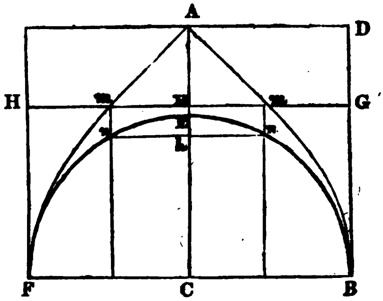
Let ACBD be a parallelogram circumscribing the quadrant of the revoloidal surface ABC, and let EBC be a quadrant of the circle, then will the area of the quadrant of the revoloidal surface ABC be to that of a parallelogram ACBD, as the sum of an indefinite series of sines of the quadrant BCE, to the sum of an equal series of radii.

For, since all ordinates *vb*, *d*H, &c., cut the surfaces of those figures in relation of their magnitudes in the sections through which such ordinates pass; and since (Prop. XI,) if ordinates be drawn through those figures equidistant from each other, the portions of the ordinates intercepted by the curve and axis, are equal to the sum of a series of sines of arcs in arithmetical progression for the whole quadrant equal in number to the number of the ordinates, and if these ordinates are equidistant from each other, the sum of the portions passing through either surface, drawn into their common distance, may be taken for the surface; and since the distance of the ordinates is equal by hypothesis, both for the parallelogram and revoloidal surface, the portions of the ordinates intercepted by each, will be in relation to their surfaces respectively, when their number is indefinitely increased, and their distance becomes indefinitely small. Hence, as the sum of a series of sines of arcs of the whole quadrant taken in arithmetical progression, is to the area of a quadrant ACB of the revoloidal surface, so is the sum of an equal series of radii to the area of the parallelogram ACBD; and what has been shown for one quadrant of the revoloidal surface, is also true for the whole.

*Cor.* Hence, the space BDA without the revoloidal surface, is to the revoloidal surface, as a sum of an indefinite series of arcs in arithmetical progression to the sum of a similar series of sines.

### PROPOSITION XII. THEOREM.

*If the quadrant* AB *of a revoloidal curve be made to revolve about its axis* AC, *and if a plane hemisphere of a quadrangular revoloid be described about the solid so generated, having the same axis* AC, *then the revoloid will be to its circumscribing prism, as the sum of the squares of a series of sines of the quadrant, to the sum of the squares of an equal series of radii.*



Let an indefinite number of planes be passed through the revoloid perpendicular to the axis, and at equal distances from each other, and the sections made by these planes will all be squares, (Prop. 1, Cor. 4 B. III.) and their sides will all be equal to the ordinates H*m*, *hn*, &c., drawn through the intersection of such planes with the vertical sections; and hence the side of each parallel section, is equal to twice the sine *nh* of the quadrant corresponding to such section, being = *nn*. Now let each of those parallel planes be extended to HG passing through the prism, and it is evident that each of the sections of the prism will be squares, whose sides are severally equal to twice the radius CB, of an inscribed circle. Now the magnitudes of these solids through each section, are evidently in the relation of the magnitudes of such sections; and if the number of these equidistant planes are indefinitely increased, and they are indefinitely near together, their sum will represent the whole of each of the solids in the relation of their whole magnitudes, and since each conjugate section of the revoloid is the square described on double the sine, answering to the ordinate in reference to the quadrant CEB, and each section of the prism is the square described on the line HG, equal twice the radius, it follows that the solidity of the revoloid, is to that of the circumscribing prism, as the sum of the squares of a series of the sines of the quadrant, to the sum of the squares of an equal series of radii.

*Cor.* Hence, the solidity of the space between the surfaces he revoloid described as above, and that of its circumscribing prism, is to that of the revoloid, as the sum of the squares of a series of versed sines of the quadrant, to the sum of the squares of an equal series of sines, and this space is to that of the prism, as the sum of the squares of the versed sines, to the sum of the squares of an equal series of radii.

*Cor.* 2. Let the prism and also the revoloids, be divided into four quadrants by planes through the vertical axis, and passing through the centres of the opposite sides; and the solid so described. will be truly represented by the value of the conjugate parallel sections, passing through them, viz: the segment of the revoloid, will be represented by the sum of the squares of the sines; the space between the revoloid and surface of the prism, by the sum of the squares of an equal series of versed sines, and the prism by the sum of the squares of an equal series of radii.

*Cor.* 3. Since the revoloid has the same ratio to its circumscribing prism, as the solid of revolution about which it is described, has to its circumscribing cylinder; the solid formed by the revolution of the revoloidal quadrant AB, will be to its circumscribing cylinder, as the sum of the squares of a series of sines of the quadrant, to the sum of the squares of an equal series of radii.

### PROPOSITION XIII. PROBLEM.

*Let it be required to find the circumference of the circle from the ratio of the sum of the series of sines for every minute of the quadrant to the sum of an equal series of radii.*

The number of the series of sines to every minute is 5400 = the number of minutes in the quadrant, which is the number of radii to be compared with the series of sines, and if the radius = 1, then 5400 is the sum of the series of radii; and the sum of the series of sines to every minute is by Trigonometry = 3438.2467465.

And (Prop. XI,) the area of the revoloidal surface is to that of its circumscribing parallelogram, as the sum of an indefinite series of sines to the sum of an equal series of radii; but the series of sines to every minute being a definite number, and such that the surfaces between the lines may be rendered appreciable, they do not represent those spaces or trapeziums in their exact ratios, but represent the longest sides of those trapeziums, making up the revoloidal surface, but in order that they may be true indices of those trapeziums, they should be such as pass through the centres, when each would be reduced, by a quantity equal to half the difference between itself and the next greater one, and as the sum of all their differences, is evidently equal to the radius, half of the sum of their several differences is equal to half the radius; therefore, the sum of the natural sines must be reduced by that quantity, viz: 3438.2467165 — ,5=3437.7467465, when if we make

$r =$ radius

and $x = \frac{1}{4}$ of the circumference, we shall have, (Prop. IX., Sch. 3.)

$r : x :: 3437.7467465 : 5400$

Hence, $x=1.570.796337= \frac{1}{4}$ the circumference when the diameter is 2, which is true to 8 places of figures, viz: to 1,5707963, but the 9th figure should be 2 instead of 3.

*Cor.* 1. Because the cosine of 60° is equal to half the radius, and because the surface of any conjugate section of the revoloid is equal to the radius multiplied by the cosine corresponding to each section, (Prop. XI, Cor. 1.) the sum of the sines for 60° is equal to half the sum for 90° or the whole quadrant; and consequently, is equal to the sum of the series, for the arc from 60° to 90°.

*Cor.* 2. Hence, by proceeding as in the proposition, using the sum of the sines for an arc of 60° in comparison with a corresponding portion of the circumscribing parallelogram, the ratio of the circumference of the circle to its radius, may be determined by this arc, in the same manner as by the whole quadrant.

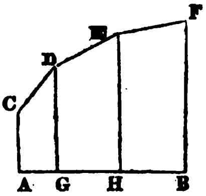
*Scholium.* It will be perceived that the sum of all the natural sines of the quadrant, to any number denoting the series, may be calculated by reversing the operation, viz: $x : r :: nr :$ sum of series of sines minus $\frac{1}{2}r$, when $n =$ the number denoting the series: and this may evidently be effected for the whole quadrant or any portion of it.

### PROPOSITION XIV. THEOREM.

*If there be any number of equidistant ordinates of different lengths drawn from a right line* AB, *and terminated by the vertices of a polygon, then the area comprehended between the greatest and least ordinate, and the right line* AB *and polygonal line* CDEF *is equal to the sum of all the middle ordinates + half the sum of the extreme ordinates drawn into the common distance* AG.

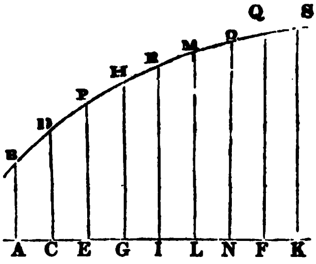
For the quadrilateral ACDG is $= \frac{1}{2}$ (AC+GD) AG, the quadrilateral EDGH = $\frac{1}{2}$ (G*D*+HE) AG or GH, and the quadrilateral HEFB = $\frac{1}{2}$ (HE+BF) HB ; hence by addition we have

$$(\tfrac{1}{2}\,AC + GD + EH + \tfrac{1}{2}\,BF)\ AG.$$



### PROPOSITION XV. THEOREM.

*If a right line* AK *be divided into any even number of equal parts* AC, CE, EG, *&c.; and at the points of division their be erected perpendicular ordinates* AB, CD, EP, *&c., terminated by any curve* BDPS ; *and if* a *be put for the sum of the first and last ordinate* AB, SK, b *for the sum of the even ordinates* CD, GH, LM, FQ ; *and* c *for the sum of all the rest,* EP, IR *and* NO ; *then* (a+4b+2c) × $\frac{1}{3}$ *of the common distance* AC *will be the area,* ABSK *very nearly*



Through the first three points BDP, let a parabola be conceived to be drawn, having its axis parallel to the ordinates; the parabolic area ABPE, (Prop. VI. *Schol* .B. I.) will be (AB+4CD+EP) × $\frac{1}{6}$ AE =(AB+4CD+EP) × $\frac{1}{3}$ AC; and when the points of BDP are at no great distance from each other the parabolic curve will very nearly cioncide with any other regular curve, drawn through the same points.

Let us now take the ordinates EP, GH, IR ; then will (EP+4GH+IR)× $\frac{1}{3}$ EG=the area EPRI; and (IR+4LM+NO)

$\times \frac{1}{3}$ IL=area IRON. Also (NO+4FQ+KS)$\times\frac{1}{3}$ NF=area NOSK, whence by addition, we have [(AB+KS) + (4DC+4GH+4LM+4FQ)+(2EP+2IR+2NO)]$\times\frac{1}{3}$CA=$(a+4b+2C)\times\frac{1}{3}$ AC.

*Cor.* This theorem may be applied for computing the contents of solids, by using parallel sections instead of the ordinates, as will appear in Prop IV., Corollaries and Scholium, B. I.

*Schol.* It is evident that the greater the number of ordinates and the nearer the points DBP, &c., are to each other, the more nearly will any curve, drawn through them, agree with the parabola, and hence the greater accuracy will be obtained; the same remark will also apply to solids.

*Cor.* 2. Hence if the area of any space AB, KS is known, and the ordinates AB, IR, KS, the value of the line AK may be determined.

For if AB + KS = $a$ and IK = $b$, we have, by considering the curve as a parabola, whose axis is parallel to KS, the area A = $(a + 4b) \times \frac{1}{3}$ AK, let AK = $p$, hence we have

$$\tfrac{1}{3}p = \frac{A}{a+4b} \text{ or } p = \frac{\frac{1}{3}A}{a+4b} \quad - \quad - \quad - \quad - \quad - \qquad (1)$$

Or, if the number of ordinates is increased, we have by the proposition, A = $(a + 4b + 2c)\ \frac{1}{3}p \div n$, $n$ being the number of divisions in the line AK, or the number of ordinates less 1, and we have $p = \dfrac{3nA}{a+4b+2c}$ - - - (2)

### PROPOSITION XVI. PROBLEM.

*To find eleven equidistant ordinates to hyperbola between the asymptotes, and by means of those ordinates to find the area.*

Taking the equation $a^2=xy$, and assuming $a = 10$, and the first value of $x$, or the distance from the centre to the first ordinates = 10, and if the lower distance is 1 = $d$, we shall have for the ordinates.

$$\frac{10}{10}\quad \frac{10}{11}\quad \frac{10}{12}\quad \frac{10}{13}\quad \frac{10}{14}\quad \frac{10}{15}\quad \frac{10}{16}\quad \frac{10}{17}\quad \frac{10}{18}\quad \frac{10}{19}\quad \frac{10}{20}$$

the sum of the first and last or $a=\frac{10}{10}+\frac{10}{20}=1{,}5$

the sum of the even ordinates or $b$

$$=\frac{10}{11}+\frac{10}{13}+\frac{10}{15}+\frac{10}{17}+\frac{10}{19}=3.4595393$$

The sum of all the recurring ordinates or $c$

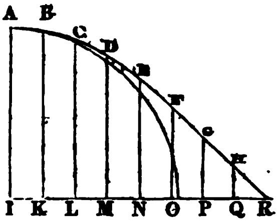
$$=\frac{10}{12}+\frac{10}{14}+\frac{10}{16}+\frac{10}{18}+=2.7281744\tfrac{4}{5}$$

Therefore, by the proposition,

$(a+4b+2c)\frac{1}{3}d=6{,}9315021$, is the area required.

### PROPOSITION XVII. PROBLEM.

*To find the value of π by equi-distant ordinates to the revoloidal curve.*



If A = the area of a quadrant, AIR of the revoloidal curve, and $a$ = the minor semi-axis, AI = the radius of a circle inscribed in the curve, and $\frac{1}{2}\pi$ = the semi-major axis, IR; and if $a$ = an ordinate EN to the major axis, equi-distant from I to the vertex R, then will the area of the curve, considering it as a parabola, be $= a+4\,b\times\frac{1}{6}\frac{\pi}{2}$; and if A equal the area of the revoloidal quadrant, the semi-transverse $=\frac{\pi}{2}=\frac{A}{\frac{1}{6}(a+4b)}=\frac{6A}{a+4b}$. (1)

Let $a=1$, and $b=\sqrt{\frac{1}{2}}=70712$, &c.

and A will $=r^2=1$,

hence we have $\frac{A}{a+4b}=1.5672239=\frac{\pi}{2}$

this is true to two places of figures only, but the third should be 7 instead of 6.

Let now two other ordinates, CL, GP, be taken in addition to this, equidistant therefrom, and from the extremities of the axis, and we have by Prop. XVI, Cor. 2, $\frac{3An}{a+4b-2c}=\frac{1}{2}\pi$

Let the two ordinates $b$ be the sines of 22° 30′ = .382683, and 67° 30′ = .923880, and $c$ = sine of 45° = .707107, and we have $a=1$, $4a=5.226252$, $2c=1.41424$, $A=r^2=1$.

Hence $\frac{3nA}{a+4b+2c}=\frac{12}{7.640466}=1.5705848$, &c. $=\frac{\pi}{2}$

which is true to four places, viz: 1,570, but the fifth should be 7 instead of 5.

Let, now, seven ordinates be taken between the two extremes, and the distance of the ordinates will be reduced $\frac{1}{2}$.

Thus the ordinates will be the sines of 11° 15′, 22° 30′, 33° 45, 45°, 56° 15′, 67° 30′, 78° 45′, 90° = .195090. .382683. 555570, .707107, .831470, .823880, .980785, 1.

$$a = \left\{\begin{array}{l} \sin.\ 0^\circ = 0 \\ \sin.\ 90^\circ = +1 \end{array}\right\},\ b = \left\{\begin{array}{r} \sin.\ 11^\circ\ 15' = .195090 \\ 33^\circ\ 45' = .555570 \\ 56^\circ\ 15' = .831470 \\ 78^\circ\ 45' = .980785 \\ \hline 2.562915 \end{array}\right.$$

$$c = \left\{\begin{array}{r} \sin.\ 22^\circ\ 30' = .382683 \\ 45^\circ\ \ 0' = .707107 \\ 67^\circ\ 30' = .923880 \\ \hline 20.13670 \end{array}\right.$$

Hence, $a=1$, $4b=10.251660$, $2c=4027340$, and $n=8$.

Therefore, in the formula $\frac{3nA}{a+4b+2c} = \frac{\pi}{2}$ we have $\frac{1}{2}\pi =$ 1.5707833, which is true to five places, but the sixth should be 9 instead of 8.

By comparing each of the results obtained above with the true numbers, we shall have the ratio of its approximation.

Thus, the difference between the first result and the true number is, .0035724

that of the second, .0002115

that of the third .0000130.

Hence, it will be seen that the result approximates in nearly a geometrical progression, to the true value of $\pi$ as we increase the number of ordinates, or as the distance between the ordinates is decreased. We may, therefore, determine the number of ordinates that must be taken, in order to give an accurate result to any number of decimal places; for it will be perceived that the ratio of the above variations are nearly 16 to 1. Hence, we may safely infer, that it will approximate at the rate of three decimal figures in every two subdivisions. Instead of computing the value of $\frac{1}{2}\pi$ from the ordinates drawn in the whole quadrant, we may take any small arc of the quadrant, and having found its quadrature let it be called A′; and if we proceed as before, by drawing one ordinate equidistant from its extremes, we shall have, according to the formula, $\frac{\frac{1}{6}A'}{a+4b} = \pi' =$ the assumed arc, which, multiplied by the number of times this arc is contained in the quadrant, will give the same result as though the ordinates are taken for the whole quadrant.

Let the formula $\frac{\frac{1}{2}A'}{a+4b} = \pi'$ be applied to a segment of the revoloidal surface whose arc is 30°, and whose greater ordinate is the radius.

Here, $A' = \sin. 30° \times r = \frac{1}{2}$; hence, $A' = \frac{1}{2}A$,

$$a = r + \sin. 60° = 1.866025$$
$$b = \sin. 75° = .965926$$
$$\frac{6A'}{a+4b} = \frac{3A}{5.729709} = .52358670 = \pi' = \text{arc of } 30°,$$

radius 1; hence the arc of 90°, or $\frac{1}{2}\pi = 1.57076010$; which is true to five places of figures.

Let there be two other ordinates drawn across the segment, then by the formula, $\frac{3nA'}{a+4b+2c} = \pi'$, since $\pi = \frac{1}{2}\pi \div 3$, we shall have $\frac{9nA'}{a+4b+2c} = \frac{1}{2}\pi$, and $\pi = 4$.

$$a = \left\{ \begin{array}{l} \sin. 60° = .866025 \\ \sin. 90° = .100000 \end{array} \right\} = 1.866025$$
$$b = \left\{ \begin{array}{l} \sin. 67° \ 30' = .923880 \\ \sin. 82° \ 30' = .991445 \end{array} \right\} = 1.915325$$
$$c = \sin. 75° = .965926$$

Hence, $a+4b+2c = 11.459157$

and, $9nA' = 18,$

$$\frac{9nA'}{a+4b+2c} = 1.57079616.$$

which is true to seven places, which is as far as the sines are calculated, on which the value is predicated.

Let the first and second results be compared, and we shall have the difference of the first result and the true value,

$$\left. \begin{array}{r} 1.57079632 \\ -1.57076010 \end{array} \right\} = .00003622$$

that of the second,

$$\left. \begin{array}{r} 1.57079632 \\ -1.57079616 \end{array} \right\} = .00000016$$

Divide the first error by the last, and we have the quotient = 226 = the ratio of the approximation, or the proportional accuracy of determination, by varying the number of the ordinates. It will be perceived, therefore, that this portion of the revoloidal curve approximates much faster than the whole quadrant, and is, therefore, more nearly similar to a parabola than the whole curve; it may hence be inferred, that if any small segment of the revoloidal surface is taken adjacent to the conjugate diameter, such segment will be very nearly a portion of a parabola, and that, by so considering it, the value of the length of the arc may be determined with any required degree of accuracy.

For, in taking the whole quadrant, we found the ratio of convergency, by doubling the number of the ordinates, to be 1 to 16; and in the segment embracing the arc of 30°, adjacent to the conjugate diameter, we find the ratio of convergency to be 1 to 226; and if an arc is taken still smaller, the ratio of convergency will become proportionally greater.

Let any segment of the revoloidal surface be taken, and if the value of its arc, or the value of $\pi$, be computed by any number of ordinates, and if the number of ordinates is then increased so that the common distance is reduced one-half, and the value of $\pi$ is again estimated by the increased number of ordinates, and if the variations of the two results from the true value be compared with each other, they will show the ratio of convergency of the process for that arc by increasing the number of its ordinates, or the rate of approximation by any specific increase of the ordinates for such arc or segment.

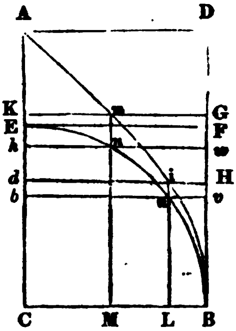
Hence, we may at all times determine the value of $\pi$ to any required degree of exactness; for if we wish to determine its value to any given number of decimal places, we have only to assume some given arc and find its rate of convergency, then take such an arc as, according to this rate, will give the required result.

The arc of 90° gave the true result only to 2 places, that of 30° to 5 places, with three ordinates; and we may expect a still greater ratio of convergence for a smaller arc; let us take an arc of 10°; we may, according to this ratio, only have the value to 8 places, and by proceeding to decrease the arc, we should, by taking 1°, have the value to 16 places; but since the curve approaches more and more to a similarity with that of a parabola as we approach the vertices of the conjugate axis, the ratio of convergency increases also rapidly as we approach that point, or as the arc assumed is decreased; so that, by taking an arc of one minute of a degree, the accuracy of determination would extend to many places of decimals; and if the arc should be reduced still further, to seconds and fractions of a second, the result would come out true to several hundred decimal places; all of which is manifest by pursuing the investigation.

Let a distance be taken on the axis equal the arc of 2 minutes of a degree from the conjugate diameter. Then having
the sine 89° 58′ = .9999998308
that of 89° 59 = .9999999577

### PROPOSITION XI. THEOREM.

*The area of a plane revoloidal surface is to that of its circumscribed parallelogram, as the sum of an indefinite series of sines in the circle, to the sum of an equal series of radii.*



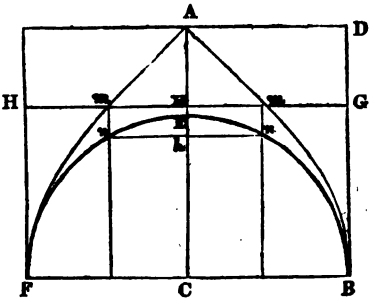
Let ACBD be a parallelogram circumscribing the quadrant of the revoloidal surface ABC, and let EBC be a quadrant of the circle, then will the area of the quadrant of the revoloidal surface ABC be to that of a parallelogram ACBD, as the sum of an indefinite series of sines of the quadrant BCE, to the sum of an equal series of radii.

For, since all ordinates *vb*, *d*H, &c., cut the surfaces of those figures in relation of their magnitudes in the sections through which such ordinates pass; and since (Prop. XI,) if ordinates be drawn through those figures equidistant from each other, the portions of the ordinates intercepted by the curve and axis, are equal to the sum of a series of sines of arcs in arithmetical progression for the whole quadrant equal in number to the number of the ordinates, and if these ordinates are equidistant from each other, the sum of the portions passing through either surface, drawn into their common distance, may be taken for the surface; and since the distance of the ordinates is equal by hypothesis, both for the parallelogram and revoloidal surface, the portions of the ordinates intercepted by each, will be in relation to their surfaces respectively, when their number is indefinitely increased, and their distance becomes indefinitely small. Hence, as the sum of a series of sines of arcs of the whole quadrant taken in arithmetical progression, is to the area of a quadrant ACB of the revoloidal surface, so is the sum of an equal series of radii to the area of the parallelogram ACBD; and what has been shown for one quadrant of the revoloidal surface, is also true for the whole.

*Cor.* Hence, the space BDA without the revoloidal surface, is to the revoloidal surface, as a sum of an indefinite series of arcs in arithmetical progression to the sum of a similar series of sines.

### PROPOSITION XII. THEOREM.

*If the quadrant* AB *of a revoloidal curve be made to revolve about its axis* AC, *and if a plane hemisphere of a quadrangular revoloid be described about the solid so generated, having the same axis* AC, *then the revoloid will be to its circumscribing prism, as the sum of the squares of a series of sines of the quadrant, to the sum of the squares of an equal series of radii.*



Let an indefinite number of planes be passed through the revoloid perpendicular to the axis, and at equal distances from each other, and the sections made by these planes will all be squares, (Prop. 1, Cor. 4 B. III.) and their sides will all be equal to the ordinates H*m*, *hn*, &c., drawn through the intersection of such planes with the vertical sections; and hence the side of each parallel section, is equal to twice the sine *nh* of the quadrant corresponding to such section, being = *nn*. Now let each of those parallel planes be extended to HG passing through the prism, and it is evident that each of the sections of the prism will be squares, whose sides are severally equal to twice the radius CB, of an inscribed circle. Now the magnitudes of these solids through each section, are evidently in the relation of the magnitudes of such sections; and if the number of these equidistant planes are indefinitely increased, and they are indefinitely near together, their sum will represent the whole of each of the solids in the relation of their whole magnitudes, and since each conjugate section of the revoloid is the square described on double the sine, answering to the ordinate in reference to the quadrant CEB, and each section of the prism is the square described on the line HG, equal twice the radius, it follows that the solidity of the revoloid, is to that of the circumscribing prism, as the sum of the squares of a series of the sines of the quadrant, to the sum of the squares of an equal series of radii.

*Cor.* Hence, the solidity of the space between the surfaces he revoloid described as above, and that of its circum-

*El. Geom.*); but the altitude of the triangle AHC is HI, equal to LC, equal to half the altitude CD of the triangle ACD; therefore, the triangle ACH is equal to half the triangle ACD; but the triangle ACD is equal to the quadrant ACE; therefore, the triangle ACH is equal to half that quadrant, or is equal to the sector ACT, which is the first branch of the proposition.

Now, because the triangle ACH is equal the sector ACT, the segment A*b* cut off from the sector by the line AH, is equal to the portion HT*b* included in the triangle and without the sector, (Prop. XVIII. Cor.) which is the second branch of the proposition.

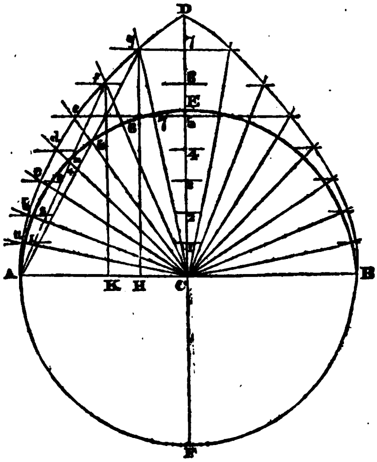
And because the line CD is equal to the arc AE of the quadrant, and because the arc AT is equal to half the arc AE, it is also equal to half the line CD=CL=HI, which is the last branch of the proposition.

*Cor.* If BA be extended to *a*, so that C*a* shall be equal CH, and if a line H*a* be drawn and bisected by C*e*, and a line be drawn from *e* to A, forming the triangle AC*e*, this triangle so formed will be equal to the sector of the circle intercepted by the lines CA and C*e*; and the line A*e* will cut off a segment of the circle without the triangle equal to the space included in the triangle without the sector, and a perpendicular *ei* let fall from *e* on the radius AC, is equal to the arc intercepted by the lines CA and C*e*. And the same may be inferred from any further divisions or subdivisions of the circle.

### PROPOSITION XX. THEOREM.

*With the radius* CA *let there be described a circle* AEBF, *and from the centre* C *draw the line* CD *perpendicular to the diameter* AB, *or radius* CA, *and equal in length to the arc of the quadrant* AE, *let the line* CD *be divided into any number of equal parts, as* 1, 2, 3, 4, *&c., and let the arc of the quadrant be divided in like manner into a similar number of equal parts* 1, 2, 3, 4, *&c. From the divisions on the line* CD *draw the lines* 1a, 2b, 3c, *&c., parallel to the radius* CA; *and through the divisions* 1, 2, 3, *&c., on the arc draw the radial lines* Ca, Cb, Cc, *&c. Then, if from the points of intersection* a, b, c, *&c., of the radial lines, with their respective parallel lines according to their respective numbers, lines be drawn as* fA, *to the extremity of the radius* CA, *then will this line, together with* CA *and* Cf *form a triangle which is equal to the sector* CA6, *included within the radial line* Cf *and the radius* CA.

*And the segment* Am *cut off from the sector by the line* fA *is equal to the portion* fm6 *included in the triangle without the sector; and if lines be drawn from the several points of intersection of the parallel and radial lines perpendicular to the radius as* fK, *the lines so drawn will be respectively equal to the arcs intercepted by the radial lines from the extremities of which they are drawn, and their radius* CA.

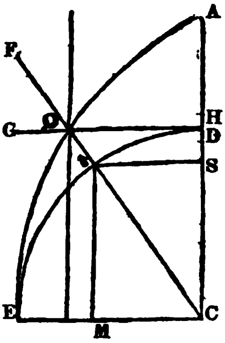


For each of the radial lines C*a*, C*b*, &c., cut the arc AE in the same ratio that the corresponding parallel lines 1*a* 2*b*, &c., cut the perpendicular CD; thus the radial line C*g*, passing through the point or division 7 on the arc, cuts off one division from the arc, and intercepts with the radius CA all the rest, and the corresponding line 7*g* parallel to the radius CA cuts off D7 on the perpendicular CD, so that if the whole line CD is equal to the arc AE, the portion C7 of the line CD = H*g* is equal to the arc A7 of the circle. And the radial line C6*f* cuts the arc AE in the same ratio as the line *f*6 parallel to CA, cuts the line CD, viz., the radial line C6*f* cuts the circle through the division marked 6, and the line *f*6 parallel to CA, cuts the line CD in the division marked 6, so that if the whole line CD equal the arc AE, then will the portion C6 of that line=*f*K be equal to the portion A6 of the arc. Now, the area of every sector of the arc AC6, is equal to the arc of the sector multiplied by half the radius CA; but the triangle A*f*C is equal to the line *f*K multiplied by half the base or radius CA; therefore, the triangle A*f*C is equal to the sector AC6, and hence the segment A*m*, cut off from the sector by the line A*f*, is equal to the portion *fm*6 included within the triangle, but without the circle, and, as has been shown, *f*K=C6 is equal to the arc A6. And as the same holds true in each of the points of intersection *a*, *b*, *c*, *d*, &c., it follows that the result corroborates the affirmation expressed in the proposition.

*Cor.* If a curve line be drawn through the several points *a*,*b*, *c*,*d*, &c., and a triangle be formed by two lines from any point in the curve drawn to the two extremities of the radius, the triangle so formed, will be equal to the sector included between the radius and the other line terminating in C, and the same relation of areas and lines will exist with regard to the triangles and lines drawn from any point of the curve, as though they were drawn through the points *a*, *b*, *c*, *d*, &c.

*Scholium* 1. The curve BD may be described about the quadrant BE in a similar manner; and since from any point in this curve, if a triangle is constructed on the radius as a base, this triangle is equal to the sector of the circle included between the radius and one of its sides, the curve may be called the curve of the circle's quadrature. This curve varies from the revoloidal curve, inasmuch as the revoloidal curve is formed by drawing a line through the points of intersection of a series of lines parallel to the radius drawn through their respective divisions on the perpendicular, with an equal series of lines perpendicular to the former, drawn through their respective divisions on the arc (see Prop. II.)

*Scholium* 2. This curve is generated by the locus of the intersection Q. of the two right lines CF,HG; HG being made to pass uniformly along from A to C, being always perpendicular to AC, while CF revolves about the centre through the arc ED.



Let the origin be at A.

Let AH $= x$, HQ $= y$

AC $=$ DE $= \pi,'$ angle ACQ $= \theta$, $t$S $=$ sine $\theta = s$, $t$M $=$ *cosine* $\theta = c$

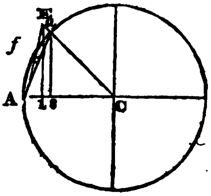
Then $c : s :: \pi' - x : y$

Hence $y = \dfrac{s\pi' - sx}{c}$ which is the equation to the curve.

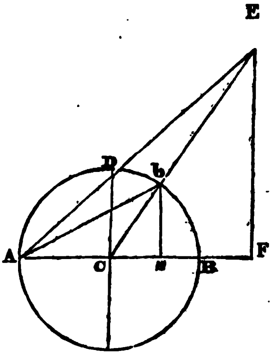
PROPOSITION XXI. THEOREM.

*If from the extremity of the radius of a circle, a chord be drawn cutting off any segment less than a semi-circle, and if from the centre of the circle a secant be drawn through the opposite extremity of the segment, and if the secant be produced so that a line drawn from its extremity, perpendicular, to meet the diameter produced, shall be equal to the arc of the segment, then the area of the segment will be equal to that of a triangle formed by the chord of the segment, and the part of this secant line without the circle, and a line joining the opposite extremities of this line with that of the chord.*

For it is evident from the converse of Prop. XVIII Cor., that if CE be so drawn that the perpendicular E*t* shall be equal to the arc A*a*, then the triangle CEA will be equivalent to the quadrant A*a*C; take away the triangle AC*a*, then there will remain the triangle A*a*E = the segment A*fa*.



First, let A*a* be a chord cutting off the segment A*fa*; from the centre C through the extremity of the segment at *a*, draw a secant line C*a* produced to E, so that the perpendicular E*t* on the diameter, shall be equal to the arc A*fa* cut off by such secant; then will the area of the segment A*fa* be equal to that of the triangle A*a*E formed by the chord A*a*, with the part *a*E of the secant without the circle, and the line AE joining the opposite extremities.



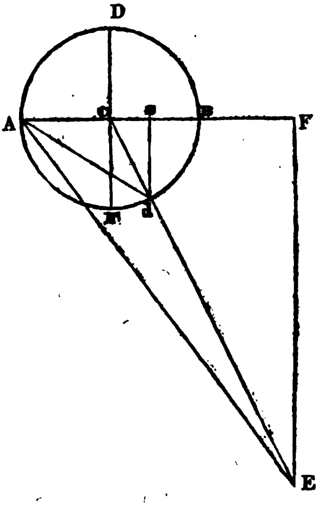
Secondly, let the chord A*b* extend into the second quadrant, cutting off the segment AD*b*, draw C*b*, and extend it to E, so that the perpendicular EF drawn from the point E to the diameter AB, produced, shall be equal to the arc AD*b* cut off by the chord A*b*, or the secant CE, and the area of the segment AD*b* will be equal to that of the triangle A*b*E formed by the chord A*b*, the part of the secant *b*E, and the line AE joining their opposite extremities.

For, in the triangle ACE, the area is equal to the base AC, the radius of the circle multiplied by half the altitude EF, or the arc AD*b* of the segment, but the sector AD*b*C is equal to the radius AC multiplied by half the arc AD*b*; hence the triangle ACE = the sector AD*b*C, therefore, if we take the triangle AC*b* from each, we shall have the segment AD*b* = the triangle A*b*E.

### PROPOSITION XXII. THEOREM.

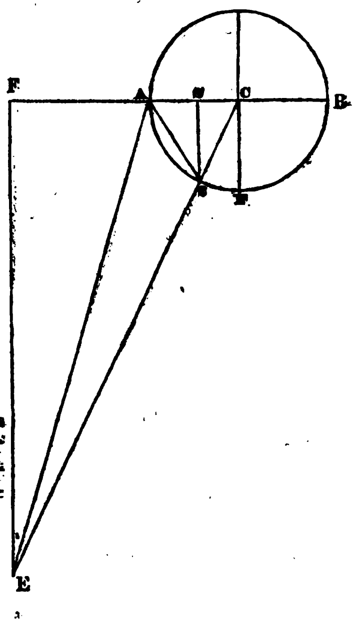
*If from the extremity of the radius of a circle, a chord be drawn cutting off a segment greater than a semi-circle, and if through the opposite extremity of the segment, a secant be drawn from the centre of the circle, and if the secant be produced, so that a line be drawn from its extremity on the diameter, produced, if necessary, shall be equal to the arc of the segment, then the area of the segment will be equivalent to the triangle formed by the radius, secant line, and a line joining the opposite extremities of these lines,* plus *a tri-*

***angle formed by the radius and the chord, with the line joining the opposite extremities of those lines.***



For, let the chord extend into the third quadrant, cutting off the segment ADB*d*A, greater than a semi circle, draw C*d*, and produce it to E, so that the perpendicular EF, on the diameter AB produced, shall be equal to the arc ADB*d* of the segment, and join AE, and the area of the triangle ACE, *plus* AC*d* will be equal to the segment ADB*d*A.

For the area of the triangle ACE is=the base AC, or radius, multiplied by half the perpendicular FA, which is equal to the arc ADB*d*, by hypothesis; but the area of the sector ADB*d*CA is equal to the radius CA multiplied by half the arc ADB*d*, hence the triangle ACE = the sector ADB*d*CA. Add to both the triangle AC*d*, and we have the segment ADBCA = the triangle ACE + the triangle AC*d*.



Secondly, let the chord A*e* extend into the fourth quadrant, cutting off the segment ADBFE. Through the point *e* draw the line CE produced, so that the line EF perpendicular to BA produced, shall be equal to the arc ADBF*e*. Then the segment ADBF*e* will be equal to the trian. ACE + the trian. A*e*C.

For the area of the triangle ACE is equal to the base AC, or radius multiplied by half the perpendicular FE, which is, by hypothesis, equal the arc ADBF*e*; but the area of the sector CADBF*e*C is equal to the radius AC multiplied by half the arc ADBF*e*; hence the triangle ACE is equal the sector

CADBF*e*C. Add to each the triangle AC*e*, and we have the segment ADBF*e* = the triangle ACE + the triangle AC*e*.

*Scholium*. It may be observed, that, as the termination of the segment approaches the point B, or as the segment becomes equal to the semi-circle, its equivalent triangle becomes infinitely extended in the line AE, and at the same time the sine becomes infinitely small, and while it passes the point B, the sine is equal to 0, and AE is infinite. The same may be said as it approaches the point A on the fourth quadrant, and becomes equal to a complete circle.

### PROPOSITION XXIII. THEOREM.

*The area of a segment of a circle is equal to half the product of the difference between the arc of the segment and its sine multiplied by the radius.*

First, let A*fa* (see first diagram to Prop. XXI) be a segment of a circle cut off by the radial line C*a*, and produce it to E, so that the perpendicular E*t* on the radius will be equal to the arc of the segment; join EA and (Prop. XXI.) the area of the triangle AE*a* will be equal to the segment A*fa*; from the point *a* draw the perpendicular *as*, which is the sine of the arc A*a* of the segment, and the area of the triangle AE*a* will be equal to (E*t*—*as*) $\frac{1}{2}$AC (Prop. XXXI, B. IV, *El. Geom.*) Hence, the segment being equal to the triangle, AE*a* is equal to half the product of the difference of the arc, and its sine multiplied by the radius.

Secondly, let the chord A*b* (see second diagram to Prop. XXI) extend into the second quadrant, draw CE, making EF = the arc AD*b*; draw AE, and the triangle AE*b* will be equal to the segment AD*b*; draw *bs*, the sine of the arc AD*b* of the segment; then will the area of their triangle AE*b* be equal to (EF -- *bs*) $\frac{1}{2}$AC; (Proposition XXXI, Book IV, *Elements Geom.*); hence the segment, being equal to the triangle, is equal to half the difference of the arc and its sine multiplied by the radius.

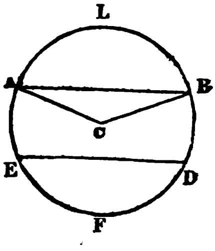
Thirdly, let the chord A*d* (see first diagram to last prop.) extend into the third quadrant, cutting off the segment ADB*d*, greater than a semi-circle.

Draw C*d* and extend it to E, making EF equal to the arc ADB*d*; join EA, and the area of the segment ADB*d* will be equal to the triangle ACE+AC*d*, (Prop. XXII); draw *ds* perpendicular to the radius CB, and this line will be the sine of the arc ADB*d*; and since it is a sine of an arc greater than a semi-circle, its value is to be considered negative, by Trigonometry.

From EF substract — $dS$, and we have EF + $dS$, which, if we multiply by AC, will give the area of the triangle AEC +AC*d*, which (Prop. XII.) is also equal to the segment ADB*d*. Hence the area of the segment ADB*d* is equal to the difference of the arc, and its sine multiplied by the radius.

Fourthly, let the chord A*e* (see 2d fig. last prop.) extend into the fourth quadrant, cutting off the segment ADBF*e*; draw EC so that EF shall be equal to the arc ADBF*e* of the segment join EA, and the area of the segment ADBF*e* will be equal to that of the triangle ACE + the triangle AC*e* (Prop. XII); draw *es*, the sine of the arc ADBF*e*, which, by Trigonometry, is negative, being below the diameter, and in the fourth quadrant, which sine substract from the line EF, and we have EF + *es*, which, multiplied by half AC, gives the value of the segment ADBF*e*. Hence we have, as before, the segment equal to half the product of the difference of the arc of the segment, and its sine multiplied by the radius.

Hence we may infer, that a circular zone or a portion of the circle included between two segments is equal to half the product of the difference of the arc of the zone, and one of the segments included, and the sine of such arc, — the difference between the arc of the included segment, and its sine multiplied by the radius.



For if ABDE be a zone, and ABL be a segment of the circle C, the segment EALBD equal to the zone ABDE, and segment ABL; but these segments are respectively equal to the excess of their several arcs above the sines multiplied by half the radius. Therefore, the zone ABDFE, is equal to the difference of the excess of the arcs EALBD, and ALB, above their respective sines, multiplied by half the radius. The same may be shown in relation to the area ABDFE in reference to its external segment EDF, or AEFDB, regard being had to the positive and negative value of the sines.

*Scholium.* Let A represent the area of the segment of a circle; let $x$ = the arc of the segment, and $s$ = sine of that arc, and let $r$ = radius, and the segment may be expressed as in the following formula

$$A = \tfrac{1}{2}(rx - rs) \quad - \quad - \quad - \quad (1)$$

If $r = 1$, then the expression becomes

$$A = \tfrac{1}{2}(x - s) \quad - \quad - \quad - \quad (2)$$

whence the segment is expressed in terms of the arc and sine

Let $A' =$ the area of a circular zone ABDE
and $x' =$ the arc EALBD
$s' =$ the sine of that arc.

Then will the area of the zone be expressed by

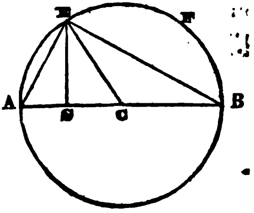
$A'=\frac{1}{2}(rx'—rs')—\frac{1}{2}(rx—rs)$ - - - - (3)

If $r=1$ $A'=\frac{1}{2}(x'—s')—\frac{1}{2}(x—s)$ - - - - - (4)

The arc may be expressed in terms of the numbers representing the segment, and the sine of the arc, by the formula, $x = 2A+s$ - - - - - - - - - (5)

And the sine, in terms of the arc and segment by

$s = x—2A$ - - - - (6

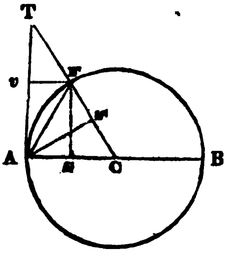


*Sch.* That the area of the segment of a circle is equal to half the difference of the arc of the segment, and its sine × by the radius may also be shown as follows. Let AE be the segment of a circle, whose radius is AC, draw ES perpendicular to the radius, and ES will represent the sine of the arc AE; let AC be represented by $r$, and ES by $s$, and the arc AE by $x$, then will $\frac{1}{2}rx =$ the sector ACE, and $\frac{1}{2}rs =$ the triangle AEC, and $\frac{1}{2}rx — \frac{1}{2}rs =$ the difference of the sector and triangle = the segment AE, viz., the seg. is = half the difference of the arc, and its sine multiplied by the radius.

Also, in the segment EFB let CB, be represented by $r$, and the arc EFB by $x$, and ES will be the sine of the arc EFB, $rx =$ the sector ECBF, and $\frac{1}{2}rs =$ the triangle ECB. $\frac{1}{2}rx —\frac{1}{2}rs =$ their difference = the segment EBF.

### PROPOSITION XXIV. THEOREM.

*The area of the space intercepted by a tangent and secant without a circle, is equal to half the product of the difference of the tangent and arc, intercepted by the secant, multiplied by the radius.*



Let ATF be the space intercepted by the tangent AT, and the secant CT, without the circle; and the area of that space will be equal to half the product of the difference of AT and the arc AF, multiplied by the radius AC.

For the arc of the triangle ATC is equal to $\frac{1}{2}$AT × AC, and the area of the sector ACF is equal to half the arc AF × AC.

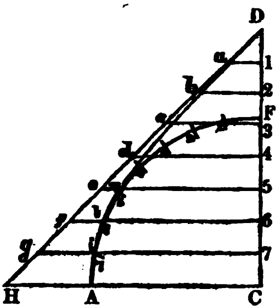
Let $t=$the tangent AT, and $x=$the arc AF, $r=$AC.

Then we have for the triangle ACT, $\frac{1}{2}rt$, and for the sector AFC, $\frac{1}{2}rx$; if from the triangle we take the sector, we have $\frac{1}{2}rt-\frac{1}{2}rx=$the space AFT, hence as in proposition.

*Scholium.* Draw F*s*, FA, and F*v*; then F*s* will represent the sine of the angle ACF, and *s*A or T*v*, the versed sine; hence the traingle ATF equals one-half the product of the difference of the tangent and sine multiplied by radius equals one-half the product of the tangent and versed sine; draw A*s'* perpendicular to CF, and this will also represent the sine of the angle ACF on the radius CF; hence, also, the triangle ATF is equal to one-half the difference of the secant CT and radius multiplied by the sine.

### PROPOSITION XXV. THEOREM.

*If about a plane revoloidal surface from a quadrangular revoloid, a square be described, the space enclosed by the square, but without the revoloidal surface, will be to that contained by the square, as the sum of an infinite series of segments of the quadrant of the circle, whose arcs are in arithmetical progression, to the sum of a similar series of sectors subtended by the same arcs.*



Let ACD be a quadrant of a plane revoloidal surface, and let HCD be a triangle forming one-fourth of a square circumscribing the whole revoloidal plane surface. Divide the axis CD into any number of equal parts as 1, 2, 3, 4, &c., and through the points of division draw ordinates 1*a*, 2*b*, 3*c*, &c., parallel to HC, and these ordinates will cut the surface of the quadrant of the revoloid and its circumscribing triangle, in the relations of their magnitudes, through the portions where such ordinates pass; and if the number of those ordinates be indefinitely increased, the sum of those drawn across the triangle HCD, will be to the sum of those drawn through the revoloidal surface ADC, as the area of the triangle to the area of the revoloidal surface. Now, the ordinates *a*1, *b*2, *c*3, &c., are equal to the arcs represented by numbers on the arc AF of the quadrant, corresponding to those on the axis, (Prop. II, and Cors.); and the portions *i*7, *l*6, *m*5, &c., of those ordinates, are severally equal to the sines of those arcs, (Prop. XIII, Cor. 1,); hence the portions of those ordinates, intercepted by the curve AD and line DH, are severally equal to the difference of the arcs and sines represented by these ordinates. And since we have shown (Prop. XXIII.) that the area of the segment of a circle is equal to half the difference of the arc of the segment and its sine multiplied by radius, and since the area of the sector of a circle is equal to half the arc of the sector multiplied by radius, it follows that the sector and segment containing the same arc, are to each other, as the arc is to the difference of the arc and sine.

Let $a$ equal the arc, $s$ equal the sine, $r$ equal the radius.

Then $\frac{1}{2}ra=$the sector,
and $\frac{1}{2}ra-\frac{1}{2}rs=$the segment containing the same arc, which are to each other in the ratio of

$$\tfrac{1}{2}ra : \tfrac{1}{2}ra-\tfrac{1}{2}rs, \text{ or of } a : a-s.$$

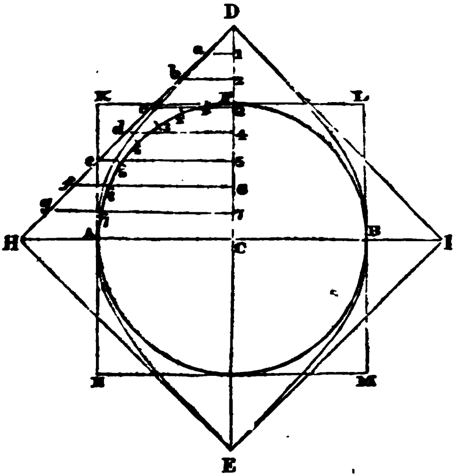
And since this is true for a segment and sector, contained by any arc in that quadrant, that is, the sector of a circle is to the segment containing the same arc as the arc is to the arc minus the sine, or as the ordinates $g7$ to $gi$, or as $f6 : f\,l$, &c.; and as this relation evidently exists in reference to each of the parallel ordinates, and as the ordinates represent arcs of the circle in arithmetical progression, it follows that those ordinates drawn across the triangle, may also represent a series of sectors of a circle, while the portion of those ordinates intercepted by the curve AD and line HD, may represent a similar series of segments of the circle. And as we have shown above that the surfaces HCD and HAD are to each other in the relation of the ordinates passing through each, it follows that the trilinear space ADH will be to the triangle HCD as the sum of an infinite series of segments of the circle whose arcs are in arithmetical progression, whose first term is equal to the common difference, and whose last term is the quadrant of the circumference to the sum of a similar series of sectors with the same arcs. And since the whole plane revoloidal surface consists of four quadrants, ADC, and its circumscribing square consists of four triangles, HCD, it follows that the whole revoloidal surface will be to the whole circumscribing square in the same ratio.

*Cor.* The revoloidal quadrant ACD, may be represented by a similar series of isosceles triangles formed by the chords of the series of segments with two radii drawn from the extremities of those chords. For we have shown that the series of sectors may be represented by the area HCD, while the series of segments are represented by HAD; if we take the segments from the sectors, we shall have the triangles, evidently equal HCD — HAD=ACD.

### PROPOSITION XXVI. THEOREM.

*The area of a quadrangular revoloidal surface, from a right quadrangular revoloid, is to that of its circumsribing square, as the sum of an infinite series of sines of the quadrant whose arcs are taken in arithmetical progression, and whose common difference is equal to the first term, to a similar series of arcs of such sines.*

Let DHEI be a square circumscribing the plane revoloidal surface DAEB, and let ADC be a quadrant of that surface, the triangle HDC being its corresponding portion of the square; divide the semi-axis CD into any number of equal parts, as 1, 2, 3, 4, &c., and through the points of division draw the ordinates 1*a*, 2*b*, 3*c*, &c., parallel to CH, those lines will each be equal to such arc of the quadrant AF as corresponds to the point of division on the line CD, in reference to similar divisions of the quadrant AF; thus the line 1*a* equal the arc F1 on the quadrant, since it is evidently equal to 1D on the axis CD; and (Prop. II, Cor.) 1D=1F; also, if we take the line 2*b*=2D, this is also, for similar reasons, equal to the arc 2F; and hence each line parallel to CH, terminated by the lines CD, HD, are equal to the arc, corresponding to the divisions on the line CD, through which it passes.



Again, the portion of these lines or ordinates intercepted by the axis CD, and the curve AD, (Prop. X, Cor.) are severally equal to the sines of the arcs, corresponding to the divisions from which they are drawn on the axis. And since, by hypothesis, the arcs are taken in arithmetical progression, they must be equidistant; hence, if the number of such ordinates are indefinitely increased, the sum of the portions of them intercepted by the axis CD, will represent the area or the surface ADC, as the whole ordinates drawn across the triangle HDC represent the area HDC. Therefore the surface ADC is to the surface HDC as the sum of the series of sines, to the sum of a similar series of arcs taken as above. Now, since this is the case with one quadrant, it must also be true in relation to the whole revoloidal surface and its circumscribing square.

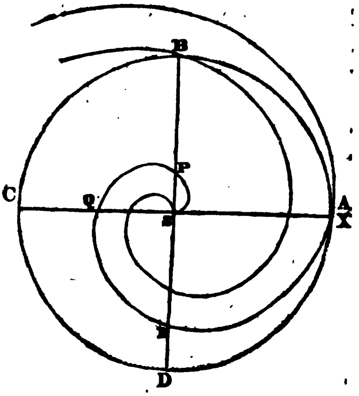
*Cor.* 1. Hence, if a square KLMN be described on the diameter of the inscribed circle, the square so described will be to the square DHEI as the sum of the series of sines of the quadrant, to the sum of a similar series of arcs of the sines; since the square KLMN (Prop. III, B. III,) is equal to the area of the revoloidal surface DAEB.

*Cor.* 2. Each of the ordinates *a*1, *b*2, *c*3, &c., drawn across the triangle HCD parallel to HC, is equal to the arc of the circle represented by the corresponding numbers in the divisions of the quadrant AF. Thus, *a*1 equal the arc 1F, *b*2 equal the arc 2F, and *g*7 equal the arc 7F or the quadrant.

## ON SPIRALS.

There is one class of curves which are called spirals, from their peculiar twisting form. They were invented by the ancient geometricians, and were much used in architectural ornaments. Of these curves, the most important as well as the most simple, is the spiral invented by the celebrated Archimedes.

This spiral is thus generated: Let a straight line SP of an indefinite length move uniformly round a fixed point S, and from a fixed line SX, and let a point P move uniformly also along the line SP, starting from S, at the same time that the line SP commences its motion from SX, then the point will evidently trace out a curve line SPQRA, commencing at S, and gradually extending further from S.



When the line SP has made one revolution, P will have got to a certain point A, and SP still continuing to turn as before, we shall have the curve proceeding on regularly through a series of turnings, and extending further from S.

To examine the form and properties of this curve, we must express this method of generation by means of an equation between polar ordinates.

$$\text{Let } SP = r, SA = b, ASP = \theta;$$

then since the increase of $r$ and $\theta$ is uniform, we have

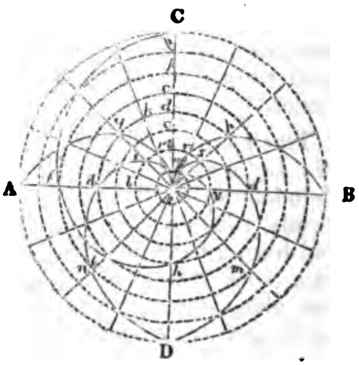
$$SP : SA :: \text{angle } ASP : \text{four right angles} :: \theta : 2\pi$$

$$\therefore r = \frac{b\theta}{2\pi} = a\theta, \text{if } a = \frac{b}{2\pi}.$$

From this equation it appears that when SP has made two revolutions or $\theta = 4\pi$, we have $r = 2b$, or the curve cuts the axis SX again at a distance, 2SA ; and similarly after 3, 4, $n$ revolutions it meets the axis SX at distances 3 ; 4, $n$ times SA.

Let any number of concentric circles be described, whose radii IA, IC are in arithmetical progression, and if the circumference of the outer circle is divided into any number of equal parts, and radii are drawn from each of the points of division in the circumference of the circle to the centre I, lines drawn from the centre in such manner as to pass through the points of intersections, of the several radii with the curves in consecutive order, will be spirals similar to those of Archimedes.

If the curve line pass from I, through *rp*B, or through *rq*A, the curves I*rq*A, I*rp*B will be spirals ; so also the lines I*sdm*D, I*sdn*D, and I*vhf*C and spirals ; all generated by the same laws, but with different ratios of their angular, compared with their rectilenear motion ; or their circular, with their radial motion. The two spirals I*sdm*D, I*sdn*D, commencing on the line IC, and terminating at D, form the heart-like figure I*sdm*D*nds*I.



These spirals, and especially the one with a heart-like form, are extensively used in mechanical operations ; to communicate a uniform rectilenear, reciprocating, from a rotary motion; it is therefore important that they receive some consideration.

The area of any portion included between the spiral and its circumscribing arc of the circumference terminated by the radius, its origin, is equal to two-thirds the sector, having the same arc of the circumference as its base.

For let I$a$, I$b$, I$c$, &c., be the radii of circles in arithmetical progression, then will the arcs of these circles intercepted by two radii, be also in arithmetical progression, and since the value of $\theta$ also increases uniformly in arithmetical progression along with $r$ or $\pi$, hence the value of the intercepted parts of the several arcs will be a series of arithmeticals multiplied into another corresponding series of arithmeticals, therefore their products will be a series, of the squares or a series of numbers proportional to a series of squares of a series of arithmeticals.

It has been shown (Prop. IV, Cor. 3, B. I) that the sum of an infinite series of the square of a series of numbers in arithmetical progression increasing from 0, is equal to $\frac{1}{3}$ of the last term multiplied by the number of terms.

But the arc intercepted by the spiral with the radius, as its origin is the last term, and the radius represents the number of terms; hence the area I$rp$BCI is equal to $\frac{1}{3}$ of the product of the arc BC × IC; and the area I$sdm$DBCI is = $\frac{1}{3}$ arc CBD × IC; also the area I$vhf$CADBCI is = $\frac{1}{3}$ circumference ACBD × IC. But the area of the whole sector IBC, ICBD, or ICBDAC in either case is equal to half the product of their respective arcs, multiplied by the radius; hence the space intercepted by the spiral in each case is $\frac{1}{3}$ that of their respective sections.

*Cor.* 1. Hence the space I$rp$BI, is = $\frac{1}{3}$ the sector ICB; the area I$sdm$DI is = $\frac{1}{3}$ the sector ICBD or $\frac{1}{3}$ the semi-circle; and the area I$vhf$CI is = $\frac{1}{3}$ of the whole circle,

Also the heart I$sdm$D$nds$I = $\frac{1}{3}$ of the whole circle.

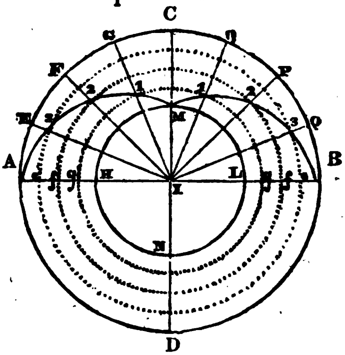
*Scholium* 1. The spiral of Archimedes is sometimes used for the volutes of the capitals of columns, and in that case the following description by points is useful. (See first diagram.)

Let a circle ABCD be described on the diameter CSA, and draw the diameter BD at right angles to AC; divide the radius SC into four equal parts, and in SB take SP=$\frac{1}{4}$SC, in SA take SQ=$^{1}$SC, and in SD take SR=$\frac{3}{4}$SC; then from the equation to the curve these points belong to the spiral; by subdividing the radius SC and the angles in each quadrant we may obtain other points as in the figure. In order to complete the raised part in the volute, another spiral commences from SB.

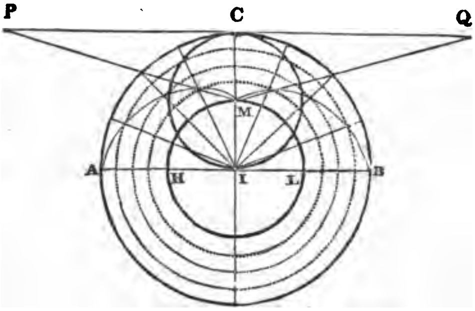
*Scholium.* These spirals are of the same kind, as those formed by winding a chord around a conical spire, from the vertex to the base, in such manner, as to encircle the spire at equal distances; the exact length of such curve is of difficult determination.

The same spiral would be represented by the convolutions of a conical screw; also, by a screw represented on a disc.

If the origin of the spiral is at any point M, not in the centre of the concentric circles, then the area AFCM123A between the spiral and the outer circumference is$=\frac{1}{3}$ of the product of the arc ACB through which the curve would have passed from the centre I, multiplied by the radius$-\frac{1}{3}$ of the arc LM×IM $-\frac{1}{2}$ (arc LM + arc BC) ×MC.



If PQM be a triangle, whose base PQ = the semi-circumference ACB = the angular space passed through by the two spirals MA, MB, then either portion PCM, QCM of the triangle may be expressed by $\frac{1}{2}$ PC or $\frac{1}{2}$ AC×CM = $\frac{1}{2}$ arc CB× IC$-\frac{1}{2}$ arc LM×IM$-\frac{1}{2}$(arc LM + arc BC) × MC, from this



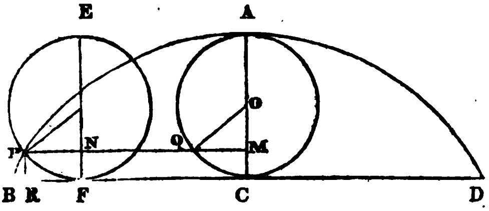
substract the expression for the area included within the spiral, and the arc AB, and we have $\frac{1}{6}$ ACB × IC $-\frac{1}{6}$LM × IM = the difference of the areas.

## THE CYCLOID.

If a circle EPF be made to roll in a given plane upon a straight line BCD, the point in the circumference which was in contact with B at the commencement of the motion, will, in a revolution of the circle, describe a curve BPAD, which is called the cycloid.

This is the curve which a nail in the rim of a carriage-wheel describes during the motion of the carriage on a level road. The curve derives its name from two Greek words signifying "circle formed."

The line BD which the circle passes over in one revolution is called the base of the cycloid; if AQC be the position of the generating circle in the middle of its course, A is called the vertex and AC the axis of the curve. The description of the curve shows that the line BD is equal to the circumference of the circle, and that BC is equal to half that circumference. Hence also if EPF be the position of the generating circle, and P the generating point, then every point in the circular arc PF, having coincided with BF, we have the line BF = the arc PF, and FC = the arc EP or CQ;



Draw PNQM parallel to the base BD.
Let A be the origin of the rectangular axes,
AC the axis of $x$, and O the centre of the circle AQC.
Let AM $= x$, AO $= a$,
MP $= y$, angle AOQ $= \theta$:

then by the similarity of the position of the two circles, we have

$$\text{PN} = \text{QM}, \text{ond PQ} = \text{NM};$$

$$\therefore \text{MP} = \text{PQ} + \text{QM} = \text{NM} + \text{QM} = \text{FC} + \text{QM} = \text{arc CQ}$$
$$+ \text{QM that is, } y = a\theta + a \sin.\theta = a(\theta + \sin.\theta) \qquad (1)$$
$$x = a - a\cos.\theta = a \text{ vers. } \theta \quad - \quad (2)$$

The equation between $y$ and $x$ is found by eliminating $\theta$ between (1) and (2)

$$\cos.\theta = \frac{a-x}{a} \quad \therefore \sin.\theta = \frac{\sqrt{2ax-x^2}}{a}$$

$$\text{and } y = a\theta + a\sin.\theta$$

$$= a\cos.^{-1}\left(\frac{a-x}{a}\right) + \sqrt{ax-x^2}$$

But we can obtain an equation between $x$ and $y$ from (1) alone; that is from the equation, AP = arc CQ + QM.

For arc CQ = a circular arc whose radius is $a$ and versed sine $x$

$$= a \left\{ \text{a circular arc whose radius is unity and vers. sin.} \frac{x}{a} \right\}$$

$$= a \text{ vers.}^{-1}\frac{x}{a}$$

$$\therefore y = a \text{ vers.}^{-1}\frac{x}{a} + \sqrt{2ax-x^2}$$

If the origin is at B, BR = $x$ and RP = $y$, the equations are

$$x = a\theta - a\sin.\theta$$

$$y = a - a\cos.\theta.$$

We shall not discuss these equations at length, as the mechanical description of the curve sufficiently indicates its form.

The cycloid, if not first imagined by Galileo, was first examined by him; and it is remarkable for having occupied the attention of the most eminent mathematicians of the seventeenth century.

Of the many properties of this curve the most curious are, that the whole area is three times that of the generating circle, that the arc CP is double of the chord of CQ, and that the tangent at P is parallel to the same chord. Also that if the figure be inverted, a body will fall from any point P on the curve to the lowest point C in the same time; and if a body falls from one point to another point, not in the same vertical line, its path of quickest descent is not the straight line joining the two points, but the arc of a cycloid, the concavity or hollow side being placed upwards.

# BOOK V.

## ON THE PRODUCTION AND RESOLUTION OF GEOMETRICAL MAGNITUDES, CONSIDERED AS LINES, SURFACES, AND SOLIDS, EXISTING IN THEIR SPECIFIC RELATIONS OF FORM AND PROPORTIONS.

## CHAPTER 1.

### DEFINITIONS AND PRINCIPLES.

ART. 1. We have hitherto referred lines, surfaces and solids, in all their varieties of figures and species, to some specific quantities and relations which were cognizable in such magnitudes, and whose properties were rendered evident to our consideration. Magnitude we have compared with magnitude; figure with figure; and we have thereby established their relations, under arbitrary considerations.

We will now consider magnitudes in the relation of their organization, or in the relation of their laws of production; and instead of referring magnitudes to specific magnitudes arbitrarily chosen, we will refer them to others, only in the relation of their laws of generation.

2. Since a point by definition is locality without extension, any number of associated points cannot possess magnitude, hence a magnitude is not a multiple of one, or any number of points.

3. Neither can any number of lines, however associated, constitute a surface, since lines are supposed to possess no breadth or thickness, one of which is essential to a surface; for if one line does not possess breadth, neither can any number of associated lines; and if a line be multiplied by any abstract number, since it is expressed only in relation to its length, it can only be multiplied or increased in that relation.

4. So, also, if a surface be multiplied by any number, in itself considered, the product cannot be a solid; for since the surface possesses no thickness, it does not possess the characteristic of the solid, and hence any number of such surfaces, or multiple of such surfaces in themselves considered, cannot be a solid.

5. The distance between any two points is a line. For a point being locality without extension, if there be two locali-

ties, they must be seperate from each other, and their distance from each other is necessarily extension in space, which agrees with our definition of a line, viz., "extension in one dimension."

6. Space is a medium in which all positive objects, and all local relations exist; its existence is only indicated by its universal property of extension; it is infinitely divisible in each or all its three dimensions of extensions, and infinitely extensible.

7. Any definite portion of space, or any extension in space, is magnitude.

Magnitude may possess extension in one, two, or three dimensions, but space can properly exist only in its three dimensions of extension; if it can be divested of extension in one dimension, it can in another, and so on till its extension is extinguished. Magnitude may be properly applied to extension in whatever degree it exists; but space cannot properly exist independent of its three dimensions, which are its essential properties.

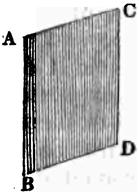
8. If there be two points A, B, the first point A, drawn through the distance AB, produces or describes the line AB; that is, the distance from A to B in the portion of space passed through by the point A, or the locality occupied by the point A in its passage, is the line AB.

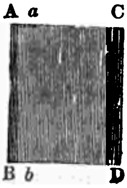
A———B

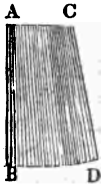
9. If a line be moved through any space in a direction not agreeing with its length or extension, the locality passed through by the line is a surface.

Thus, if a line AB, be moved from its position AB to CD, it will, by that means, generate the surface ABDC, for the line will have occupied every portion of the extension between the two lines AB and CD, which cannot be said of any limited number of lines placed in juxtaposition across the figure ABDC.

If, in this motion, the line AB always maintains its parallel position, and if any point A in the line, describes a right line AC, the surface will be a rhomboid or parallelogram; and if, in addition to this, the line AC, or the direction of its motion is perpendicular to AB, then will the rhomboid be a rectangle.







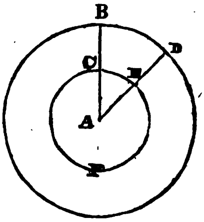
But if the line AB in its motion should not preserve its parallel position, or if the distance BD, passed through by the point B, is greater than AC, then the figure generated will depend on the nature of the lines which serve as its boundaries, but in general, in such case, one or both of the lines AC, BD will be curves, the nature of which will depend on the specific variation of the course and inclination of the generating line.

If instead of one line AB being drawn through the whole distane AC, a series of parallel lines AB, *ab*, &c., are drawn through their respective distances from each other, the result will be similar; or the same surface will be described by the series drawn through their several small distances, as by the single line drawn through the greater distance.

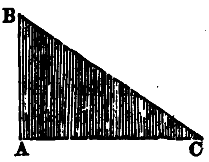
The geometrical operation of drawing a line through a given distance to produce a surface, is equivalent to that of multiplying a line by a line, the product of which we have shown in the elements of geometry to be a surface.

The measure of the surface generated by the motion of a line, is the length of the line multiplied by the distance passed through by the line; which distance may always be regarded as the distance moved by the centre of the line.

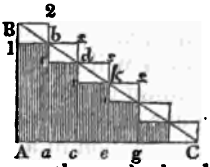
10. If the point A remains fixed while the line revolves around it till it, comes again into the position AB from whence it started, the surface generated is a circle; and the line described by the point B, will be the circumference of the circle. If it moves only from the position AB to AD, the surface will be a sector of a circle, and the line described by the point B, is the arc BC of the circumference, and because the circumference CEFC described by the centre C of the revolving line represents the whole motion of the line AB in its revolution; hence, AB drawn into the circumference CEFC, represents the whole surface generated.



11. If the line AB be conceived to decrease uniformly, as it is moved forward, always in a parallel position, till it terminates in a point, the figure generated by the motion of the decreasing line AB, will be a triangle ABC.



We may, instead of supposing the triangle to be generated by the line AB, decreasing as it advances toward C, suppose it to generated by an infinite series of decreasing ordinates, parallel to AB; each of which may be supposed to be drawn through the distance, between itself and the next one. Thus, let *ab*, *ef*, &c., be a series of decreasing ordinates situated equidistant from each other on the line AC, and let *ab*, be drawn into the position A1 on the line AB, *cd*, into the position *a*1, *ef*, into the



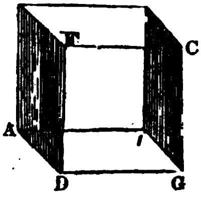
position *c*1, &c., through the whole series. Now, since at the extremity of each of the consecutive ordinates a triangle B1*b*, *b*1*d*, &c., is left without the triangle ABC, which has not been described or passed over by the ordinates, the triangle is therefore not perfectly described by the decreasing ordinates; but if the number of the ordinates are infinitely increased, those spaces become indefinitely small, and hence may be omitted as being of no appreciable value. Moreover, if the motion of the ordinates is reversed, or if they are moved from A toward C, taking AB as the first ordinate. the surface thus described would exceed the triangle itself, in the same amount as it would fall short in the former case; thus, let AB be brought in the position *a*2, *ab*, into the position *c*2, &c., and the small triangles formed above the hypothenuse BC, by this means, would be equal to those falling below it in the former case, and half their sum would be a correction to be added or subtracted in either case; but since this correction is equivalent to the surface generated by half the line AB drawn through the distance A*a*, it follows that the sum of all the ordinates *ab*, *ac*, &c., +half the line AB drawn into the common distance A*a*, generates an area equivalent to the triangle ABC.

* But when the number of ordinates are indefinitely increased one-half, AB is infinitely small in regard to their sum, and hence may be omitted.

12. Hence, if an infinite series of equidistant and parallel ordinates to a right line AD, decrease uniformly from AB to AC, then will the surface generated by drawing those ordinates through or into their common distance, be a trapezium.

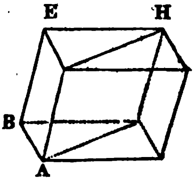
13. If a surface be drawn through any space not in the direction of a line parallel to the surface, the product will be a solid.

Thus, if a surface ABFD be drawn through the distance DG, till it comes into the position IEGC, it will thereby generate a solid AC.



If the surface is always parallel to its first position, and if any point A, moves through a right line AI, the solid will be a prism, which will also assume a character according to the figure of the generating surface: thus, a rectangular, a triangular, or a polygonal prism, may be generated by the motion of a rectangle, a triangle, or a polygon; and in the same manner may a circular prism, or cylinder, be generated by the motion of a circle, always parallel to its first position, and in a direction perpendicular to such generating surface, If instead of one surface being drawn through the whole distance AB, there be a series of similar surfaces parallel to each other, and if each be drawn through their respective distances, the same solid would thereby be generated.

The same principles may also apply to solids of revolution of any figure about a fixed axis, and the partial revolution of a series of similar figures, one of whose several sides is the common axis.



14. If a plane quadrilateral surface ABGF, be conceived to move uniformly along in a direction perpendicular to itself, and to decrease uniformly in one of its dimensions during its motion, till it terminates in a line; it will, by that means, generate a wedge or a triangular prism, ABEFHD.

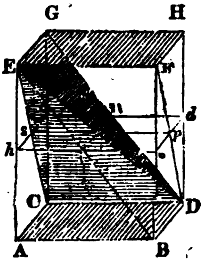
If, instead of the decreasing plane, there be an infinite series of quadrilateral planes equidistant from each other and decreasing in one of their dimensions in consecutive order, and in arithmetical progression till one of them terminates in a line, then the series of planes drawn through their infinitely small distance will generate a wedge or a triangular prism; all of which becomes evident by reference to Art. 11, for the same reasoning will apply here as in that case, since this prism may be conceived to be generated by the perpendicular motion of the triangle, which was there found to be the product of an infinite series of decreasing lines.

15. Hence, if across any plane figures, A, B, C, an infinite number of parallel and equidistant ordinates are drawn, the sum of the ordinates intercepted by each may be regarded as a measure of their surfaces when compared with each other; although lines, however associated, cannot represent surface in absolute terms, yet an infinite number of parallel lines drawn equidistant across two or more surfaces, will represent the ratios of those surfaces to each other, and may hence represent those surfaces in relation to their forms and comparative magnitudes.

Hence, for the purposes of investigation, the properties of geometrical magnitudes and their relations, an infinite series of parallel ordinates drawn across any plane figure, may be regarded as the measure of that figure.

And also, for the same reasons, may an infinite series of parallel and equidistant planes passed through a solid, represent the capacity or value of the solid.

Neither will this mode of investigation lead to any error, seeing, we do not thereby establish any absolute measure for the surface or solidity, in terms of superficial and solid units, but only the relations of certain surfaces or solids through this medium, to certain other surfaces or solids with which they are compared; and since they are compared with each other under the same circumstances, the result of that comparison must hence be correct.

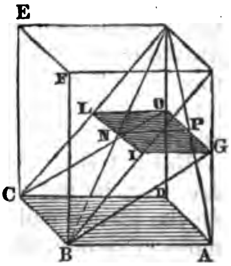


16. Let the rectangular prism AH, whose base ABDC is supposed to be a square, be divided as at Prop. IV. B. I., into the pyramids ABDCE, EFHGD, CDGE, BEFD, through which let a plane *hodc* be passed, cutting the several pyramids in the sections *higs*, *sgnc*, *iopg*, *gpdn*, parallel to the base ABDC. Let *hi* or $hs=a$, and *ia* or $dn=b$, then may the section *hode* be expressed by $a^2+2ab+b^2$, that is, the section $higs=a^2$, $(sgnc+iopg)=2ab$, and $gpdn=b^2$; $\sqrt{a^2+ab}$ is a mean proportional between $a$ and $a+b$. Let an indefinite number of planes be passed through the solid parallel to the base, and each section may be expressed in the same manner, but the value of $a$, it will be seen, is constantly decreasing in arithmetical progression, as we ascend from the base to the vertex E; and hence, represents successively a series in arithmetical progression; the value of $b$ is also increasing in the same order during the successive ascent of the series of planes; but $a+b$ is a constant quantity during the whole change of the relative values of $a$ and $b$, and hence $a+b$ represents a successive series which is constant. Now, because $a^2+2ab$, the square of the proportional mean between $a$, one of the series of arithmeticals, and $a+b$, one of the series of constants, which repsesents a section *hine* through the two pyramids ABDCE, CDFE; it follows that since an infinite series of parallel sections represent the whole of these pyramids, that the sum of the squares of the whole series of geometrical means will represent the whole of the solid ABDCEG. But it has been shown, (Prop. IV. B. I,) that the solid ABDCGE is equal to one-sixth of the product of the sum of the squares of the two bases *plus* four times a middle section drawn into its altitude. Hence, if there be any infinite series of quantities, such that the terms are severally geometrical means between the several terms of a series of arithmeticals, and of a similar series of constant quantities; then will one-sixth of the sum of the square of the first and last terms *plus* four times the square of the middle term drawn into the series, be equal to the sum of the squares of the series.

Hence, if an infinite series of quantities varying in arithmetical progression, be drawn into a similar series of constant quantities, then will the sum of the series of rectangles be equal to the product of the first, *plus* the last term, *plus* four times the middle term drawn into one-sixth of the series.

17. It will appear that since the solid ABDCGE represents the sum of all the, $a^2+ab$, viz., the squares of the whole series of mean proportionals between the corresponding terms of two other series; if there be a series of quantities, decreasing from $z$ to 0 in arithmetical progression, and another equal series of $z$ a constant quantity, the sum of the squares of a series consisting of mean proportionals between the corresponding terms of the two series, will be equal to half the sum of the squares of the series of $z$, or equal to half the square of $z$ drawn into the series.

18. It will also appear, that if there be an infinite series of quantities decreasing from $z$ to $p$ in arithmetical progression, and another similar series of constant quantities, $z$, the sum of the squares of a series of mean proportionals between the corresponding terms of the two former series, will be equal to half the sum of the squares of the series of $z$, *plus* one-half the sum of a similar series of rectangles of $p\times z$.

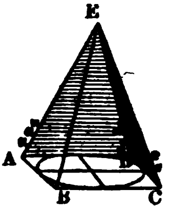


For, let AE be a prism, the length AB or AC, of whose side is equal $z$, and if we make IG and LO$=p$, and construct the plane LIBD, we shall have the prismoid ABDOCGIL, which may be represented by the sum of the squares of a series of geometrical means between the terms of the series of the decreasing arithmeticals, and those of the constant quantities. But this prismoid may be further divided by the plane GOCB into the wedge ABCDOG, which is equal to half the sum of the squares of the series of $z$, and the wedge GOLIBC, which is equal to half of a similar series of the rectangles of $z$ into $p$.

19. Let a series of $s$, be the series of constant quantities, (and let $s=$AD,) while another series varies from $z$ to $p$ in arithmetical progression, (making $z=$AB, and $p=$GI.) Then will the sum of the squares of the series of geometrical means between the terms of the two former series be equal to half the sum of a similar series of rectangles of $s\times z+$a series of rectangles of $s\times p$.

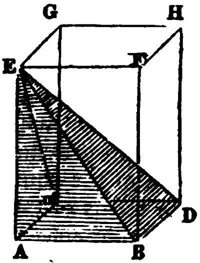
For the prismoid ABCDOGIL equals the series of mean proportionals as before, and the wedge ABDCOG will be equal to half a similar series of rectangles of the series $s$ with a similar series of $z$, and the wedge IGHLCB will be equal to half the sum of a similar series of $s$, drawn into an equal series of $p$.

Hence, a wedge may be conceived to be generated from a series of squares of mean proportionals between a series of constant quantities, *s* or *z*, and an infinite series of lines in arithmetical progression from *z* to *o*, or from *o* to *z*.



20. Let AD, *ad*, *ad* &c., be an infinite series of lines, increasing in arithmetical progression from 0 to *z*, or from E to BA, and the arithmetricals will generate or constitute the triangle ABE, and the sum of their squares will generate or describe the pyramid ABCDE. For if an infinite number of parallel lines or ordinates be drawn across a plane figure, those ordinates will represent the figure in the relation of its magnitude and form; (Art. 12,) but the lines or ordinates, *ad*, *ad*, are infinite in number by hypothesis, and increase from 0 to *z*, or they commence at E and terminate at AB; hence the sum of the ordinates constitute, or are a function of the triangle. Also if an infinite number of parallel planes be passed through a solid, the sum of those planes drawn into their distance, may be regarded as the solid when compared; for the solid consists of an infinite number of parallel planes drawn into their infinitely small distance; (Art. 16) and hence this series of planes would represent the solid in the relation of its magnitude, as the ordinates represent the triangle.

But if CB = AB, then each of the parallel planes, *adc*, *adc*, &c., will be the square of its corresponding ordinate, *ad*, and all the planes will be the sum of the squares of all the ordinates. Hence, the sum of the squares of the series represents the pyramid, ABCDE, in the same manner as the sum of the ordinates represents the triangle, ABE.

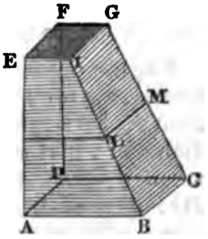


21. If the series of lines be a constant series of *z*, then in the same manner as the series of arithmetricals would describe or generate the triangle ABE, the constant series of *z* would generate the circumscribing rectangle ABFE, and the series of squares of *z*, would generate in like manner the prism ABCDEFHG, circumscribing the pyramid.

If instead of the squares of the several terms of the series, in the proposition generating the pyramid, there be taken the series of circles described on the several terms as diameters, those circles would generate a cone = the cone inscribed in the pyramid. And if instead of the squares constituting the prism, Art. 1, there be taken a similar series of circles, then would the solid generated be a cylinder equal to a cylinder inscribed in the prism. Also, if we take any other figure, which we constitute a similar function of each successive term of the series, we may generate pyramids, prisms, or frusta, of different character, but of the same general species.

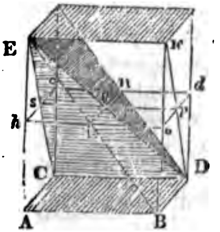
21. If a line be made to increase from *p* to *z*, and if its variation be represented by an infinite series of arithmeticals, then the series will truly represent a trapezium, and the series of the squares of the first, may represent a frustum of a pyramid.

22. If *a* be made to pass successively through all the values from 0 to *z*, while *b* is made to pass in like manner through all the values from 0 to *p*, then the two series drawn into each other will generate a pyramid with a rectangular base.



23. If there are two variable lines *a* and *b*, and if *a* be made to pass in succession from *p* to *z*, or from *z* to *p*, in an infinite series, while *b* remains equal to *z* or *s*, representing a similar series of constant quantities, then will the solid produced by drawing the corresponding terms of these series into each other represent a prismoid AG; but in this case the prismoid will have two parallel sides ABIE, CDFG. But if, while *a* is passing from *z* to *p*, *b* at the same time varies successively from *z* or *s* to *f*, then will the solid generated by the series of rectangles of the corresponding terms of the variable quantities, be a prismoid, neither of whose sides would be parallel except the two bases.

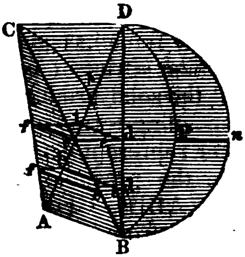
24. If one of the variable magnitudes should be made to pass successively in an infinite series from 0 to *z*, while the other should pass from *z* or *s* to *p*, then the solid generated by the rectangle of the corresponding terms of the series would be a wedge.



If *a* be an infinite series of lines, varying from 0 to *z* in arithmetrical progression, and *h* be a like series varying from *s* or *z* to 0, then if the corresponding terms of the series be drawn into each other, their product will be a triangular pyramid which may be resolved into the two wedges *scng*CD, and *scng*EG; all of which is evident by inspection.

Let the ordinates drawn across the triangle CD, parallel to its base, be a series of ordinates increasing from 0, at the vertice E, to *z*=CD; and let this series be called *a*: let the ordinates drawn across the triangle EGD, be a series decreasing in the same time from EG or $s$ to 0; call this series $b$: then the series represented by $a$, drawn into that represented by $b$, will generate the solid EGCD.

25. Let the ordinates drawn across the triangle ABC be a series decreasing in arithmetical progression from $z$, or AB to 0, and let those drawn across the triangle BCD be a similar series of ordinates increasing from 0 to $z$, then may the series of proportional means between the corresponding terms, represent a series of equidistant ordinates, drawn across a semi-circle, whose diameter is $z$=DB, and the rectangle of the corresponding terms may represent a similar series of equidistant ordinates, drawn across such portion of a parabola whose axis is AB, as is intercepted by the curve, and a diagonal from the vertex B to the extremity of the base, the axis of the parabola being equal to $z$.

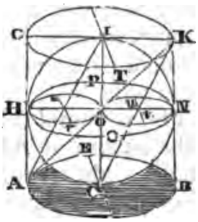


For the ordinates *dn* drawn across a semi-circle are severally mean proportionals, between the abscissæ of the diameter; that is, any ordinate *dn*, is a mean proportional between *d*B and *d*D; now if their be a series of abscissæ *d*B taken in arithmetical progression increasing, then their corresponding abscissæ *d*D, *d*D, &c., will be a series of decreasing arithmeticals; and if BD is equal to CD or AB, then will the abscissæ *d*D, *d*B be severally = to their corresponding ordinates *de*, *ef*, in whatever position they are taken; hence the series of ordinates, across the semicircle is a series of proportional means between the several corresponding terms of the increasing and decreasing series.

Again, it has been shown (Prop. VII, *Sch.* B. I,) that the expression for any ordinate *ec*, drawn across the parabola, CB*h*, between the curve and the diagonal, parallel to the axis, is equivalent to the rectangle *de* × *ef*, and since this is true of every parallel position of the ordinate *ec*; hence the sum of the series of ordinates, is equivalent to the sum of the series of rectangles of the series of variables, = the sum of the series of squares, of a series of ordinates *dn*, &c., across the semi-circle.

Let DB be greater than CD, or AB and the ordinates *dn*, will be equivalent to those drawn across a semi-ellipse, whose major axis is BD, and minor axis CD; but if CD is greater than BD, then will BD be the minor axis of the ellipse, &c.

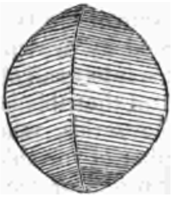
Hence the solid ABCD formed by drawing the corresponding terms of the increasing and decreasing lines into each other is equivalent in its expression to the area CB*h* of the parabola, and the semi-circle DB*n* is equivalent to the sum of a series of square roots of an infinite series of parallel ordinates *ec* across the parabola, or of parallel sections *defi* through the solid.



26. The elements contained in the foregoing propositions, may be applied also to solids of revolution; for instead of a prism and its inscribed pyramids in Art. 16, we may substitute a cylinder, and its inscribed cones, &c.; and if planes HPNQ be passed through the cylinder, the cones and their complements, these will be cut in the relation of their magnitudes, respectively. in each section; which will be in all cases, the same as that of the prism, and the inscribed pyramids, of equal base and altitude.

If the semi-circle or semi-ellipse CIN be made to revolve about the axis CI, and by its revolution to produce a sphere, or spheroid, then, because every section described by the parallel ordinate ON, would be proportional to the squares of their circumscribing lines respectively, the sum of the sections described, would be proportional to the sum of the squares of the generating lines; hence the sum of an infinite series of those sections, and consequently the solid generated by the semi-circle would be equal to four times a middle section, multiplied by $\frac{1}{6}$ of the series. And any segment or zone of the sphere or spheroid, will be equal to the sum of the bases + four times a middle section × $\frac{1}{6}$ its altitude.

27. If every section *defi* through the pyramid ABCB (Art. 25) should be contracted in one of its dimensions till it becomes a square, and if the edge DB should continue to be a right line, then will the sides BCD, ABD, be plane surfaces, and the side ACD, ABC will become curved, and the whole solid will be equal and similar to a quadrant of a revoloid; and four of such pyramids, would constitute a perfect right revoloid; moreover the sides ABC, and ACD would become semi-circles.



For it has been shown that the square root of every section *defi*, through the solid, is equal to an ordinate drawn across the semi-circle D*n*B through the same parallel; but the square root of the section, is the side of a square equivalent to the section; hence, if the sides of the sections are severally the ordinates belonging to a semi-circle, then the solid must acquire a form, similar to that of a quadrant of a rectangular revoloid, such as would be formed by passing planes through the axis, bisecting its oppo-sides.

If, when the sections of the solid become squares, its vertices are conceived to be at the extremities of an axis, passing through its centre; then the solid would become an elliptical revoloid; its conjugate axis being = to $\frac{1}{2}$ its vertical or transverse axis.

The pyramid ABCD is equal to $\frac{1}{6}$ of the prism, whose base is the square of AB, and altitude BD; four such solids, or the whole right revoloid is $= \frac{4}{6} = \frac{2}{3}$ the circumscribing prism. Also, the prism, circumscribing the elliptical revoloid, whose base = *defi*, and altitude BD, being $= \frac{1}{4}$ of the former prism, is the prism circumscribing the elliptical revoloid; hence, the elliptical revoloid is $= \frac{2}{3}$ its circumscribing prism.

If from a cylinder of equal base and altitude, two equal cones be taken, one on either base, and of an altitude equal to that of the cylinder, the two remaining portions would be each equivalent to the cylinder; and every section through each of these portions, by planes parallel to the cylinder's base, would be equivalent to a corresponding section through the quadrant of the sphere; each of these portions are $= \frac{1}{6}$ of the cylinder; hence, four of these portions = the sphere, are $= \frac{2}{3}$ the circumscribed prism as found in the Elements of Geometry.

*Scholium.* We may, from the preceding investigations, draw the following deductions and conclusions,

First, that any series of quantities in arithmetical progression, varying from $z$ to 0, drawn into any series of constant quantities, will produce a quantity whose value is $= \frac{1}{6}$ the sum of the base or maximum product of the variable + four times the value of the product of half the maximum value drawn into the series. Also, that the value of any multiple of this series may in like manner be determined.

Second, that if any series of numbers varying from $z$ to 0, in arithmetical progression be drawn into another similar series, direct or reciprocal, the value of the product is $= \frac{1}{6}$ the sum of the two bases produced + four times a middle base formed by drawing $\frac{1}{2}$ the maximum values of the terms of the series into each other.

And that this is true for any multiple, or power of the variable series, whose exponent is an integer or when, any number of variable quantities are drawn into each other, whether direct or reciprocal, and this is the basis of the *Integeral Calculus*, as will appear in the subsequent pages.

## CHAPTER II.

### ON THE CONSTRTCTION OF QUANTITIES WHOSE ELEMENTS ARE A SERIES OF CONSTANT OR VARIABLE QUANTITIES.

Art. 1. Having proceeded thus far, in analyzing the production of geometrical magnitudes, showing the manner and law of their generation, we are enabled, by having the elements and the law of the production of any magnitudes, to give a geometrical construction of such magnitudes.

We were taught, in the application of algebra to geometry, the mode of constructing integral algebraic quantities or expressions geometrically; we are now to represent a series of quantities, under a single construction; or to construct quantities whose elements are a series, either of constant, or variable quantities.

2. In considering the relations which exist between different quantities, those which, during the whole of any investigation are supposed to retain the same value, are called *constant quantities;* those to which different values are assigned, are called *variable quantities*: constant quantities are usually represented by the former letters of the alphabet, as $a$, $b$, $c$, &c., and variable quantities by the latter, as $uxyz$, &c.

3. When two or more variable quantities are connected in such a manner, that the value of one of them is determined by the value assigned to the other, the former is said to be a function of the other variables.

Thus, in the equation $y=ax+bx^2+c$, where the value of $y$ depends on the value assigned to $x$; $y$ is said to be a function of $x$, which is usually expressed by $f'(x)$, $\varphi(x)$, $\psi(x)$, or similar abbreviations.

Also, if an infinite series of equidistant ordinates are drawn across a surface, the sum of those ordinates is a function of the surface. So, also, the sum of an infinite series of planes through a solid may be regarded as a function of the solid, or the solid or surface a function of the planes or ordinates.

4. When any quantity or magnitude as the element of other quantities or magnitudes is variable, the sum of a series of the variable quantity, within its variable limits, may be represented by a dash drawn below or above the letter representing the variable quantity.

Thus $\underline{z}$ or $\bar{z}$ may represent a series of the variable quantity, $z$. If the incipient value of the variable is $z$, and the series is decreasing to $s$, or 0, the dash must be placed below the letter; but if its incipient value is 0 or $s$, and its terminate value is $z$, being an increasing series, the dash must be placed above. Thus $\underline{z}$ indicates a series decreasing from $z$ to $s$ or 0; and $\bar{z}$ indicates the series increasing from 0 or $s$ to $z$.

5. If it is required to express a series of quantities in arithmetical progression from $x$ to $s$, or from $s$ to $x$, it may be thus written—$\underline{x} \cdot\cdot s$, or $\bar{x} \cdot\cdot s$; or which is the same, $\underline{x} \cdot\cdot x'$ or $\bar{x} \cdot\cdot x'$, $x$ being the incipient and $x'$ the terminate value of the series.

The condition of questions involving these variables, generally indicate the incipient and terminate values of the increasing or decreasing variables.

6. The expression $\underline{z}$ is equivalent to that of $\bar{z}$, when considered independent of other variables; but the product arising from drawing $\underline{z}$ into $\bar{z}$ is not equivalent to that of drawing $\underline{z}$ into $\underline{z}$; for $\underline{z}\bar{z}$ indicates that the greatest value of $z$ is drawn into its least value, and consecutively; and $\underline{z}z$ indicates that the greatest value of $z$ is drawn into the greatest, and so on through the series.

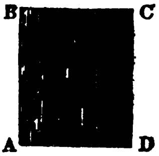
A series of the squares of $\underline{z}$ or $\bar{z}$, is represented by $\underline{z}^2$ or $\bar{z}^2$; a series of roots by $\sqrt{\underline{z}}$ or $\sqrt{\bar{z}}$.

7. When it is designed to express, a series either of constant or variable quantities, without regard to their progression or law of variation, a small capital, of the letter denoting a single term of the series may be used.

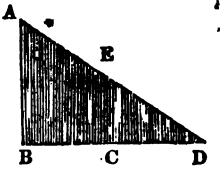
Thus in the equation $y=\sqrt{dx}$, if we would express a series of $y$, it may be written $\mathrm{Y}$, and the equation will be $\mathrm{Y}=\sqrt{d\underline{x}}$; this, unless otherwise restricted, expresses an infinite series of the quantity represented by $y$. If $y=\sqrt{(dx)}$ is the equation of any figure, then $\mathrm{Y}$, is a function of the surface, or $\mathrm{Y}=\sqrt{(d\underline{x})}\underline{x}$ is the equation to the surface.

Hence the equation to a surface consists of the equation of the figure considered as a series, drawn into the axis or abscissa.

8. If $a$ be the magnitude AD of any series whose number is $n$, of lines AB, and whose length is the constant quantity $z$, then the sum of the lines will be $nz$; and their magnitude made by drawing them into a surface, will be $\frac{a}{n} zn = az$.



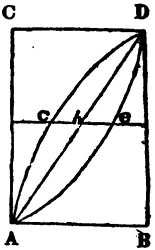
For we have shown that if an infinite series of lines are drawn into their respective distance, the product is a surface; hence, we have the following construction, viz: a rectangle ABDC.



9. Let $a$ be a series of lines $\underline{z}$, constantly decreasing in arithmetrical progression from $z$ to 0, or from the line AB to the point D, while the line BD represents the number or magnitude of the series; and the series of $\underline{z}$ drawn into the quantity $a$, will be equivalent to the triangle ABD.

If $a$ be a series of lines, AB decreasing from $\underline{z}$ to $s$, or from AB to EC, then if $AD=a$ = the magnitude of the series, the construction will be the trapezium ABCE.

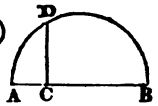
10. If $a$ be a series of $z^2$, where $z$ decreases uniformly from AB, to a point D; or from $z$ to 0, then $az^2$ may be constructed by the exterior space of a parabola A$e$DB, for while $a\underline{z}$ generates the triangle ABD, $a.z^2$ will generate the parabola A$e$DBA, (Prop. VII, Sch., B. 1,) whose axis is CD.



If $a$ be a series of $\sqrt{z}$, then may $a\sqrt{z}$ be put under the construction of a semi-parabola, A$c$DB, whose axis is $DB=a$, and whose base $AB=z$, the series of $\underline{z}$ being a series of ordinates across the triangle ABD, parallel to AB.

12. Let it be required to construct a quantity $d\sqrt{(\underline{z}.\bar{z})}$, or a series of mean proportionals between the corresponding terms of two equal increasing and decreasing series drawn into $d$, the number of the series.

If $d$ = the line AB, and if $z$ = the same line, the surface generated by drawing $d$ into $\sqrt{(\underline{z}.\bar{z})}$, would be the semicircle ABD.



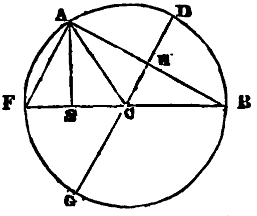
For the equation to the circle is $y^2=\sqrt{(dx-x^2)}$
or $DC=\sqrt{AC^2+CB^2}$

But if $z$ be greater or less than $d$, then the construction will be a semi-ellipse; which if $z$ is less than $d$, will have AB for its major axis; but if $z$ is greater than $d$, AB will be the minor axis.

For the ellipse by its equation, $d^2 : c :: x(d-x) : y^2$ or $d^2y^2 =c^2(dx-x^2)$ is only the circle expanded, or contracted, in the ratio of the major and minor axes.

The equation for the surface of a circle or an ellipse, may more properly be expressed by $d\sqrt{(\underline{x}\bar{x})}$ for a semi-circle or semi-ellipse, and if $x = d$, then the equation will be that of a circle; but if $d$ and $x$ are unequal, the equation becomes that of an ellipse, and $d$ will be the major or minor axis, according as it is greater or less than $x$, and $x$ will be its conjugate; and its equivalent in either case is $\frac{1}{8}dx\times\pi$, or for the whole circle or ellipse $\frac{1}{4}dx\times\pi$: hence $\pi$ is a function of $\surd(\underline{x}\bar{x})$ or $f\surd(\bar{x}\underline{x})$.

Hence we have a finite expression for the circles quadrature in algebraic terms, viz: $2d\surd(\underline{x}.\bar{x})$ or $2x\surd(\underline{x}.\bar{x})=r^2\pi$, where $x$ is equal to the diameter. And $\pi=(2x\div r^2)\surd(\underline{x}.\bar{x})=$ the circumference of any circle whose diameter is $x$; $\bar{x}$ and $\underline{x}$ being series of increasing and decreasing quantities. Hence, $\pi=2d\surd(\underline{x}\bar{x})\div\frac{1}{4}d^2=(8\div d)\surd(\underline{x}\bar{x})$, or $(8\div x)\surd(\underline{x}\bar{x})$.



13. Let ABD be the segment of a circle, the height of the segment DE being equal $z$; and if the diameter $= x$, then will the chord AB$=\surd(x'z;)$ and if EG $= x-z = x'$, then will the equation to the surface be $2Vz=2z\surd((\underline{x}\cdot\cdot x')\bar{z})$. - - - - - - - -(1)

Also, let CE $=\frac{1}{2}x-z=u$, then if AB in a decreasing series is drawn into CE, or if $(u\underline{\surd(xx)})$ be constructed, it will be equal to the triangle ABC; and the sum of the segment and triangle, $= 2z\surd((\underline{x}\cdot\cdot x')\bar{z}) + u(\underline{\surd(xz)}) =$ the sector ACBD. (2)

In the triangle ABF the side AF is equal 2CE$=x-2z$; and BF : BA : : AF : AS, or $x:\surd xz::x-2z$ : AS the sine of the angle, BCA, or sine of the arc of the segment

$$=\surd(zx)-2z\surd(zx)\div x. \quad - - - - - - - - (3)$$

Let AS$=s$, and let the arc ADB$=\pi'$ and the area of the segment, (Prop. XVIII B. IV) will be $(\pi'-s)\times\frac{1}{4}x$, hence the expression $2z\surd((\underline{x}\cdot\cdot x')\bar{z})=\frac{1}{4}x\pi'-\frac{1}{4}xs$ - - - - - (4)

Therefore the arc ADB of the segment may be expressed

$$\pi'=\frac{2z\surd((\underline{x}\cdot\cdot x')\bar{z})+s}{\frac{1}{4}x}=2z\surd(\underline{x}\cdot\cdot x')\bar{z})+\surd(xz)-\frac{2z\surd(xz)}{x} \quad (5)$$

14. If a series of variables consisting of two or more factors, as its elements, are drawn into the number denoting the series, the construction can more conveniently be represented by a solid.

Surfaces properly consist only of factors, equivalent to the second power; cubes of factors equivalent to the third power, or the product of three factors.

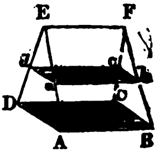
But we may, under certain conditions, construct quantities representing solids as surfaces; those whose factors are constant, by any surface arbitrarily chosen, and those which are variable by such curves as yield to the conditions of the expression; and we may thence proceed to construct such solids as would result from drawing such surfaces into a series; or such as would be the result of drawing the variable series of elements of such surface into a constant, or a variable quantity; and the quantity so constructed may, by being put under a superficial construction, become the base of another quantity or magnitude, made by drawing this quantity into another given quantity or into a series; and we may proceed, in this manner, to construct quantities geometrically, converting one construction into a base for the next, and so on.

This may be performed algebraically, and with greater facility, inasmuch as a quantity involving any power of the variable may be assumed as a base for a higher power of the series, and thus the powers and roots of variables may be extended at pleasure; and the laws of variation or their increase or decrease in value, may be determined by geometrical construction and analytical deductions.

15. If it were required to construct a series, $a$, of the quantity $bc$, drawn into each other, $b$ and $c$ being constant quantities; here it is evident that, for the purposes of construction, $bc$ may represent either a line or a surface; if the series of $bc$ be represented by a line, then $bc$, drawn into $a$, or $abc$, will represent a rectangle. But, if $bc$ represents a surface, then $abc$ will represent a prism. And for the purposes of investigation, the conditions of the quantities would be similar in either case; for the line would have the same relation to the rectangle generated by drawing the line into the series, as the surface to the prism, generated by drawing the surface into the series.

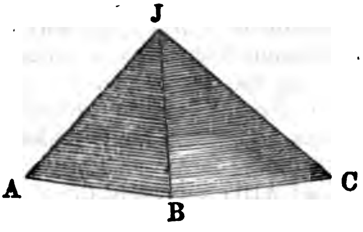
16. Let it be required to construct a series $a$, of $bz$ or $a.b\underline{z}$, where $z$ is a series decreasing uniformly to $o$.

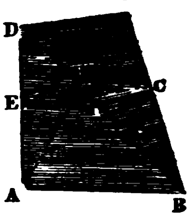
This would evidently be represented under the form of a wedge ABCDEF, whose base ABCD is represented by $bz$, and the perpendicular ED represents the value of $a$.



If $\underline{z}$ decrease only to $s$, then the construction would be the prismoid ABCD $abcd$, whose greater base, ABCD, is represented by $bz$, and whose lesser base, $abcd$, is represented by $bs$.

17. If $u$ be also a quantity decreasing uniformly with $z$, in such ratio as to reach $o$ at the same time, then the expression $a.\underline{uz}$ would generate a pyramid ABCJ; which will be its proper construction; and if $u=z$, or if the expression can be but under the form $a.\underline{z}^2$, then the pyramid will have a square base.





18. Let a series $a.\bar{z}\underline{z}$, be constructed. This quantity, representing the products of the corresponding terms of two series, one increasing from *o* to *z*, while the other decreases from *z* to *o*, drawn into *a*, the quantity denoting the number of the series, generates the double wedge ABGD, where AB, or GD, is equal to *z*, and the perpendicular AD$=a$.

If $\underline{u}$ represent a series of ordinates drawn across the triangle ADG, then $au$ will represent the triangle; and if these ordinates are severally drawn into a series of $u$, the result would be the same as before.

19. Let it be required to construct $Y^2=p\underline{x}$, which is the equation to the vertical parabolic revoloid, $y=\surd(px,)$ being the equation to the parabola, $p$ being the parameter, is a constant quantity, we have $x$ variable.

First, we may draw $p$ into a series of $\underline{x}$, which gives us the triangle DEA (see diagram to art. 16,)$=p\underline{x}$, $p$ being equal to DA, $x=$the axis of the revoloid$=$DE the altitude; and if this triangle, as a base, is also drawn into $x$, the axis of the revoloid, we shall have the solid ABCDEF.

Or, more properly, if we first draw the axis $x$ into the series of the variables, we shall have the triangle as before, with a base AD; and if we multiply this by $p=$DC, we shall have the solid described as before, then will any section *abcd* of this solid parallel to the base ABCD, be equivalent to a similar section through a parabolic revoloid, or pyramoid, which it is designed to represent.

The expression for a cube may at all times be constructed on a surface by means of curves; but every different species of solid requires some peculiar construction, according to the equations for the ordinates or sections of the solid, the expressions for which may be transferred to expressions for ordinates to a surface, or equations to some curve; and since any multiple, or power of a series $\underline{x}$, whose exponent is an integer, is known, when the series or root is known; we can hence discover an innumerable variety of curves, which are quadrable; but, in general, in descending powers, it is not certain that the series of roots of a given series may be so. Thus, the circle is equivalent to a series of square roots, of the ordinates to a parabola; the series of the parabala is quadrable, but not the circle, except in certain functions of given quantities; we may get an expression for its value, but it will be under an incommensurable form.

Let ABD represent the vertical segment of a rectangular spherical revoloid; and its equation considered in relation to its figure, will be $y^2 = (x\,..\,x')z$ - - - - (1)

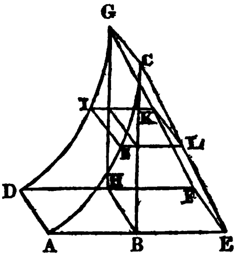
the equation considered as a solid, will $Y^2z = z(\underline{x} \,..\, x')\overline{z}$ (2)
which from its organization will readily be discovered to be cubable, for its value is $\frac{4z}{6}$ $(x'z + 4\,(\tfrac{1}{2}x + \tfrac{1}{2}x')\,\tfrac{1}{2}z) = \tfrac{2}{3}xz^2$ $+\tfrac{4}{3}x'z^2$, - - - - - - - - - (3)
its convex surface will be $4xz$ - - - - - (4)

Hence, the solidity of a spherical segment will be =
$(\tfrac{2}{3}xz^2+\tfrac{4}{3}x'z^2)\tfrac{1}{2}\pi = \tfrac{1}{6}xz^2\pi + \tfrac{1}{3}x'z^2\pi$ - - - - (5)
and its surface will be $\pi xz$ - - - - - (6)

20. Let it be required to construct $Y^2=\frac{c^2}{d^2}(dx+x^2,)$ the equation to an hyperbolic revoloid, $Y^2$ being a series of parallel planes, being the squares of the ordinates to the axis.

Because this equation involves powers higher than the square; that is, because the terms $(dx+x^2,)$ multiplyed by $\frac{c^2}{d^2}$, produces a solid, the series represented by these quantities drawn into $a$, would be of a higher power than a cube; hence, in order that our construction shall be under a cubic form, $a$ $(d\overline{x}+\underline{x}^2)$ must be represented on a plane. Let $a$=CB.

Hence, we have $a.\underline{x}^2$=the exterior parabolic surface ABC, AB being the axis of the parabola, and also $d\overline{x}$ = CBE; now, if this is drawn into $\frac{c^2}{d^2}$, we shall have the solid ABECFDG, for the construction of a solid equivalent to the quadrant of an hyperbolic pyramoid or revoloid; every section NIKL through this solid is equivalent to a corresponding section through a hyperbolic pyramoid of the same altitude and equivalent base.



Hence, it will be perceived that the hyperbolic revoloid is cubable; for this, its representative is equal to one-sixth of the product of the sum of the base AEFD *plus* four times a middle section NIKL drawn into the altitude BC. This is also true in relation to the parabolic revoloid.

And since a paraboloid or hyperboloid is to its respective circumscribed revoloid as a circle to its circumscribed square, or as $r^2\pi : 4r^2$, it follows that the paraboloid and hyperboloid, are also cubable in terms of $\pi$.

21. It may be perceived that in constructing the quantity $a\underline{x}$, where $\underline{x}$ is a series decreasing to 0, we have a triangle; hence the value of $a\underline{x}$ is $\tfrac{1}{2}ax$; or the value of $x\underline{x}$ may, in like manner be shown to be $\tfrac{1}{2}x^2$.

Also, since $\underline{x}^2x$ is equivalent to a pyramid whose base is $x^2$, and whose altitude is $x$; hence, $\underline{x}^2x$ is equal to $\tfrac{1}{3}x^3$.

And if we have a series of $\underline{x}^3$ to be drawn into $x$ or $a$, we may also construct $\underline{x}^3$ by art. 20, and we shall here have $\underline{x}^3x=\frac{1}{4}x^4$; also $x^nx=\frac{1}{n+1}x^{n+1}$

Hence, the law of the progression of the powers of the variable quantities is manifest, for the value of the continued product of any series of power, of a series of variable quantities drawn into any constant quantity, is equal to the same power of the maximum value of the variable drawn into the same quantity, divided by a number denoted by the index of the power *plus* 1.

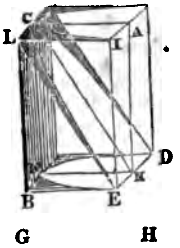
If the variable has a fractional exponent, the same consideration will also apply. Thus, $\underline{x}^{\frac{1}{2}}x$ is equivalent to $\frac{2}{3}x^{1\frac{1}{2}}$, and $\underline{x}^{\frac{1}{3}}x$ is equivalent to $\frac{3}{4}x^{1\frac{1}{3}}$ and $\underline{x}^{\frac{1}{n}}x$ equivalent to $\frac{1}{n+1}x^{\frac{1}{n}+1}$ &c.

Where the product of any power of a variable quantity is made the base of another construction, according to its determinate value, the magnitude constructed on this base, is no longer considered as a function of the variables, entering into the former constructions as elements of this base; but this new construction is subject to the laws of its own organization; and if raised to any power, or multiplied by any number of factors, either as constants or variables, the different powers depend on, or are a function of the base assumed, and not of its original constituents.

Thus, if we have the quantity $\underline{x}^3x$, its value is $\frac{1}{4}x^4$; now, if $\frac{1}{4}\underline{x}^4$ is assumed as the base of another construction, as $(\frac{1}{4}\underline{x}^4)x$, or $\frac{1}{4}\overline{x}^4x$ the product is not $=\frac{1}{20}x^6$, but is $=\frac{1}{8}x^6$.

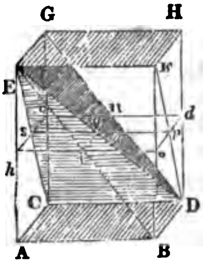
If a series $\underline{uxz}$ are drawn into $a$, the product is equivalent to $\frac{1}{4}uxza$; the co-fficient being the samc as that for the product of $\underline{x}^3x$, which gives $\frac{1}{4}x^4$ as its equivalent.

When the terms of the variable series denoted by $x$ or $z$, are drawn into each other according to their reciprocal value, or where an increasing series is drawn into a decreasing series, the value of some power of the product of the series may be obtained; and the other powers and roots of this value are subject to peculiar laws for their development. For the continued products of these series, are not subject to the same ratio, or the same order in the variation in their successive changes from one power to another, as is observed in the products of such as are either all increasing or all decreasing. Thus, if we have $x\sqrt{\underline{x}\overline{x}}$, let this quantity be involved till its radical is removed, and we have $x^2\underline{x}\overline{x}$, the value of which may be determined by a construction. Thus, $\underline{x}\overline{x}.x$, which is equivalent to a double wedge, (Art. 24, Chap. I,) or which will be more available for our present purpose, it may be put under the construction of the parabola cut off by a diagonal, (Art. 25, Chap. 1,) the value of which is evidently equal $\frac{1}{4}x^2\times4\times\frac{1}{6}x=\frac{1}{6}x^3$, which is equal to one-half the value of $\underline{x}^2x$; and if this value is again put into a series, and drawn into $\underline{x}$, we shall have $\frac{1}{2}\underline{x}^2.x$, or $\underline{x}^2\overline{x}.x$, whose proper geometrical construction will be the solid CELI, which is the solid of a prism circumscribing a parabolic complemental ungula, which (Prop. XVIII, Cor. 2, B. II.) is equal to one-fifth of the prism CANELI $=\frac{1}{10}x^4$.



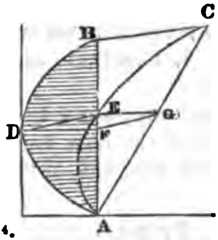
Let there be a series of $\underline{x}^2x$, a series of $\overline{x}^2x$, and two series $\overline{x}\underline{x}x$, and the sum of the series will be equivalent to $x^3$, as may be represented by the prism AH.

For $\underline{x}^2x$ equals the pyramid ABCDE, $\overline{x}^2x$ equals the pyramid EFHGD, and $2\overline{x}\underline{x}x$ equals two wedges CDEG completing the prism whose value is $x^3$.



Let there be a series $a$ of $\underline{z}$ drawn into a reciprocating series of $\overline{x}$, the terminate value of $z$ being $z'$, and the incipient value of $x$ being 0. Since the product of these series $a.(\underline{z}\cdot\cdot z')\overline{x}$ gives us for a construction a solid *hino*CD, whose value is equal to $\frac{1}{6}ax(2z'+z)$; and this also represents a segment of a quadrant of a revoloid where $z$ and $x$ increase and decrease in the same arithmetical ratio, equal to a segment of a cylindrical ungula.

Also, if we have a series of the products of reciprocating variables, or $\overline{z}\underline{z}$ drawn into a series of $h$ or $z$, its value has been found equal to $\frac{1}{6}hz^2$ or $\frac{1}{6}z^3$, let this be drawn into another series of $\overline{z}$, and we have the following construction, which is a parabola whose vertice is D; and we have $z\underline{z}\overline{z}^2$ equal the parabolic ungula ADBC equal $\frac{2}{3}\text{AB}\times\text{HA}\times\frac{1}{2}\text{BC}=\frac{2}{3}(z\times(\frac{1}{2}z\times\frac{1}{2}z)\times\frac{1}{2}z)=\frac{1}{12}z^4$; hence, the series $z\underline{z}\overline{z}^2$ is equivalent to $\frac{1}{12}z^4$.



Let this be drawn into another series of $\overline{z}$, and we shall have $z\underline{z}\overline{z}^3=\frac{1}{6}z(z^2+4(\frac{1}{8}z^3\times\frac{1}{2}z))=\frac{1}{6}z^4+\frac{1}{24}z^6$; if this is drawn into another series of $\overline{z}$, we shall have $z\underline{z}\overline{z}^4=\frac{1}{6}z(z^3+4(\frac{1}{16}z^4\times\frac{1}{2}z))=\frac{1}{6}z^4+\frac{1}{48}z^6$; hence the law of progression is manifest.

If we have a series of $z\underline{z}^2\overline{z}^2$, its value will be $\frac{1}{6}z(2z^3+\frac{1}{4}z^4)=\frac{1}{3}z^3+\frac{1}{24}z^6$. Hence, the law of the progession of this series will easily be discovered.

*Scholium.* The results obtained by the preceding notations and geometrical constructions, are similar to those obtained by the integral calculus, the same principles as here used serving as the foundation of that science; and in order that a comparison between the two modes of notation may be instituted, we will present some of the elementary principles of the calculus, as the subject of the next chapter.

## CHAPTER III.

### DIFFERENTIAL AND INTEGRAL CALCULUS.

Art. 1. As a basis of the deferential *calculus*, we may premise that if $y$ be a function of $x$, and if a change takes place in the value of $f(x)$ so that $x$ becomes $x+h$, $x$ being quite indeterminate, and $h$ any quantity whatever, either positive or negative, a corresponding change must take place in the value of $y$, which may then be represented by $y'$. If the quantity $f(x+h)$ be now developed in a series of the form

$$f(x)+\mathrm{A}h+\mathrm{B}h^2+\mathrm{C}h^3+\ldots\ldots$$

which is always practicable, in which the first term is the original function $f(x)$, and the other terms ascend regularly by positive and integral powers of $h$, and A, B, C, &c., are independent of $h$; then the co-efficient of the simple power of $h$ in this series is called *the first differential co-efficient of $y$ or $f(x)$*. This is the fundamental definition of the differential calculus.

2. Let us now examine the change which takes place in the function for any change that may be made in the value of the variable on which it depends.

Let us take, as a first example,

$$u = ax^2,$$

and suppose $x$ to be increased by any quantity $h$. Designate by $u'$ the new value which $u$ assumes, under this supposition, and we shall have

$$u' = a(x+h)^2,$$

or by developing

$$u' = ax^2+2axh+ah^2.$$

If we subtract the first equation from the last, we shall have

$$u'-u=2axh+ah^2;$$

hence, if the variable $x$ be increased by $h$, the function will be increased by $2axh+ah^2$.

If both members of the last equation be divided by $h$, we shall have

$$\frac{u'-u}{h} = 2ax+ah, \; (1)$$

which expresses the ratio of the increment of the function to that of the variable.

The value of the ratio of the increment of the function to that of the variable is composed of two parts, $2ax$ and $ah$.

If now, we suppose $h$ to diminish continually, the value of the ratio will approach to that of $2ax$, to which it will become equal when $h=0$. The part $2ax$, which is independent of $h$, is therefore the *limit* of the ratio of the increment of the function to that of the variable. The term, *limit of the ratio* designates the ratio at the time $h$ becomes equal to 0. This ratio is called the *differential co-efficient* of $u$ regarded as a function of $x$.

3. Then let $y$ be a function of $x$, such that

$$y=ax^2$$

Let $x$ become $x+h$ and $y$ become $y'$

$$y'=a(x+h)^2$$

expanding $\quad =ax^2+2ax\,.\,h+ah$

This, it will be perceived, is a series of the required form, the first term $ax^2$ is the original function $y$, and the other terms ascend by integral and positive powers of $h$; hence, according to our definition, $2ax$ the co-efficient of the simple power of $h$ in this series, is the first differential co-efficient of $y$ or $f\,(x)$

4. Again, let

$$y=x^3$$

Let $x$ become $x+h$ and $y$ become $y'$

$$y'=(x+h)^3$$

expanding $\quad =x^3+3x^2\,.\,h+3x\,.\,h^2+h^3$

Here, therefore, $3x^2$, the co-efficient of the simple power of $h$ is the first defferential co-efficient of $x^3$.

5. Again, let

$$y=ax^3+bx^2+cx+d$$

Let $x$ become $(x+h)$ and $y$ become $y'$

$\therefore\ y'=a(x+h)^3+b(x+h)^2+c(x+h)+d$

expanding $= ax^3+3ax^2h+3axh^2+h^3+bx^2+2bxh+bh^2+cx+ch+d$, arranging according to powers of $h$,

$$=(ax^3+bx^2+cx+d)+(3ax^2+2bx+c)h+(3ax+b)h^2+h,$$

a series of the required form, for the first term is $ax^3+bx^2+cx+d$, the original function, and the succeeding terms ascend regularly by powers of $h$.

Hence, $3ax^2+2bx+c$ the co-efficient of the simple power of $h$ in the development of $y'$ is the first differential co-efficient of $y$ or $ax^3+bx^2+cx+d$.

6. We have now to introduce a notation by which this ratio may be expressed. For this purpose we represent by $dx$ the last value of $h$, that is, *the value of* h *which cannot be diminished according to the law of change to which* h *is subjected,*

without becoming 0; and let us also represent by *du* the corresponding value of *u*: we then have

$$\frac{du}{dx} = 2ax. \quad (2)$$

The letter *d* is used merely as a characteristic, and the expressions *du*, *dx*, are read, *differential* of *u*, *differential* of *x*.

It may be difficult to understand why the value which *h* assumes in passing from equation (1) to equation (2), is represented by *dx* in the first member, and made equal to 0 in the second. We have represented by *dx* the *last* value of *h*, and this value forms no appreciable part of *h* or *x*. For, if it did, it might be diminished without becoming 0, and therefore would not be the last value of *h*. By designating this last value by *dx*, we preserve a trace of the letter *x*, and express at the same time the last change which takes place in *h*, as it becomes equal to 0.

7. Let us take as a second example,

$$u = ax^3.$$

If we give to $x$ an increment $h$, we shall have,

$$u' = a(x+h)^3 = ax^3 + 3ahx^2 + 3ah^2x + ah^3.$$

hence, $\quad u' - u = 3ahx^2 + 3ah^2x + ah^3,$

and the ratio of the increments will be

$$\frac{u'-u}{h} = 3ax^2 + 3ahx + ah^2,$$

and the *limit* of the ratio, or *differential co-efficient*,

$$\frac{du}{dx} = 3ax^2.$$

In the function

$$u = nx^4, \text{ we have } \frac{du}{dx} = 4nx^3.$$

And if $\qquad y = f(x)$

the first differential co-efficient of $y$ is denoted by the symbol $\frac{dy}{dx}$, thus in the above examples (Arts. 4 & 5.)

$$y = ax^2$$

$$\frac{hy}{dx} = 2ax$$

$$y = x^3$$

$$\frac{dy}{dx} = 3x^2$$

$$y = ax^3 + bx^2 + cx + d$$

$$\frac{dy}{dx} = 3ax^2 + 2bx + c$$

in like manner if $u=f(z)$ the first differential co-efficient of $u$ or $f(z)$ will be represented by $\frac{du}{dz}$.

*Note.*–Since a constant quantity is not susceptible of change, it is manifest that it can have no differential co-efficient, or if $y=a, \frac{dy}{dx}=0$.

### 3. *To find the first differential co-efficient of any power of a simple algebraic quantity.*

Let $y=x^n$
Let $x$ become $x+h$
$\therefore\ y'=(x+h)^n$
Expanding by the binomial

$$=x^x+nx^{n-1}h+\frac{n(n-1)}{1\,.\,2}x^{n-2}h^2+\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}x^{n-3}h^3+\&c.$$

$$\therefore\ \frac{dy}{dx}=nx^{n-1}$$

From this it is manifest that

*The first differential co-efficient of any power of a simple Algebraic quantity is found by multiplying the quantity by the index of the power, and then diminishing the exponent by unity.*

*Ex.* 1. $y=x^7$ $\qquad \frac{dy}{dx}=7x^6$

2. $y=ax^{p+q}$ $\qquad \frac{dy}{dx}=a\,(p+q)\,x^{p+q-1}$

3. $y=a+x^{-q}$ $\qquad \frac{dy}{dx}=-\,qx^{-(q+1)}$

4. $y=a+7x^{-\frac{m}{n}}$ $\qquad \frac{dy}{dx}=-\,7\frac{m}{n}x^{-(\frac{m}{n}+1)}$

## INTEGRAL CALCULUS.

The object of the Integral Calculus is to discover the primitive function from which a given defferential co-efficient has been derived.

This primitive function is called the *integral* of the proposed differential co-efficient, and is obtained by the application of the different principles established in finding differential co-efficients and by various transformations.

When we wish to indicate that we are to take the integral of a function, we prefix the symbol $S$. Thus if

$$y = ax^4.$$

We know that $dy = 4ax^3dx$

If then, the quantity $4ax^3\,dx$ be given in the course of any calculation, and we are desirous to indicate that the primitive function from which it has been derived is $ax^4$, we express this by writing

$$S\,4ax^3\,dx = ax^4$$

The characteristic $S$ signifies *integral* or *sum*. The word *sum*, was employed by those who first used the differential and and integral calculus, and who regarded the integral of $x^m dx$ as the *sum* of all the products which arise by multiplying the *m*th power of $x$, for all values of $x$, by the constant $dx$.

When constant quantities are combined with variable quantities by the signs + or — they disappear in taking the differential co-efficients, and therefore they must be restored in taking the integral.

Thus, if $y = ax^3 + b$

or, $y = ax^3 - b$

or, $y = ax^3$

In each three cases equally

$$dy = 3ax^2\,dx$$

Hence in taking the integral of any function it is proper always to add a constant quantity, which is usually represented by the symbol C. Thus, if it be required to find the integral of a quantity such as

$$dy = 3ax^2\,dx$$

$$y = S\,3ax^2\,dx$$

$$= ax^3 + C$$

where C may be either positive, negative, or 0. We cannot determine the value of C in an abstract example, but when particular problems are submitted to our investigation, they usually contain conditions by which the value of C can be ascertained.

By reversing the principles established for finding the differential co-efficients, or differentials of functions, we shall obtain an equal number of rules for ascending to the integrals from the derived functions. Recurring therefore to these we shall perceive that

1. *The integral of the sum of any number of functions is equal to the sum of the integrals of the individual terms, each term retaining the sign of its co-efficient.* Thus, if

$$dy = 4ax^3\,dx + 3bx^2\,dx - 2bx\,dx + dx$$

$$y = \int 4ax^3\,dx + \int 3bx^2\,dx - \int 2\,bx\,dx + \int dx + C$$

II. Since, if

$$y = ax^m$$

$$dy = max^{m-1}\,ax$$

it is manifest that

*The integral of a function raised to any power is obtained by adding unity to the exponent of the function, and by dividing the function by the exponent so increased, and by the differential of the function.*

*Ex.* 1. $dy = ax^n\,dx$

$$y = \frac{ax^n + 1}{n+1} + C$$

*Ex.* 2. $dy = \frac{a}{x^n}dx$

$$= ax^{-n}\,dx$$

$$y = \frac{ax^{-(n-1)}}{-(n-1)}$$

$$= -\frac{a}{(n-1)\,x^{n-1}} + C$$

*Ex.* 3. $dy = (a+x)^n\,dx$

$$y = \frac{(a+x)^{n+1}}{n+1} + C$$

$$dy = \frac{1}{(a+x)^n}\,dx$$

$$= (a+x)^{-n}\,dx$$

$$y = -\frac{1}{(n-1)\;(a+x)^{n-1}}$$

This rule applies to all functions of the form

$$dy = (a + bx^n)^m\,cx^{n-1}\,dx$$

for these can all be reduced to the form $ax^m\,dx$.

*Scholium.*

The coincidence of the results obtained by the notation, used in Chap. II, with those determined by the *Calculus*, indicates that the same principles serve as the foundations of both ; which, as was observed at the close of Chapter I, are the following, viz., that when any series of quantities vary in arithmetical progression, the sum of the series as well as their powers, or products, with a constant quantity, or with another arithmetical series, may be measured by the sum of the products of the maximum values + the minimum values + four times the products of their medium values multiplied by $\frac{1}{6}$ of the magnitude, representing the number of the series.

Any series of numbers which are not subject to these conditions, cannot be made the subjects of accurate numerical determination or calculation, by the *Calculus*; hence, we may always determine, by construction, whether any geometrical magnitude, or algebric quantity, is susceptible of accurate development numerically in terms of given quantities. One advantage possessed by the investigation of geometrical subjects, by the principles contained in Chapter II, of this subject, is, that the notation there used, is more elementary than that of the *Calculus*, and expresses the conditions of geometrical subjects in a more obvious and inteligent manner, and is more direct in its results.

Hence, instead of pursuing the method of defferentation, we express the conditions of quantities depending on variable factors according to their several conditions, or organization, and proceed to integrate or sum up the series of functions, and determine the magnitude of the production, according to the principles therein contained. Hence $S\,x^2$ expresses definitely the function of the series of squares of a series of variables, and if this series be drawn into a constant quantity $x$, and integrated, we shall have according to our notation $\frac{1}{3}x^3$.

If this is made equal to $u$, or if we have $u = \frac{1}{3}x^3$, its differential is $\frac{du}{dx} = x^2$, which, being again integrated by the calculus, we have again $\frac{1}{3}x^3$, showing similar results, from different considerations. The notation by the calculus is arbitrary, but by this inductive; and hence, by its intimate connection with the principles of the calculus, may serve to render that subject more obvious.

There are cases, however, for which the calculus is more particularly adapted than the notation here referred to; such as drawing tangents to, and rectifying curve-lines. This is accomplished by that science in a manner the most elegant and complete; but in relation to surfaces and solids, nothing can be more complete or satisfactory, than the discussions which we have introduced.

## CHAPTER IV.

### *On the centres of surfaces and solids, the method of finding them, &c.*

Article 1. The centre of surfaces and solids is susceptible of two considerations, that of magnitude and distance. Hence, we have the centre of *aggregation* and the *virtual centre.*

2. The *centre* of *aggregation* or *centre* of *magnitude* of any plane surface, is that point through which, if a line is drawn in any direction in the plane, it shall divide the surface equally.

3. The centre of *magnitude* of any solid is that point, through which if a plane is passed in any direction, the plane shall divide the solid equally.

4. The *virtual centre,* or *centre* of *gravity* of any plane is that point through which, if a line be drawn in any direction in the plane, the sum of all the points, on one side, drawn into their distances from the line, shall be equal to the sum of all the points on the other side drawn into their respective distances from the line.

5. The virtual centre of any solid is that point through which if a plane be passed in any direction, the sum of the points, on one side, drawn into their distances from the plane, shall be equal to the sum of all the points on the other side, drawn into their distance from the plane.

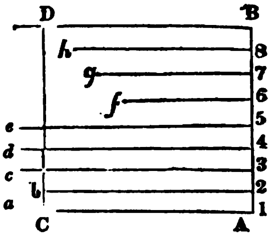
6. The virtual centre between any two points is evidently in the right line connecting the two points, and equidistant from each.

Hence the virtual centre of a right line, is in the middle of that line, or is in the centre of magnitude of the line.

7. In a system of points, their virtual centre has respect to the magnitude of the lines drawn from the several points to such centre.

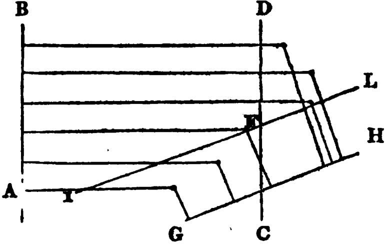
But the centre of magnitude, has regard only to the number of points.

8. If there be drawn from a system of points, lines perpendicular to a given base, the sum of those lines divided by their number, will give their average length ; which is = to the distance of the virtual centre of the system of points from the base.



Let AB be a given base line, and let *a*, *b*, *c*, *d*, *e*, &c., be a system of points; if perpendiculars *a*1, *b*2, *c*3, &c., be drawn from the points to meet the base line AB, and if the sum of these lines be divided by their number, the quotient will give their average length, or the distance AC, from which, if a line CD be drawn, parallel to AB, it shall pass through the virtual centre of the system of points.

For if the line CD is drawn, according to the conditions expressed in the proposition, the portions of the lines cut off beyond the line CD, are sufficient to extend those which fall short of that line, to the line CD. Hence, the line CD passes through the virtual centre of the system, for the lines drawn from the points on both sides of the line are equal.

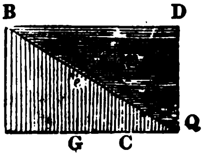


Hence, if two base lines AB, GH are drawn, not parallel to each other, and if the system of points are connected to both of the bases, and we proceed, as in the proposition to draw CD parallel to AB, and IL parallell to GH, the intersection F, of the lines CD, and IL will be the virtual centre of the system.

If the base line AB, should pass through the system of points, so as to leave one portion of them on one side, and another portion on the other, then the lines connecting the points to the base must be estimated on one side positive, and on the other negative ; and the virtual centre will fall on one side or the other, as the positive or negative signs predominate.

9. If, instead of a system of points, the virtual centre of a system of parallel lines is required. Let a base line be drawn parallel to those lines, and because the lines are to each other, as the number of equidistant points, arbitrarily taken in each ; hence, if we take the product of the several lines into their respective distance from the base, the sum of these products divided by the sum of the lengths of the lines, will determine the distance from the base, through which, if a line is drawn parallel to the base, it shall pass through the virtual centre.

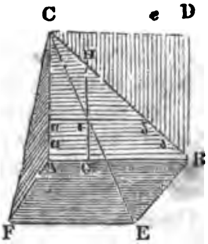
10. The virtual centre of a surface may be found by supposing parallel ordinates drawn across its surface, and assuming those ordinates, as representing the surface, and computing the distance of the virtual centre of the ordinates from a given base; and if from the properties of the figure, it cannot be readily discovered what part of the line passing through the centre, is occupied by the centre, then another base may be assumed, making an angle with the former, and if ordinates are supposed to be drawn across the figure, parallel to this base, another line may be found parallel to this base, passing through the centre; hence, the intersection of these two lines will be the centre required.



11. If AQ be a base line drawn through any point, as suppose the vertex of any body, or figure QBD, and if *a* denote any ordinate EF of the figure, $d$ = AG, its distance from the base line AQ, and S = the sum of all the ordinates, or the whole figure QBD; then the distance IC of the virtual centre from AQ, is denoted by the (sum of all the *ad*) ÷ S.

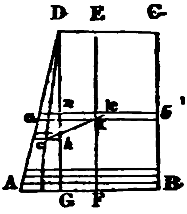
### PROBLEM I.

*To find the centre of a triangle.*



Through the vertice C of the triangle draw a base line CD, parallel to AB, let an indefinite number of ordinates *ab*, or $z$, parallel to DC, be conceived to be drawn across the triangle, each of which we may conceive to be drawn into their respective distances *be*, &c., from the line CD, which may be represented by a series of $\underline{x}'$; hence, if the series of $\underline{z}$, whose magnitude is $x$, be drawn into the series of $\underline{z}x$, the product may be represented by a pyramid, whose base ABEF is $= zx$ and whose altitude is AC, or $x$; hence, $\frac{1}{3}AB \times BD \times BD = \frac{1}{3} AB \times BD^2$ or $xz\underline{x}, = \frac{1}{2}x^2z, =$ the sum of the products, but (Art. 2) the sum of the products ÷ S, the sum of the series of ordinates, or series of $z$, whose measure is $\underline{z}x$, is equal to the distance of the centre, from the line CD. The sum of the series of ordinates may be expressed by half the rectangle of $AB \times BD$, or $\frac{1}{2}zx$. Hence, by (Art. 11.) $\dfrac{\frac{1}{3}AB^2 \times AD^2}{\frac{1}{2}AB \times HD}$ $= \frac{2}{3}AC = \frac{2}{3}DB$ or $\frac{1}{3}z^2x \div \frac{1}{2}zx = \frac{2}{3}x$, that is the virtual centre is some where in a line *ab* $\frac{2}{3}$ of the distance from the vertice C of the triangle to the base AB, in like manner, we shall find it also in a line GH, $\frac{1}{3}$ of the distance from the vertice B toward the side CA; hence, it must be in the intersection of the two lines.

### Problem 2. *Let it be required to find the virtual centre of a trapezium*, ABCD.



First, let DC, parallel to AB, be taken as the base and imagine an indefinite number of equidistant ordinates *ab*, drawn across the figure, parallel to the base, and if these are severally drawn into their respective distances from the base DC, we shall have constructed a wedge, whose base is equal to AB, drawn into EF, and whose altitude is equal to EF.

Let $z =$ AB, and $z' =$ DC, then will the series of ordinates be represented by $\underline{z}..z'$ let EF $= x$, then will the series of decreasing distances be represented by $\underline{x}$.

Then, we have the solid generated by the production of $\underline{x}\times(\underline{z}..z') = \frac{1}{6}x(2z + z')x = \frac{1}{6}x^2(2z + z')$, let this be divided by $\frac{1}{2}$ $(z + z')\,x$, and we have $\frac{x\ (2z+z')}{3(z+z')} =$ the distance of the virtual centre I from the line DC.

Now, in order to find in what part of the line A*b* is the centre, we may proceed as before to construct a quantity on CB, DG, or any line parallel to CB, consisting of ordinates EF, &c., across the figure, drawn into their respective distances from the bases, and dividing the quantity thus constructed by the sum of all the ordinates, or the area of the trapeziums.

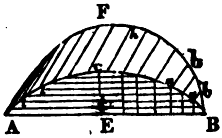
Or, we may divide the trapezium into the triangle ADG, and the parallelogram GDCB, and proceed to find the virtual centres of each of these figures; and it is then evident, that if the distance between the two points, thus found, is divided in the alternate ratio of the two figures, the result will determine the centre required.

Thus, the distance *hc* of the triangle, from the side DG is $\frac{1}{3}$ *a*H, (Prob. 1,) and the distance *ne* of the centre of the parallelogram from the line DG is evidently $= \frac{1}{2}$ *nb*.

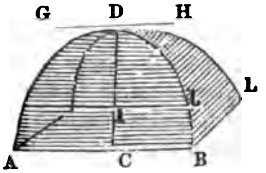
Let $t =$ the area of the triangle, and $p =$ that of the parallelogram, then as *ce* the distance of the two centres, is to $t + p$, so is $p$, to the distance *e*I, of the vertical centre of the trapezium from the point *e*.

Any rectilinear figure may, if necessary be divided into triangles, and the virtual centres determined for each, and then finding the common centre of every two of these, till they are all reduced to one only, which will be the virtual centre of the whole.

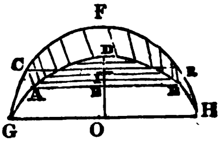
*Problem* 3. To find the virtual centre of a segment of a circle ABD.



Let parallel ordinates *ab*, &c., be drawn across the segment parallel to its base AB, and let each of these ordinates be conceived to be drawn into their respective distances from AB; then by their properties we shall have generated an ungula ABCF; divide the solidity of this ungula by the area of the segment, and the quotient is the distance EF of the virtual centre from the chord AB.

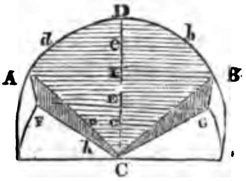


Or we may let GH be the origin, passing through the vertice D of the segment, parallel to AB, and if the ordinates *ab*, &c., are drawn into their distances from this line GH, or the vertice D, the product will be the ungula ABDL, which is the compliment of the former ungula, and the distance DI of the centre will be found by dividing the solidity of the ungula by the area ABD.



If we make the origin at the centre of the circle, we shall have the ungula ADBRCF, which is a segment of the ungula GDHF, and the solidity of this segment, divided by the surface of the segment ABD, will give the distance OI, of the centre required.

*Problem* 4, Let it be required to find the virtual centre of a sector of a circle CADB.



Let C be the origin, and if an indefinite number of equidistant ordinates *db*, parallel to the chord AB, are drawn into their respective distances C*e*, &c., we shall have produced the sectoral ungula, FADBGC, which if we divide by the surface, CADB, we shall have the distance CI of the centre.

Let $r$ = CD the radius of the circle, and the chord AB = $c$ then will $rc$ = the convex surface AFDGB of the ungula, and $\frac{1}{3}r^2c$ = the solidity of the ungula. Let $\pi'$ = the arc ADB, then we have the area CADB $= \frac{1}{2}\pi' r$; hence the distance CI $= \dfrac{\frac{1}{3}r^2c}{\frac{1}{2}\pi' r} = \dfrac{2rc}{3\pi'}$,

Resolving this into a proportion, we have, $3\pi : 2r :: c :$ the distance CI. Hence, generally, the distance of the virtual centre of any sector of a circle is = the fourth proportional to three times the arc of the sector, twice the radius, and the chord of the arc.

Let CE $= \sqrt{(r^2 - \frac{1}{4}c^2,)}$ hence we have $\frac{1}{3}r^2c - \frac{1}{2}c^3 =$ the solidity of the pyramid ABGFC; subtract this from the sectoral ungula, and we have $\frac{1}{12}c^3$ for the solidity of the segment ABGFD, divide this by $a$, the area of the segment ABD and we have the distance of the virtual centre of the segment from the centre C of the circle $= \dfrac{\frac{1}{12}c^3}{a}$.

*Problem*. 5. To find the virtual centre of the parabola.

Let its distance be estimated from the vertex.

The equation to the parabola is $y^2 = \sqrt{px}$, if this be drawn into a series denoted by $x$, the surface of the parabola will become $y\ x\sqrt{dx}$.

Let this be drawn into a series $x$, and we have $x\underline{x}\sqrt{p\underline{x}}$, and since the quantity $p$ will not affect the result, it may be omitted, and we have by removing the radical sign $\underline{x}x^{1\frac{1}{2}} = \frac{2}{5}x^{\frac{5}{2}}$; let this be divided by $x\underline{x}^{\frac{1}{2}} = \frac{2}{3}x^{\frac{3}{2}}$ and we have

$\dfrac{\frac{2}{5}x^{\frac{2}{5}}}{\frac{1}{2}x^{\frac{2}{3}}} = \frac{3}{5}x =$ the distance of the virtual centre from the vertex.

*Problem* 6. To find the distance of the virtual centre of a semi-parabola from the axis.

Its equation considering the origin at the extremity of the axis is $y=(d-x)^2+2x(d-x)$; $d=$ the maximum value of $x=$ the axis; or its equation may be expressed $\underline{x}^2 + 2\underline{x}\overline{x}$. This being put into a series, whose measure is $a$, represented by the base, we shall have its area, which may be expressed $a\underline{x}^2+2a\underline{x}\ \overline{x} = d\mathrm{Y}$, which being integrated, we have $\frac{1}{3}ax^2 + \frac{1}{3}a\underline{x}^2 = \frac{2}{3}a\underline{x}^2$.

Let the expression $a\underline{x}^2 + 2a\underline{x}\overline{x}$, be drawn into $\underline{z}$, $z$ being = the base, and we have $az\underline{x}^2 + 2a\underline{z}\underline{x}\overline{x}$, let this be integrated, and we have $4(\frac{1}{8}zx^2 + \frac{1}{4}zx^2)\frac{1}{5}a = \frac{1}{6}a(1\frac{1}{2}zx^2) = \frac{1}{4}azx^2$.

Hence, $\dfrac{\frac{1}{4}azx^2}{\frac{2}{3}ax^2} = \frac{3}{8}a =$ the distance of the virtual centre from the axis.

*Prob.* 7. Let it be required to find the virtual centre of a solid. If the solid be a pyramid, its solidity may be expressed $\underline{x}^2x$. If we estimate the centre in relation to its distance from the vertex, we may draw this quantity into another series of $\underline{x}$, and we have $\underline{x}^3x$, let this be divided by $\underline{x}^2x$, which represents the base of the latter construction, and we have $\frac{\underline{x}^3x}{\underline{x}^2x}=\frac{\frac{1}{4}x^4}{\frac{1}{3}x^3}=\frac{3}{4}x$, that is the vertical centre is $\frac{3}{4}$ the distance from the vertex of the pyramid to the base.

*Problem.* 8. To find the virtual centre of a vertical segment, of a spherical revoloid, or the virtual centre of a segment of a sphere.

The equation of the reveloid is $4y^2 = 4dx—x^2$, let this be drawn into a series of $\underline{x}$, and we have $4x\underline{x}\ (d—x)$, the segment, $x$ being the altitude of the segment, and $d$ the diameter of the sphere, the value of this is

$$\tfrac{1}{6}(2dx^2-2x^3)+\tfrac{1}{6}(4dx^2—2x^3) = dx—\tfrac{2}{3}x^3.$$

Let the former series be drawn into another series of $\underline{x}$, and its value will be $4x(dx^2+\bar{x}\underline{x}^2 = \frac{2}{3}dx^3—\frac{1}{2}x^4$;

hence, $\frac{\frac{2}{3}dx^3-\frac{1}{2}x^4}{dx^2-\frac{2}{3}x^3}=\frac{4dx—3x^2}{6d—4x}$ = the distance of the virtual centre of the segment from the vertex.

This construction will serve for a segment of a sphere, a spheroid, a spherical, or an elliptical revoloid.

*Problem.* 9. To find the virtual centre of a vertical parabolic revoloid pyramoid, or of a parabolic conoid.

The equation to the revoloid is $y^2=px$, let this be put into a series, whose measure is the axis, and we have its solidity $xp\underline{x}$; let this, as a base, be also drawn into $\underline{x}$, and we have $px\underline{x}^2$ equivalent to $\frac{1}{3}px^3$;

$xp\underline{x}$ is equivalent to $\frac{1}{2}px^2$;

hence $\frac{\frac{1}{3}px^3}{\frac{1}{2}px^2}=\frac{2}{3}x=\frac{2}{3}$. the distance from the vertex to the base.

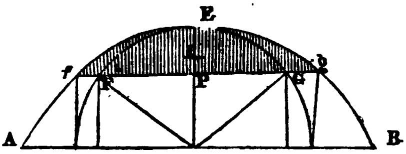
*Problem.* 10. To find the virtual centre of an hyperbolic conoid, pyramoid or vertical revoloid.

Its equation is $4y^2 = 4\frac{c^2}{d^2}\ (dx + x^2)$: since $4\frac{c^2}{d^2}$ is a constant quantity, and a factor of the whole expression, it may be omitted, without affecting the result; then the expression will become $dx + x^2$. Let this be drawn into a series of the form $x(d+\underline{x})$, which expresses the segment, whose value is, $\frac{1}{6}(dx+x^2)+\frac{1}{6}(\frac{1}{2}dx+\frac{1}{4}x^2)=\frac{1}{6}x(3dx+2x^2)=\frac{1}{6}(3dx^2+2x^3)$

Let the former series be drawn into another series of $\underline{x}$, and there will be produced $x(d + x)\underline{x}^2 = \frac{1}{6}x(dx^2 + x^3) + \frac{1}{6}x\times(\frac{1}{4}dx^2+\frac{1}{3}x^3)4=\frac{1}{6}x(2dx^2+1\frac{1}{2}x^3)=\frac{1}{6}(2dx^3+1\frac{1}{2}x^4)$, hence, we have $\frac{\frac{1}{6}(2dx^3 + 1\frac{1}{2}x^4)}{\frac{1}{6}(3dx^2 + 2x^3} = \frac{4dx + 3x^2}{6d + 4x}$ the distance of the centre from the vertex.

*Problem.* 11. To find the virtual centre of the arc of a circle.

The virtual centre of an arc of a circle, is the same in reference to the centre of the circle, as that of the segment of a revoloidal curve, whose conjugate diameter is the same as that of the circle, and the base of whose segment is equal to the given arc of the circle.



For, if a revoloidal curve AEB circumscribe the semi-circle DEH, and if any chord FG, be produced to *fg*, so as to meet the curve, the ordinate *fg* will be equal to the arc FEG. Draw across the segment *f*E*gf*, equidistant ordinates, perpendicular to *fg*, and they will represent the distance of the several points in the arc, from the line *fg*, since they are supposed to be drawn from points equidistant from each other on the line *fg*, or the arc FEG; and, since if there is an infinite number of ordinates, they may be regarded as the area of the segment, it therefore, follows that if this area is divided by the arc, the quotient is the distance of the centre, from the line FG; also, if the area *f*E*g*HD is divided by the arc FEG or line *fg*, the quotient is the virtual centre of the arc *f*E*g* from the axis DH.

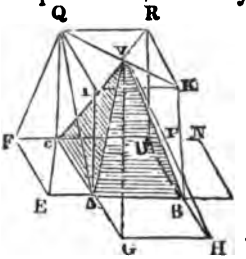
Let $c$ = the chord FG$a$ = the arc FEG, and $r$ = CE, then will $cr$ = area D*f*E*g*H, (Prop. III, *Cor.* 4, B. III.,) hence, we have $\frac{cr}{a}$ = the distance EI of the centre.

This is also the virtual centre of the segment *f*E*gf* of the revoloidal curve, and it may also be shown that if a series of equidistant ordinates to the axis CE, are drawn through the revoidal surface, or any segment of it parallel to *fg*, the series of ordinates so constructed, drawn into their several distances from any given line, parallel to such ordinates, will determine the virtual centre of the segment *fgef*, or of the arc FEG, by proceeding as before.

Art. 12. If it be required to find the virtual centre of the arc of an ellipse, a parabola, or an hyperbola, it may be done in a similar manner, by taking a portion of the surface of the elliptic, parabolic, or hyperbolic revoloid, and proceeding as for the arc of the circle, which also gives the virtual centre of the segment of the revoloidal surface pertaining thereto.

*Problem* 3. To find the virtual centre of the surface of a solid.

Let the proposed surface be the convex surface of a pyramid ABCDV; and because any portion of the convex surface, included between any two sections, by planes parallel to the base, is proportional to the portion of a vertical triangle through the pyramid included between the same planes; it follows that the vertual centre of the convex surface, is the same as that of the vertual triangle; hence the same process will determine both. If it is required to find the virtual centre of the whole surface of a pyramid, including its base, we have only to imagine an infinite number of ordinates to be drawn across the several triangular sides parallel to their several bases, and also a similar series of parallel ordinates across the base, and if each of these ordinates are severally drawn into their respective perpendicular distances from the vertex of the given pyramid, we shall have produced, as many new pyramids AEFCQ, whose bases ACFE, ABHG, &c, are, severally equal to the bases of the sides of the pyramid multiplied by IA, the distance of the base from the vertex, as the given pyramid has sides, and also a prism ABDCQIKR, formed by drawing every line in the base, or the whole surface of the base into the distance of the base from the vertex. And the sum of the imaginary solids so generated, divided by the whole surface of the pyramid, will give the distance of the virtual centre from the vertex.



Let AB$=x$ and if the pyramid is generated from $\underline{x}^2$ or $h\underline{x}^2$, or if the base of the pyramid is a square, the perimeter of the base will be $4x$; and since each pyramid ACFEQ, is equal to $\frac{1}{3}$ the prism ABCDQRKI: hence the four pyramids generated by the series drawn into the four sides of the base are $= \frac{4}{3}$ the prism, and if $h =$ the altitude IA of the pyramid, $hx^2+\frac{4}{3}hx^2= \frac{7}{3}hx^2 =$ the sum of the four pyramids + the prism ABDCQIKR; the surface of the given pyramid is $= x^2+2x\surd(h^2+\frac{1}{2}x^2)$

Hence we have, $$\frac{\frac{7}{3}hx^2}{x^2+2x\surd(h^2+\frac{1}{4}x^2} = \frac{\frac{7}{3}hx}{x+2\surd h^2+\frac{1}{4}x^2}$$

equal the distance of the virtual centre from the vertex.

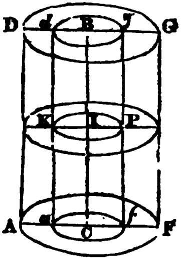
## BOOK VII.

## CHAPTER V.

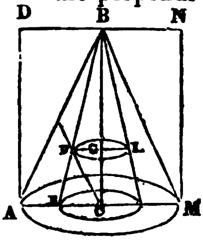
### ON THE RELATIONS OF LINES, SURFACES, AND SOLIDS, GENERATED BY MOTION.

The capacity of any solid generated by the motion of a surface perpendicular to itself, is measured by the generating surface drawn into the distance moved; which distance is always equal to the distance passed through by the virtual centre of such surface.

If the motion of the generating surface is such, as that it always maintains a parallel position, and moves in a direction perpendicular to itself, the proposition is sufficiently manifest.

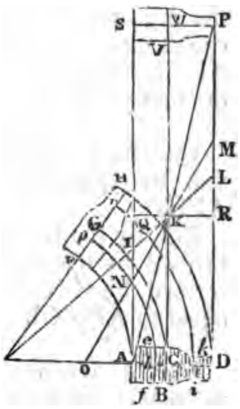


Let now the rectangle ACBD revolve about the side BC, which remains fixed, and the product will be the cylinder DF, whose solidity is equal to the surface ACBD drawn into the circumference PK. described by the virtual centre K, of the plane, which centre is in this case also the centre of aggregation.



If a right-angled triangle ABC revolve about the perpendicular BC, so as to describe the cone ABD, this is also measured by the triangle ABC drawn into the circumference FL, described by the virtual centre of the triangle. The virtual centre of the triangle we have shown to be situated at the point F, on the line BE, from the vertex bisecting the base at a distance from B $=\frac{2}{3}$ its length. Let the surface ABC be multiplied by the circumference described by the centre F; and since the radius FG$=\frac{1}{3}$EC, hence the circumference FG$=\frac{1}{3}$ the circumference EC, and because the triangle ABC$=\frac{1}{2}$its circumscribing rectangle ADBC, which generates a cylinder ADNM, the generating surface of the triangle ABC drawn into the circumference, is equal to one-third the cylinder generated by the rectangle, or one-third the rectangle drawn into the circumference EC, as it ought to be.

And in general, let any plane figure be revolved about any line or axis without the figure, but always in the same plane, and the solid generated will be measured by the generating surface drawn into the arc described by the virtual centre of the surface.



Let AFHD be a solid generated by the plane ABD; through C, the virtual centre of which, draw DCAE, perpendicular to the axis of rotation, and meeting HGFE in E, let an indefinite number of parallel ordinates, *ef*, *ik*, &c., be drawn across the generating surface, parallel to the axis about which it revolves; and the solid generated is equal to all of those ordinates, drawn into the distances passed through by each; viz., the ordinate drawn across the point A×the arc AF+the ordinate *ab* drawn through *c*×the arc CG+, &c., through the whole series. And because EA, EC, ED, &c., are as the arcs AF, CG, DH, &c. Hence E*h*×*ef*, and E*l*×*ik*, &c., are as *ph*×*ef*, and *rl*×*ik*, and because EC drawn into all the *ef*, *ik*, &c., is equal to all, the E*h*×*ef*, E*l*×*ik*, &c., it follows that CG×all the *ef*, *ik*, &c., is equal to all the *ph*×*ef*, *rl*×*ik*, &c.; or that the solid ABDHF is equal to the generating surface ABD*e* drawn into the line described by the virtual centre of the surface.

*Cor.* 1. Hence, if any curve or any line be made to revolve about any axis exterior to such curve, but in the same plane, the surface described by its motion will be equal to the line or curve drawn into the distance passed through by the virtual centre of such line or curve.

For, let the perimeter of the figure generating the solid above, be the generating line, and let us suppose its virtual centre the same as before; let every point in this perimeter be reduced to the line AD by means of perpendiculars thereto; and the figure generated by its revolution about the axis, is equal to all the *ph*, *rl*, &c., described by every point; but we have seen that all the *ph*, *rl*, &c., are as all the E*h*, E*l*, &c.; and since the sum of all the E*h*, E*l*, &c., is equal to as many times EC, therefore the sum of all the *ph*, *rl*, &c., is equal to as many times CG, or equal to ABD*e*×CG, that is, the surface described by the perimeter ABD*e*, is equal to ABD*e* drawn into the line described by its virtual centre C.

*Cor.* 2. From E draw EIKL, cutting the upright prismatic figure erected on the given base ABD, so as that any perpendicular AI may be equal to the corresponding arc AF. Then will the figure AILD be equal to the figure AFHD.

For, by similar figures, all the AF, CG, DH, &c., are as all the AI, CK, DL, &c., each to each; and as one of each are equal, therefore they are all equal, each to each; viz., all the AI, CK, DL, &c., equal to all the AF, CG, DH, &c.; that is, the figure AILD equal to the figure AFHD.

*Cor.* 3. Through K draw MKNO; then the figure ANMD will be equal to the figure AIKLD, or equal to the figure AFHD.

For, by the last corallary, AFMD is equal to the figure described by the base AD, revolving about O, till the arc described by C be equal to CK; which, by the proposition, is equal to AD×CK, or AD×CG.

*Cor.* 4. Hence, all the upright figures AQKRD, AIKL, ANKMD, AKPD, &c., of the same base, and bounded at the top by lines or planes cutting the upright sides, and passing through the extremity K, of the line CK, erected on the virtual centre of the base, are equal to one another; and the value of each will be equal to the base drawn into the line CK.

Hence, also, all figures described by the rotation of the same line or plane about different centres or axes, will be equal to one another, when the arcs described by the virtual centre are equal. But if those arcs be not equal, the figures generated will be as the arcs. And in general, the figures generated, will be to one another, as the revolving lines or planes drawn into the arcs described by their respective virtual centres.

*Cor.* 5. Moreover, the opposite parts NIK, MLK, of any two of these figures, are equal to each other.

*Cor.* 6. The figure ASPD is to the figure APD, as AS to CK; for, by similar triangles, they will be as AD to AC.

For ASPD is equal to AD×AS, and APD equal to AD ×CK.

*Cor.* 7. If the line or plane be supposed to be at an infinite distance from the centre about which it revolves, the figure generated will be an upright surface or prism, the altitude being the line described by the virtual centre; so that the base drawn into the said line will be equal to the base drawn into the altitude, as it ought for all upright figures, whose sections parallel to the base are all equal to each other.

*Scholium.* If a right line, or parallelogram, revolve about a line perpendicular to the length, there will be described a ring either superficial or solid; and as the virtual centre of the describing line, or parallelogram, is also the centre of magnitudes, it follows, therefore, that such surfaces or solids, are equal to the generating magnitude drawn into the distance passed through by the centre of magnitude.

When the centre of rotation is in the end of the line, the line will describe a circle whose radius is the said describing line, and whose circumference is double the circumference described by the virtual centre; consequently, the radius drawn into half the circumference, will be the area of the circle.

If a semi-circle revolve about a diameter, and describe the surface of a sphere, then will the surface of the sphere be equal to the revoloid arc × by the circumference described by the virtual centre $= \frac{1}{2}\pi \times \frac{cr}{\frac{1}{2}\pi} \times \pi = 2r\pi =$ the circumference into the diameter.

And for the solidity of the sphere, we shall have the distance of the virtual centre equal $\frac{d^3}{12a}$, where $d$ is the diameter, and $a$ the area of the segment; let twice this distance be multiplied by $\pi$, and also by $a$; and we have $\frac{2d^3\pi}{12} = \frac{d^3\pi}{6}$ the solidity of the sphere, as before found in the Elements of Geometry.

For the solidity of the parabolic spindle: putting $b =$ the base, and $a =$ the altitude, or axis of the generating parabola.

We have found that $\frac{2}{5}a$ is the distance of the centre of gravity from the base, and consequently $\frac{4}{5}a\pi =$ the line described by the centre of gravity; but $\frac{2}{3}ab$ is $=$ the revolving area; therefore $\frac{4}{5}a\pi \times \frac{2}{3}ab \div$ the $\frac{8}{15}a^2b\pi$ will be the content, which is $\frac{8}{15}$ of the circumscribed cylinder.

For the paraboloid. Making the notation as in the last example, and making $n =$ the area of a circle, whose diameter is 1; $\frac{3}{8}b$ will be the distance of the centre of gravity of the semi-parabola from the axis, consequently $\frac{3}{8}b \times 8n \times \frac{2}{3}ab = 2ab^2n =$ the solidity $=$ half the circumscribed cylinder.

# MENSURATION.

Having, in the elementary parts of the work, introduced such subjects of mensuration as depend on principles therein discussed, it only remains for us now to present the higher branches of the subject, or such subjects in mensuration, as depend on the higher branches of geometry.

The subject of mensuration admits of three general divisions: lines, superficies, and solids; but since the mensuration of lines is so intimately connected with that of surfaces, we shall make but two general divisions, termed superficies, and solids.

## PART I.

### MENSURATION OF SUPERFICIES.

#### PROBLEM I.

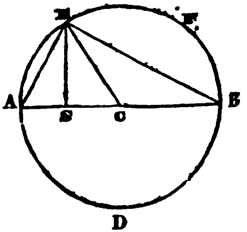
*To find the area of a segment of a circle.*

#### CASE I.

*When the arc, sine, and radius are given.*

Rule.—Multiply the difference between the arc of the segment and its sine by half the radius. (Prop. XIII, B. IV.)

Let $\pi'$ = the arc, $s$ = the sine, $r$ = the radius, and A the area of the segment; and $A = \frac{1}{2}\pi' r - \frac{1}{2}sr$.



Ex. 1. What is the area of a segment AE, whose arc AE is 2,09438, and whose sine ES = 1,73205, the radius = 2. ?

$$\left.\begin{array}{r}2.09438\\-1.73205\end{array}\right\}\tfrac{1}{2}r = .36233\times1 = .38233$$

the area of the segment AE.

Ex. 2. What is the area of the segment EADB, whose arc EADB = 4.18878 and sine ES = .86602 ?

In this example because the sine is considered negative, by Trigonometry, the arc being greater than that of a semi-circle, we shall have by the rule

$$\left.\begin{array}{r}4.18878\\+\ .86602\end{array}\right\}\tfrac{1}{2}r = 1.66138 = \text{the area required.}$$

Ex. 3. What is the area of the segment whose arc is 6.9813, and whose sine is 6,4278, the radius being 10?

*Scholium.* If the arc and radius, or the sine and radius only are given, the other parts may be taken from the table of natural sines, and the area of the segment calculated by the rule. The arc of any segment less than a semi-circle may be found approximately by formula 3, (Prop. IX, B. IV.) $\frac{1}{2}x = \sqrt{(4vr+\frac{1}{4}s^2)} - \frac{1}{2}s$. Where $v$ is the versed sine or height of the segment, $r$ the radius, and $s$ the sine of the half arc, or the $\frac{1}{2}$ chord of the segment, $x$ being = the arc of the segment.

It will appear that this gives the value of the arc to a great degree of exactness when the segment is small.

Let us see how near the truth this comes for a semi-circle.

In this case, the sine and versed sine are each equal to the radius, which suppose = 1.

Whence we have $\frac{1}{2}x = \sqrt{4\frac{1}{4}} - \frac{1}{2} = 1,56155$

And $x = 3.12310$

the true number being $3.14159$

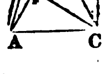
The difference of which is .01849, the error in a segment = the semi-circle.

Let us assume that the error in any smaller segment, is proportional to the sixth power of $v$ or $v^6$, then if we correct this by deducting $v^6 \times .01849$ therefrom, the result will, in this case be correct, and if our hypothesis is correct, it will give the proper result for any smaller arc.

### CASE II.

*When the arc, chord, versed sine, and radius are given.*

Rule.—Multiply the difference between the chord and arc by half the radius, to which add half the product of the chord and versed sine.



*Investigation.* Let ABD be a segment; this is composed of the segments AD + DB + triangle ADB, but the segment AD = DB = (arc AD — AF) $\times\frac{1}{2}r$; segment AD+DB = (arc ADB — chord AB) $\times\frac{1}{2}r$ and triangle ADB = $\frac{1}{2}$(AB×DF).

Ex. What is the area of the segment ABD, whose arc ADB, is 10,4719, and whose chord AB=10, the versed sine DF or height of the segment =1.3398?

$$\left.\begin{array}{r}10,4719\\ -10\end{array}\right\} 5+\frac{10\times 1.3398}{2} = .9058 = \text{the area of the segment.}$$

### CASE III.

### *When the radius of the circle, and the degrees of the arc only are given*

Rule.—Find by Trigonometry, or by the table of natural sines, the sine of the given arc for a circle whose radius is 1, observing that the same sine answers for an arc and complement, multiply this by the given radius, which gives the sine of the given arc. Then say, as 180° is to the given arc, so is $\pi$ to $\pi'$, so is the semi-circumference of a circle whose diameter is 2, to the length of the given arc; then proceed as in Rule 1.

Ex. 1. What is the area of the segment AE, (see diagram to Case 1,) whose arc AE = 60°; the radius EC being 1?

$$\sqrt{(EC^2 - SC^2)} = ES = \sqrt{(r - \tfrac{1}{2}r)} = \sqrt{(\tfrac{1}{2})} = .86602 = s$$

And $180° : 60° :: \pi : \pi' :: 3.14159 : 1.04719$ = the arc AE $= \pi'$. Hence, $(1.04719 - .86602) \times \frac{1}{2} = .09058$ = the area of the segment AE.

Ex. 2. What is the area of the segment EBF, whose arc EFB is 120°, the other quantities remaining the same as before?

The sine of the arc AE is also the sine of the arc EFB. The arc EFB = 2 arc AE=2.09438; hence $(2.09438 - 86602) \times \frac{1}{2} = .61418$ = the area of the segment EBF.

Ex. 3. What is the area of the segment EADB, whose arc EADB is 240°, the other quantities remaining the same?

As $180° : 240 :: 3.14159 : 4.18878$=the arc EADB.

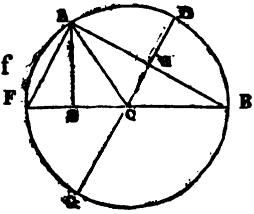
Since the segment EADB is greater than a semi-circle, its sine, ES, is considered negative by Trigonometry, we have $(4.18878 + ,86602) \times \frac{1}{2} = 2.52745$ = the area of the segment EADB.

Ex. 4. What is the area of the segment ADBFE, whose arc is 300°?

*Scholium.* The difference of the segments EBF, and the segment AE is equal to the sector ACE; the segment ADC — segment EBF = sector ACE + triangle ECB.

### CASE IV.

*When the chord of the segment, its height or versed sine and radius are given.*



Rule.—As the radius is to half the chord, so is twice the difference of the versed sine and radius, to the sine of the arc of the segment; divide this by the radius, reducing it to the sine of an arc of a circle, whose radius is 1.

Then in the table of natural sines, take out the arc answering to that sine in degrees, and proceed as in Rule III, to find its length; then proceed as in rule 1st., to find the area of the segment.

*Investigation.* In the right angled triangle FAB, we have FA=2CE, and because the triangle ASB is similar to FAB or CEB—hence, CB : EB : : FA : AS.

Or without finding the sine AS, of the arc ADB, proceed to take out from a table of natural sines the arcs AB, DB, answering to AE, BE, the sines of those arcs respectively; and after finding their lengths as in Rule III, proceed by Rule II, to find the area.

Ex. 1. What is the area of a segment ABD, whose chord AB=17,3205, and whose height ED=5, the radius being 10?

In this example we have FB=20, AB=17,3205 and AF=(10—5)2=10, to find AS.

Or, we may make the triangle CEB, whose side CB=10, EB=8.6602, and CE = 5, and FA=2CE to find AS, by the rule.

Hence, 10 : 8.6602 : : (10—5)2 : 8.6602 = the sine; and 8.6602÷10=.86602 = the tabular sine: the arc answering thereto is that of 60°, but this segment being greater than a quadrant, the arc must be the supplement of 60°=120°. Then 180° : 120° : : 3.14159 : 2.09439 = the length of the arc of 120° in a circle whose radius is 1.

Hence, 2.09439×10=20.9439 = the arc of the given segment.

Then $\left.\begin{matrix}20.9439\\-8.6602\end{matrix}\right\}$ 5, = 61.418 = the area of the segment ABD.

Or, taking the same example, having found the length of the arc ADB=20.9439, we have by Rule II.

$\left.\begin{matrix}20.9439\\-17.3205\end{matrix}\right\}$ 5+17.3205+½=61.418 = the area the same as before.

*Scholium.* If two of the following parts, viz: the chord, versed sine, and radius are given, the other may be found by the formulæ (in mensuration *El. Geom.* Prop. XIII.)

*Scholium* 2. The triangle ACB is = the triangle AFC.

$$EB\times CE=\tfrac{1}{2}CB\times AS.$$

That is, the product of the sine of an arc × its cosine = $\frac{1}{2}$ sine of twice the arc × radius.

### CASE V.

*When the chord and radius only are given.*

Rule.—Divide half the chord by the radius, and the quotient will be the sine of half the arc of the segment; find the arc corresponding thereto in the table of natural sines, in degrees, and multiply it by 2; then find its length as in Rule 3, and multiply it by the radius.

Take the versed sine $= ED = CD - \sqrt{(CA - AE.)}$ Then proceed as in Rule 2 to find the area of the segment.

Ex. Taking the same example as in the last rule, having found the arc = 20.9439, we have

$$ED = 10 - \sqrt{(100 - 69{,}899905404)} = 5.$$

$$\text{Hence,}\ \left.\begin{array}{r}20.9439\\ -17.3205\end{array}\right\} 5+173205\times 2.5 = 61.418.$$

*Examples for Practice.*

Ex. 1. What is the area of a segment whose arc is 3.14159 and whose sine is .87785, the radius being 10?

Ex. 2. Required the area of the segment whose chord is = 12, the radius = 10. Ans. 16.35.

Ex. 3. What is the area of the segment whose height is 2, the chord being 20? Ans. 26,8804.

Ex. 4. What is the area of a segment of a circle whose arc is 110° the radius being 1?

Ex. 5. Required the area of the segment whose height is 5, the diameter being 8. Ans. 33.0486.

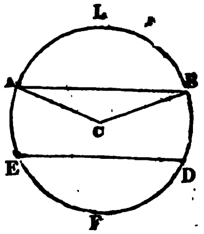
### PROBLEM II.

*To find the area of a circular zone* AEDB, *or the space included between two parallel chords* AB, ED, *and two arcs* AE, BD.

Rule. Multiply the two arcs AE, BD, of the zone by $\frac{1}{2}$ the radius, to which add the sum of the products of the sines of the external segments, with the radius, if on different sides of the centre, or add the difference of those products, if on the same side. (Prop. XXIII Schol. Formula 3, B. IV.)

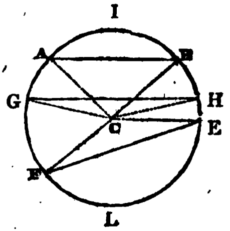
$$A = \tfrac{1}{2}(rx'-rs')-\tfrac{1}{2}(rx-rs,)$$

*Ex.* What is the area of a zone ABDE, the sum of whose arcs AE+BD is=6,28318 the sine of the arc ALB=2,57178, and the sine of the arc EFD = 2,62386, the radius being = 3 ?



$$\left.\begin{array}{r}6{,}28314\\+262386\\+257178\end{array}\right\}\times\tfrac{3}{2} = 17{,}31817 \text{ the area required.}$$

*Scholium.* The same rule will apply to any portion ABEF of the circle included between two chords; that are not parallel. Hence, if the sum of the axes AF, AE in this figure, is equal to the sum of AE, BD in the last, and if the arcs AIB, FLE in this are respectively = ALB, EFD in that, then the portion AFEB in this will be = the zone AEDB in that.



*Ex.* 2. What is the area of a zone AGHB whose two chords are on the same side of the centre, the sum of the arcs of the zone being 2,09436, the sines of the arcs on each side being 2,59806, and — 2,95440, the radius being = 3 ?

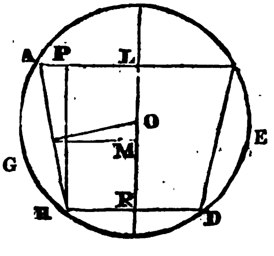
$$\left.\begin{array}{r}2{,}09436\\+2{,}59806\\-2{,}95440\end{array}\right\}\times\tfrac{3}{2} = 2{,}60703 \text{ the area required.}$$

Note. Other rules have been given for finding the areas of circular segments and zones in Mensuration, El. Geom ; formulæ are there given for finding such data as are required, for the elements of the area.

### *Examples for Practice.*

*Ex.* 1. Required the area of the zone ABEDHGH, the greater chord AB = 136 feet, the less chord HD = 68 feet, and the distance LP = 248 feet.

Ans. 55655.1965159 sq. feet.



*Ex.* 2. Suppose the zone to have its parallel chords equally distant from the centre of the circle O, each chord AB and HD = 12.49 feet and their distance LP = 10 feet; required the area of the zone. Ans. 148.86672 sq. feet.

*Ex.* 3. Supposing the circular zone ABEDHGA, having its greater parallel chord = 40 yards, being equal to the diameter of the circle, the less chord = 20 yards, and their distance LP = 17.310508 yards; required the area of the zone.
Ans. 592.08244 sq. yards.

*Ex.* 4. Required the area of the zone ABEDHGA, the parallel chords AB, and HD, being 16 feet and 12 feet, and their distance HP = 14 feet. Ans. 253.0792 sq. feet.

*Ex.* 5. The zone, whose parallel chords AB = 40, HD = 30, and the breadth = 35; required the area of the zone.
Ans. 1581.745.

*Ex.* 6. Suppose the two parallel chords AB and HD = 80 feet and 60 feet, and the perpendicular distance from each other = 70 feet; it is required to find the distance of the greater chord AB from the centre at O; and also to find the radius of the circle.

The distance OP . . . . . = 30 feet = 1st Ans.
The radius of the circle OF = 50 feet = 2nd Ans.

*Ex.* 7. Required the area of the zone ABEDHGA, whose arcs AGH, BED are together = 160°, the arc AOB being 110°, and the radius of the circle 10 feet.

*Ex.* 8. Required the area of the portion ABEF, included between the two oblique chords AB, FE, (see diagram to Scholium above,) whose arc AF = 60°, BE = 30°, and AIB = 110°, the radius being 20.

### PROBLEM III.

### *To find the circumference of a circle approximately.*

#### CASE I.

#### *When the radius sine, and cosine of any small arc is given.*

Rule 1. Divide $1\frac{1}{2}$ times the product of the radius and sine by the sum of the radius, and half the cosine, which will give the length of the arc, multiply this by the number of such arc in the whole circumference, and the product is the circumference of the circle; the accuracy of which depends in the smallness of the arc.

Let $s$ = the sine,
$c$ = the cosine,

and $\pi'$ = the arc corresponding to these functions,

$\pi$ = being the semi circumference, and $r$ the radius of the circle = 1.

$\pi' = \frac{\pi}{n}$ ; $n$ being the number of parts each = $\pi'$ that the semi-circle is supposed to be divided into. Then (Prop. IX, B., IV. Formula 1.)

$$\pi' = \frac{\frac{3}{2}rs}{r+\frac{1}{2}c}$$

Let $\pi' = \frac{\pi}{20000}$

$$\begin{aligned} \text{Then } s &= ,000157079632033 \\ c &= ,999999987462994 \\ \text{and } \tfrac{3}{2}rs &= ,000235619448050 \\ r+\tfrac{1}{2}c &= 1,499999993831497 \end{aligned}$$

$$\text{Hence } \frac{\frac{3}{2}rs}{r+\frac{1}{2}c} = ,000157079632676$$

and, 000157079632676 × 20000 = 3,1415926535,2 = the circumference of the circle, whose radius is 1, which is true to the last figure, which should be 8 instead of 2.

*Scholium.* Other formula may be found for determining the circumference of a circle at Prop. IX B. XIV, viz., formulæ 2 and 3, to which the student is referred ; at which place, will also be found some important trigonometrical formulæ for finding the value of the sines and cosines, and other functions of the circle.

RULE 2. Divide 6 times the product of the radius and versed sine of a small arc, by the sum of the sine + 4 times the sine of half the arc, which will give the length of the arc, which, multiplied by $n$, the number of times this arc is contained in the circle, gives the circumference.

$\pi' = \frac{6A'}{a+2b}$ (Prop. XVII,) which may be expressed

$$\frac{6r\ .\ versin\ .\ \pi'}{sin.\ \pi' + 4\ sin.\ \frac{1}{2}\pi'} = \frac{6rv}{s+4s'}$$

*Ex.* Having the sine and cosine of an arc=to $\frac{1}{100}$ part of a quadrant= ,0157073173118 = $s$ and ,9998766324816 = $c$ ; it is required from these data to find the arc $\pi'$ the radius being 1.

The versed sine = 1 — ,9998766324816
= ,0001233675183 = $v$

Hence, $6rv$ = 0007402051098

And by Trigonometry we shall find $s'$ = ,0078539008887.

Hence, $s + 4s'$ = ,0471229208666.

And ,0007402051098÷,0471229208666 $=\frac{6rv}{s+4s'}$ $=$ ,015707963267,5 $= \pi'$ which is true to the last figure, which should be 9 instead of 5.

*Scholium*. The expression $\frac{6A'}{a+4b}$ may also be put under the form $\frac{6rs}{\text{Cos. }\pi'+4\text{ Cos. }\frac{1}{2}\pi'}$ or $\frac{6rs}{c+4c'}$.

*Ex.* 2. Let it be required to find the value of $\pi'$ to twenty decimal places; for this purpose let $\pi'$ be an arc of $\frac{1}{10000}$ part of a quadrant, or 1 minute, according to the French centesimal divisions of the circle. The sine of this are to 21 decimal places, according to Legendre's Trigonometry, is

$=.000157079632033525563=s$

and its cosine$=.999999987662994524005=c$

And by the trigonometrical formula $\cos.\ \theta=\sqrt{\frac{1+\cos.\ \theta,}{2}}$

we have $\cos.\ \frac{1}{2}\pi' \quad =.999999996915748676195.$

Hence we have $6rs=.000157079632033525563,$

and $6+4c' \quad =.59999999975325989228785.$

Therefore, $\frac{6rs}{c+4c'}=\pi' =$ 0001570796326794896,5, which is true to the last figure, which should be 6 instead of 5.

*Scholium.* Since there is no limit to the smallness of the arc, which may be taken, and since its sine and cosine may be calculated to any number of decimal places whatever, it therefore follows, that there is no limit to the accuracy with which the circle's circumference may be calculated by this method.

The circumference of the circle, as found by M. DeLagney, to 128 decimal places, by a method furnished by the calculus, is as follows

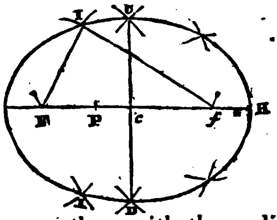
The circumference of a circle whose diameter is 1, is
3.14159265358979323846264338327950288419716939937510
58209749445923078164062862089986280348253421170679
8214808651 32723066470938446 + or 7—.

The series has more recently been extended to 154 decimal places. We might proceed by this method to verify the results obtained by the calculus, and extend the number of decimals much farther, were it worth the labor; but since we have the result already, extended beyond what is practically useful, the labor may be reserved for those who have leasure and inclination to pursue it. It may be shown that, however far, the circumference should be developed in terms of the diameter, the expression would never terminate, or in other words, the circumference and diameter of a circle arc incommensurable in terms of each other. See notes.

### PROBLEM IV.

### 1. *To describe an Ellipse.*

Let TR be the major axis, CO the minor axis, and *c* the centre. With the radius T*c* and centre C, describe an arc cutting TR in the points F, *f*; which are called the two foci of the ellipse.

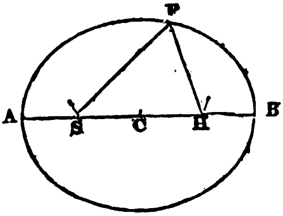


Assume any point P in the major axis; then with the radii PT, PR, and the centres F*f*, describe two arcs intersecting in I; which will be a point in the curve of the ellipse.

And thus, by assuming a number of points P in the major-axis, there will be found as many points in the curve as you please. Then with a steady hand, draw the curve through all these points.

### *Otherwise with a Thread.*

Take a thread of the length of the axis-major AB, and fasten its ends with two points in the foci, SH. Then stretch the thread, and it will reach to P in the curve: and by moving a pencil round within the thread, keeping it always stretched, it will trace out the ellipse.



There are various instruments used for the construction of this and the other conic sections. But we have not room, consistantly with our plan, to describe them here.

### PROBLEM V.

*In an ellipse having either three of the following parts given,* viz., *the major or minor-axis, the ordinate, or abscissæ, to find the fourth.*

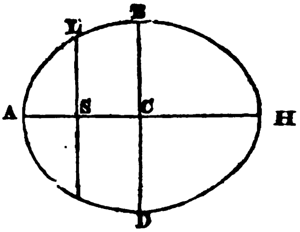
### CASE I.

### *To find the ordinate.*

*When the major axis, the minor axis and abscissæ, are given.*

Rule. As the major-axis is to the minor-axis, so is the square root of the product of the two abscissæ to the ordinate.

*Ex*. 1. In the ellipse of ABHD the major-axis AH = 70, the minor-axis BD = 50, and the two abscissæ AS = 14, HS = 56, it is required to find the length of the ordinate LS.



AH : BD : : $\surd$(AS × HS) : LS,

viz., 70 : 50 : : $\surd$(14 × 56) : 20, the length or the ordinate required.

*Ex*. 2. If the major, and minor-axes, of an ellipse are 80 and 60, the abscissæ AS = 16, what is the length of the ordinate?

Ans. 24.

CASE II.

*To find the two abscissæ.*

*When the major, and minor axes, and ordinate are given.*

RULE. As the axis-minor is to the axis-major, so is the square root of the difference of the squares of the semi-minor axis and ordinate, to the distance between the ordinate and centre; which distance, added to and subtracted from, the semi-axis major will give the two abscissæ.

*Ex*. 1. The major-axis AH = 70, its conjugate BD = 50, and the ordinate LS = 20; required the two abscissæ AS, HS.

BD : AH : : $\surd((\frac{1}{2}BD)^2—LS^2)$ : CS, viz.,

50 : 70 : : $\surd(25^2—20^2)$ : 21, the distance from the centre to the ordinate.

Hence, $\frac{1}{2}$AK ± SC = (70 ÷ 2) ± 21 = 35 ± 21 = 56 and 14 = AS, HS, the two abscissæ.

2. What are the two abscissæ AS, HS, the ordinate LS = 24, and axes AH BD = 80 and 60?

Answer 16 and 64,

3. The major-axis AH = 36, its conjugate BD = 24, and ordinate LS = 8; required two abscissæ HS, AS.

Answer 18 ± 3$\surd$2 = 18 ± 4,2426408 = 22,2426408 and 13,7573592.

### CASE III.

### *To find the major axis.*

*When the minor axis, ordinate, and abscisses, are given.*

Rule. From the square of half the minor axis, subtract the square of the ordinate; then extract the square root of the remainder. Next add this root to the semi minor axis, if the less abscissa be given, but subtract it if the greater abscissa is given, reserving the sum or difference. Then say as the square of the ordinate, is to the rectangle of the abscissa and minor axis, so is the reserved sum or difference to the major axis.

*Ex.* 1. In the ellipse ABHD, there are given the minor axis BD = 50, the ordinate LS = 20, and the less abscissa AD = 14; required the major axis AH.

*First.* $\sqrt{[(\frac{1}{2}BD)^2 - LS^2]} = \sqrt{(16^2 - 20^2)} = \sqrt{225} = \sqrt{(5^2 \times 3^2)} = 5 \times 3 = 15$, the square root of the difference of the semi-conjugate axis, and the ordinate.

Then $\frac{1}{2}BD + 12 = 25 + 15 = 40$, the sum.

*Secondly.* $LS^2 : BD \times HS :: 40 : AH$, viz., $20^2 : 50 \times 14 :: 40 : 70 = AH$, the major axis required.

*Ex.* 2. If the minor axis BD = 40, the ordinate CS = 16, and the less abscissa AS = 36; what is the length of the major axis AH. Ans. 180.

### CASE IV.

### *To find the minor axis.*

*When the major axis, ordinate, and absbissæ, are given.*

Rule. As the square root of the product of the two abscissæ is to the ordinate, so is the major axis to the minor axis.

*Ex.* 1. The major axis AH = 180, the ordinate HS = 16, and the greater abscissa HS = 144; required the length of the conjugate axis AD.

Here AH − AS = 180 — 144 = 36 = HS, the less abscissa.

Then $\sqrt{(AS \times BS)} : LS :: AH : BD$, viz., $\sqrt{(144 \times 36)} : 16 :: 180 : 40$, the conjugate axis AH.

*Ex.* 2. The major axis AB = 70, the ordinate LS = 20, and the abscissa HS = 14; required the conjugate axis BD. Ans. 50.

### PROBLEM VI.

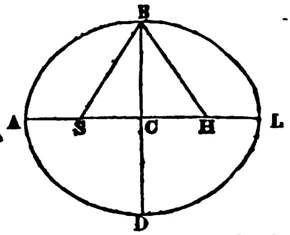
### *To find the area of an ellipse.*

#### CASE I.

#### *When the major and minor axes are given.*

RULE. Multiply the product of the semi axes by $\pi = 3{,}14159$ or the circumference of a circle whose diameter is 1, and this product will be the area.

*Ex.* 1. Required the area of an ellipse ABLD, whose axes are AL = 70, and BD = 50.
$\frac{1}{2}AL \times \frac{1}{2}BD \times \pi = 70 \times 50 \times 3{,}14150 = 2748 . 9$, the area of the ellipse required.



*Ex.* 2. What is the area of the ellipse whose major axis is 23, and the minor axis = 18 ? Ans. 339.2928.

*Ex.* 3. The major and minor axes being 61,6, and 44 respectively, required the area of the ellipse.
Ans. 2128.7481,6.

*Ex.* 4. What is the area of an ellipse, whose axes are 25 and 19 ? Ans. 373,06381.

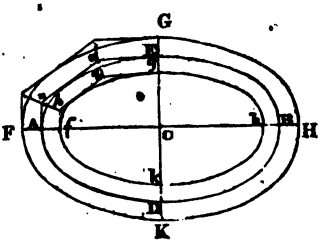
*Ex.* 5. What is the area of an ellipse whose axes are 23, and 17 respectively ? Ans. 307,09042.

*Scholium.* If there be two or more concentric ellipses FGHK, *fghk*, the area of the inner one subtracted from that of the outer one, will be the area of the elliptical ring included between them.

Hence, also as for a circular ring (Mensuration *El. Geom.*) so with the elliptical ring, its area is equal to the difference of the rectangles of the semi axes of the inner one and outer one, multiplied by $\pi = 3{,}14159$.

Let the ellipse be taken, whose axes are 25 and 19 ; 23 and 17, in the last two examples ;

and we have $\left(\frac{25}{2} \times \frac{19}{2} - \frac{23}{2} \times \frac{17}{2}\right) \times \pi = (12{,}5 \times 9{,}5 - 11{,}5 \times 8{,}5)\,\pi = (118{,}75 - 97{,}75)\,\pi = 21\,\pi = 65{,}97339 =$ he area of the ring F*f*, G*g*, H*h*, K*k*.



Let the results be taken from the two examples referred to, and the area of the ring will be found to agree with this, viz., the area of the outer ellipse is there found - - =373,06381
The area of the inner one - - - =307,09042

The difference is =65,97339
the same as found above

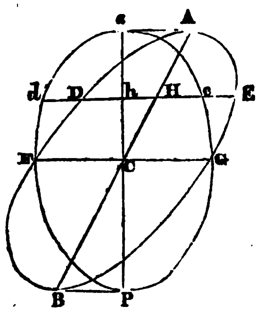
CASE II.

*When any two conjugate diameters are given.*

Rule. Multiply continually together any two semi conjugate diameters, the sine of their included angle, and $\pi$. (Prop. IV, Cor. 5, B. I.)

*Ex.* The two conjugate diameters AB, FG, of the ellipse ADFBGEA being 32 and 28, and their included angle 77° $34\frac{1}{4}'$ ; required its area.

The sine of 77° $34'\frac{1}{4}$ is ,9765625 ; therefore, 9765625 $\times$ 16 $\times$ 14 $\times$ 3,14159 = 687.225 = the area.



PROBLEM VII.

*To find the area of the segment of an ellipse, cut off by an ordinate to any diameter.*

CASE I.

*When the ordinate is perpendicular to either of the principal axes.*

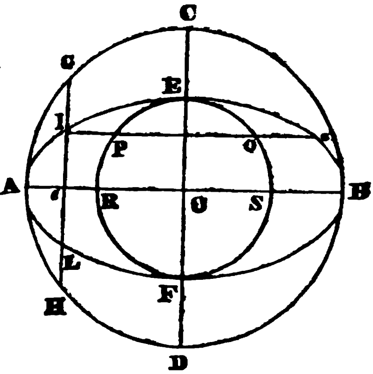
Rule. Find the corresponding segment of a circle of the same height, described on the same axis, to which the cutting line or base of the segment is an ordinate.

Then, as this axis is to its conjugate, so is the circular segment to the elliptical segment.

Or find the area of a circular segment, whose versed sine or height is equal to the quotient of the height of the elliptic segment divided by its axis. Then multiply continually together, this segment and the two axes of the ellipse, for the area of the segment required.

*Ex.* What is the area of an elliptic segment ILA, cut off by the line IL, parallel to, and at the distance of $7\frac{1}{2}$ from the minor axis EF, the axes being 35 and 25 ?

$17\frac{1}{2} - 7\frac{1}{2} = 10$ the height A*e* of the segment.



Then $2\sqrt{(Ae \times Be)}$ = GH the corresponding ordinate or chord to a segment of the circumscribing circle $= 2\sqrt{(10 \times 25)}$ $= 15{,}8113883 \times 2 = 31{,}6227766$.

Let the semi chord G*e* be divided by the semi diameter, and we shall have the corresponding sine of the arc GA of a circle, whose radius is $1 = 15{,}8113883 \div 17{,}5 = .903508$, corresponding to which, is the arc of 64° $37\frac{1}{2}'$ ; hence the arc GAH = 64° $37\frac{1}{2}' \times 2 =$ 129° 15′.

Then AB × $\pi = 35 \times 3{,}14159$ = the circumference of the circle ACBD = 109, 95565.

And 360° : 129° 15′ : : 109,95565 : 39,4771 = the arc GAH of the segment; and by Problem I, Case II,

$$(39{,}4771 - 31{,}6227)\ \tfrac{1}{2}r + 31{,}6227 \times 10 \div 2 =$$
$$7{,}8544 \times 8{,}75 + 31{,}6227 \times 5 = 68{,}726 + 158{,}1135 =$$
$$226{,}8395 = \text{the area of the circular segment GHA.}$$

Then 35 : 25, or 7 : 5 : : 226,8395 : 162, 171 = the area of the elliptic segment ILA.

*Scholium* 1. If the area of the segment ILFBE had been required, the circular arc GCBDH should have been taken instead of the arc GAH.

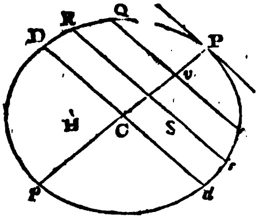
2. If the segment I*o*E, whose base is parallel to the major axis, is required, it may in like manner be found from its relation to the segment POE of the inscribed circle.

### CASE II.

*When the base of the elliptic segment is oblique to the axes.*

Rule. Divide the abscissa P$v$ by its diameter P$p$, and find a circular segment whose versed sine or height is the quotient. Then multiply continually together the area thus found, and the two axes, for the elliptic segment. Or multiply continually together the circular segment, the diameter P$p$, to which the base of the segment is a double ordinate, its conjugate diameter D$d$, and the sine of their included angle, for the area of the elliptic segment.

*Ex.* The principal axes of an ellipse being 35 and 25, it is required to find the area of a segment Q$q$P, whose base Q$q$ is an ordinate to the diameter P$p$, whose length is 33, it being divided by the ordinate into the two abscissæ P$v$ = 7, and $pv$ = 26.



FA ÷ AB = 7 ÷ 33 = $.2121\frac{7}{33}$ = the versed sine or height of the segment.

The area of a segment corresponding to this height in a circle, whose diameter is 1, is .12162866.

Hence, .12162869 × 25 × 35 = 106.4251 = the segment Q$q$P.

### PROBLEM VIII.

*To find the circumference of an ellipse.*

First, find the area of an elliptic ring included between an interior and exterior concentric ellipse, whose axes are severally the axes of the given ellipse + $n$, and the same axes − $n$; then divide this area by the average distance between the exterior and interior curves, and the quotient will be the circumference of the given ellipse. (Prop. VIII, B. IV.)

*Ex.* Let it be required to find the circumference AEBDA of an ellipse, whose axes AB, ED are 24 and 18. (See diagram to Scholium, Prob. VI.)

Let us assume two other exterior and interior concentric ellipses, whose axes FH, GK are = AB + $n$, and ED + $n$; and $fh$, $gk$, are = AB − $n$ and ED − $n$; and if $n = 2$, we shall have the area of the ring, as found in examples under

Scholium Prob. VI=65,97339. Let this be divided by the average distance as found in Proposition VII, B. IV=$\frac{9899+1}{2}$ and we have 66,3115 for the elliptical circumference AEBDA.

*Scholium* 1. If great accuracy is not required, the following approximating rules from Hutton's Mensuration may be used.

Rule 1. Multiply the sum of the semi axes by $\pi$, or 3,1416, and the product will be the circumference *nearly*.

*Ex.* Required the circumference of an ellipse, whose axes are 24 and 18.
$(12 + 9) + 314159 = 21 \times 3,14159 = 65,9735$ equal the circumference *nearly*.

Or Rule 2. Multiply the square root of half the sum of the squares of the two axes by $\pi$, and the product will be nearly = the circumference.

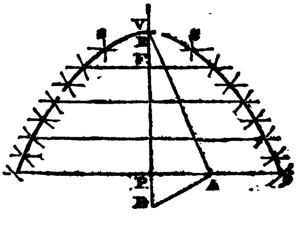
*Ex.* Taking the same example as before, we have $\sqrt{\frac{24^2 + 17^2}{2}} \times 3,14159 = 66,6433 =$ the circumference *nearly*.

It will be observed by comparing the last two results, that the former one is nearly as much in defect, as the latter is in excess; hence, if we take half the sum of the two we shall have the circumference of the ellipse more accurately.

Thus + $\left.\begin{matrix} 65,9736 \\ 66,6433 \end{matrix}\right\} \div 2 = 66,3084$, which is very near the truth.

### PROBLEM IX.

*To construct a parabola; having given any ordinate* PQ *to the axis, and abscissa* VP.



First, find the focus F thus; bisect PQ in A; draw AV, and AB perpendicular to it; take VF = PB, and F will be the focus.

*Arithmetically.* Divide the square of the ordinate by four times the abscissa, and the quotient will be focal distance VF.

Then, in the axis, produced without the vertex V, take VC = VF; draw several double ordinates SRS; then with the radii CR, and the centre F, describe arcs cutting the corresponding ordinates in the points S.

Draw the curve through all the points of intersection, and it will be the parabola required.

### PROBLEM X.

*Of any abscissa* X, *its ordinate* y, *and latus rectum, or parameter* p; *having two given, to find the third.*

#### CASE I.

*To find the latus rectum.*

Divide the square of the ordinate by its abscissa, and the quotient will be the latus rectum.

Or, take a third proportional to the abscissa and ordinate, for the latus rectum.

That is, $p = y^2 \div x$.

#### EXAMPLE.

If the abscissa be 9, and its ordinate 6

Then $6 \times 6 \div 9 = 36 \div 9 = 4$ = the latus rectum.

#### CASE II.

*To find the abscissa.*

Divide the square of the ordinate by the latus rectum, and the quotient will be the abscissa.

That is, $x = y^2 \div p$.

#### EXAMPLE.

If the ordinate be 6, and the latus rectum 4.

Then $6 \times 6 \div 4 = 36 \div 4 = 9$ = the abscissa.

#### CASE III.

*To find the ordinate.*

Multiply the latus rectum by the abscissa, and the square root of the product will be the ordinate.

That is $y = \sqrt{px}$.

### EXAMPLE.

The absciss being 9, and the latus rectum 4.
Then $\sqrt{(9 \times 4)} = \sqrt{36} = 6 =$ the ordinate.

### PROBLEM XI.

*Of any two abscisæ* A, B, *taken upon the same diameter, and their two ordinates* a, b ; *having any three given, to find the fourth.*

The abscissæ are to one another as the squares of their ordinates. That is, as any one abscissa is to the square of its ordinate, so is any other abscissa to the square of its ordinate ; and conversely. Or, as the root of one abscissa is to its ordinate, so is the root of any other abscissa, to its ordinate.

$$\text{Hence}\begin{cases} \sqrt{A} : \sqrt{B} :: a : a\sqrt{\frac{B}{A}} = \frac{a\sqrt{AB}}{A} = b \\ \sqrt{B} : \sqrt{A} :: b : b\sqrt{\frac{A}{B}} = \frac{b\sqrt{AB}}{B} = a \end{cases}$$

$$\text{And}\begin{cases} a^2 : b^2 :: A : \frac{Ab^2}{a^2} = B \\ b^2 : a^2 :: B : \frac{Ba^2}{b^2} = A \end{cases}$$

*Ex.* 1. If an abscissa = 9 correspond to an ordinate = 6, required the ordinate whose abscissa is 16.
Here $\sqrt{9} : \sqrt{16} :: 6 : 6 \times 4 \div 3 =$ the ordinate.

*Ex.* 2. Required the abscissa corresponding to the ordinate 6, the ordinate belonging to the abscissa 16 being S.
Here $8^2 : 6^2 :: 16 : 9 =$ the abscissa.

### PROBLEM XII.

*To find approximately the length of any arc of a parabola, cut off by an ordinate to the axis.*

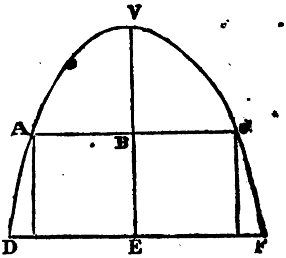
*When the abscissa and ordinate are given.*

Rule. To the square of the ordinate add four thirds of the square of the abscissa, and twice the square root of this sum will be the length of the curve, *nearly*.*

---

* See Hutton's Mensuration.

*Ex.* The abscissa VH $= 2$, and the ordinate AB $= 6$, required the length of the curve EVE.



Here $2\sqrt{(AB^2 + \frac{4}{3}VB^2)} = 2\sqrt{[6^2 + (2^2 + \frac{4}{3})]} = 2\sqrt{\frac{124}{3}} = \frac{4}{2}\sqrt{31 \times 3} = \frac{4}{3}\sqrt{93} = 9.6436508 \times \frac{4}{3} = 12.8582$, the length of the curve AVC, *nearly.*

## *Examples for Practice.*

*Ex.* 1. What is the length of the parabolic curve AVC, whose abscissa VB $= 2$, and the ordinate AB $= 8$?

Ans. 17.4356.

*Ex.* 2. Required the length of the parabolic curve DAVCF, when the abscissa VE $= 16$, and the ordinate DE $= 12$.

Ans. 42.142615.

*Ex.* 3. Required the length of the parabolic curve DAVCF, when the abscissa VE $= 8$, and the ordinate DE $= 16$.

Ans. 36.951.

### PROBLEM XIII.

### *To find the area of a parabola, when the base and height are given.*

Rule. Multiply the base by the height, and two-thirds of the product will be the area.

*Ex.* Required the area of the parabola AVCA, the abscissa VB $= 2$, and the base, or ordinate, AC $= 12$.

Here $\frac{2}{3}(AC \times VB) = \frac{2}{3}(12 \times 2) = 16$, the area of the parabola AVCA required.

## *Examples for Practice.*

*Ex.* 1. What is the area of a parabola DAVCFD, whose abscissa VE $= 10$, and the double ordinate DF $= 16$.

Ans. $106\frac{2}{3}$.

*Ex.* 2. Required the area of a parabola DAVCFD, whose base or ordinate DF = 15, and the abscissa VE = 22?
Ans. 220.

*Ex.* 3. What is the area of a parabola AVCA, the base or ordinate AC = 20, and the height or abscissa VB = 6.
Ans. 80.

### PROBLEM XIV.

*To find the area of parabolic frustum, or zone of a parabola, or of the space included between two parallel ordinates.*

*The two ordinates, and their distance being given.*

RULE. To the sum of the squares of the two ordinates, add their product, divide the result by the sum of the two ordinates, the quotient multiplied by two-thirds of the altitude of the frustum, will give the area.

*Ex.* Required the area of the parabolic frustum ACFDA, the two parallel ordinates DF, and AC = 10, and 6, and the distance BE = 4.

$$\left(\frac{(DF^2 + AC^2) + (DF \times AC)}{DF + AC}\right) \times \tfrac{2}{3}BE$$

$$= \left(\frac{10^2 + 6^2) + (10 \times 6)}{10 + 6}\right) \times 4 \times \frac{2}{3} = \frac{136 + 60}{16} \times \frac{8}{3}$$

$$= \frac{196}{8 \times 2} \times \frac{8}{3} = \frac{98}{3} = 32\tfrac{2}{3},$$ the area of the frustum ACFDA.

*Examples for Practice.*

*Ex.* 1. What is the area of the parabolic frustum ACFDA, whose two ordinates DF and AC = 10 and 6, and the distance BE = 3? Ans. $24\frac{1}{2}$.

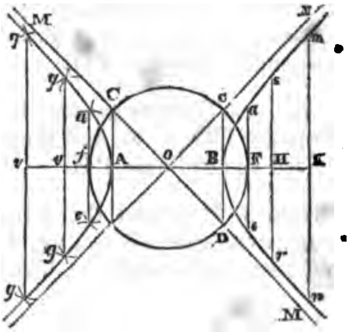
*Ex.* 2. The greater end of the frustum DF = 30, the less end AC = 20, and their distance BE = 15; required the area. Ans. 380.

*Ex.* 3. The greater end of the frustum DF = 20, the less end AC = 10, and their distance BE = 12. Ans. $186\frac{2}{3}$.

### OF THE HYPERBOLA.

### PROBLEM XV.

*To construct or describe a hyperbola.*



Let *o* be the centre of the hyperbola, or the middle of the transverse AB; and BC perpendicular to AB, and equal to half the conjugate.

With the centre *o*, and radius C*o*, describe the circle, meeting AB produced in F and *f*, which are the two foci of the hyperbola.

Then assuming several points *vv*, &c., in the transverse produced, with the radii A*v*, B*v*, and centres *f*, F, describe arcs intersecting in the several points *g*, *g*, &c., through which points draw the hyperbolic curve.

If straight lines *o*M, *o*N, be drawn from the point *o*, the middle of the transverse diameter, through C, and D, the extremities of the conjugate, they will be the *asymptotes* of the hyperbola, the property of which is to approach continually to the curve, but not to meet it, until they be infinitely produced.

### PROBLEM. XVI.

*In an hyperbola to find the transverse axis or conjugate axis, or ordinate or abscissa.*

### CASE I.

*To find the ordinate.*

*When the transverse axis, conjugate axis, and the abscissa are given.*

Rule. As the transverse axis is to the conjugate axis, so is the square root of the product of the two abscissæ to the ordinate.

Note. In the hyperbola, the less abscissa added to the axis' gives the greater abscissa.

*Ex.* If the transverse axis AB = 24, the conjugate axis CD = 21, and the less abscissa BH = 8, what is the length of the corresponding PH.

Here AB : CD : : $\surd$[(AB + BH) × BH] : : PH, viz. 24 : 21 : : $\surd$[(24 + 8) × 8] : 14, the length of the corresponding ordinate PH, required.

*Examples for Practice.*

*Ex.* 1. The transverse axis AB = 60, the conjugate axis CD = 36, and the less abscissa BH = 20, required the corresponding ordinate PH. Ans. 24.

*Ex.* 2. The transverse diameter AB = 50, the conjugate diameter CD = 40, and the greater abscissa AH = 64; required the ordinate PH. Ans. $\frac{32}{5}\surd 14$.

*Ex.* 3. Required the length of the ordinate MK, whose transverse axis AB = 609, the conjugate axis CD = 588, and the less abscissa BK = 116. Ans. 280.

### CASE II.

### *To find the two abscissæ.*

### *When the transverse axis, the conjugate axis, and the ordinate, are given.*

Rule. As the conjugate axis is to the transverse axis, so is the square root of the sum of the squares of the ordinate and semi-conjugate to the distance between the ordinate and centre, or half the sum of the abscissæ. Then will the sum of this distance and the semi-transverse be the greater abscissa, and their difference the less.

*Ex.* The transverse axis AB = 24, the conjugate axis CD = 21, and the ordinate PH = 14; required the two abscissæ AH, and BH.

Here CD : AB : : $\surd[PH^2 + (\frac{1}{2} CD)^2]$ : HO, viz.
21 : 24 : : $\surd(14^2 + 10.5^2)$ : 20.
Then the two abscissas AH and BH = HO ± $\frac{1}{2}$AB = 20 ± 12 = 32 and 8.

*Examples for Practice.*

*Ex.* 1. The transverse axis AB = 60, the conjugate axis CD = 36, required the two abscissæ AH, and BH, corresponding to the ordinate PH = 24.
Ans. AH = 80, and BH = 20.

*Ex.* 2. The transverse axis AB = 120, the conjugate axis CD = 72, and the ordinate MK = 48, required the two abscissæ AK and BK. Ans. AK = 160, and BK = 40.

### PROBLEM XVIII.

## *To find the length of any arc of an hyperbola approximately beginning at the vertex.*

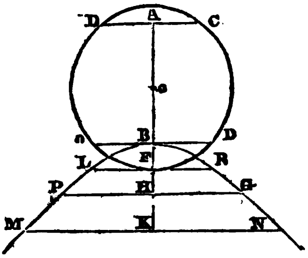
### *When the transverse and conjugate axis, the ordinate, and abscissa, are given.*

RULE. *First.* Add 21 times the the square of the conjugate to 19 times the square of the transverse, and multiply this sum by the abscissa; to this product add 15 times the transverse, multiplied by the square of the conjugate, and call this quantity the dividend.

*Secondly.* Add 21 times the square of the conjugate to 9 times the square of the transverse, and multiply this sum by the abscissa; to this product add 15 times the transverse, multiplied by the square of the conjugate, and call this quantity the divisor.

*Thirdly.* Then divide the dividend by the divisor, and multiply the quotient by the ordinate for half the length of the curve, or multiply the quotient by twice the ordinate for the length of the whole curve, *nearly.**

*Ex.* 1. Required the length of the hyperbolic curve PLBRG to the abscissa BH=2.1637, and the ordinate PH=10; the two axes AB and CD=80 and 60.



Here $\frac{(21CD^2+19AB^2)\times BH+(15AB\times CD^2)}{(21CD^2+\ \ 9AB^2)\times BH+(15AB\times CD^2)}\times 2PH$

$=\frac{[(21\times 60^2)+(19\times 80^2)]\times 2.1637+(15\times 80\times 60^2)}{[(21\times 60^2)+(\ \ 9\times 80^2)]\times 2.1637+(15\times 80\times 60^2)}\times 2\times 10$

$=\frac{(75600+121600)\times 2.1637+4320000}{(75600+\ \ 57600)\times 2.1637+4320000}\times 20$

$=\frac{426681.64+4320000}{288204.84+4320000}\times 20=\frac{4746681.64}{4608204.84}\times 20=1.03005\times 20$

$=20.601$, the length of the whole curve PLBRG required.

---

* Hutton's Mensuration.

*Ex.* 2 Required the length of the hyperbolic arc PLBRG, the abscissa BH = 20, the ordinate PH = 24, and the two axes AB and CD =60 and 36.* Ans. 62.652.

### PROBLEM XVIII.

### *To find the area of an hyperbola.*

*When the transverse axis, conjugate axis, and the abscissa, are given.*

RULE. To the product of the transverse axis and abscissa, add $\frac{5}{7}$ of the square of the abscissa, and multiply the square root of the sum by 21; to this product add 4 times the square root of the product of the transverse axis and abscissa; then multiply this sum by 4 times the product of the conjugate axis and abscissa, and divide this last product by 75 times the transverse axis, the quotient will give the area of the hyperbola, *nearly.* *

*Ex.* 1. Required the area of the hyperbola PBGP, whose abscissa BH = 10, the transverse and conjugate axis AB and CD = 30 and 18.

Here $\frac{\{21\sqrt{[(AB\times BH)+\frac{5}{7}BH^2]}+4\sqrt{(AB\times RH)}\}\times 4CD\times BH}{75\ AB}$

$=\frac{\{21\sqrt{[(30\times 10)+(\frac{5}{7}\times 10^2)]}+4\sqrt{(30\times 10)}\}\times 4\times 18\times 10}{75\times 30}$

$=\frac{(21\sqrt{371\frac{3}{7}}+4\sqrt{300})\times 8}{25}=\frac{(21\sqrt{\frac{2600}{7}}+40\sqrt{3})\times 8}{25}=$

$\frac{8}{25}\left(\frac{21}{7}\times\frac{10}{1}\sqrt{(26\times 7)}+40\sqrt{3}\right)=\frac{8}{25}\times(30\sqrt{182}+40\sqrt{3})=$

$\frac{8\times 10}{25}\times(3\sqrt{182}+4\sqrt{3})=\frac{16}{5}\times(40.4722128+6.9282032)=$

151.681328, the area of the hyperbola PBGP required.

*Ex.* 2. What is the area of the hyperbola MBNM, the abscissa BK = 25, the transverse and conjugate axis AB and CD = 50 and 30? Ans. 805.090844.

### PROBLEM XIX.

### *To find the area of any mixtilineal figure by means of equidistant ordinates, terminated by a curve on one side, and a right line as a base on the other.*

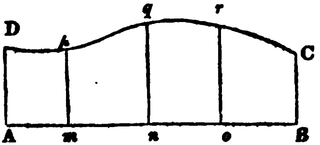
RULE. To the sum of the first and last ordinates add 4 times the sum of all the even ordinates, and twice the sum of all the odd ordinates, rejecting the first and last; and $\frac{1}{3}$ of this result, multiplied by the common distance of the ordinates, will give the area, *very nearly*. Prop. XV, B. IV.

---

* Hutton's Mensuration.

*Scholium.* This rule is absolutely true for a parabola, if the ordinates are parallel to its axis. And if the distances between the ordinates is small, it is approximately true for any other curve.

*Ex.* 1. Required the area of an irregular figure, bounded on one side by a curve line at five equidistant ordinates, the breadths being AD=8.2, $mp$ = 7.4 $nq$=9.2, $or$ = 10.2, BC = 8.6; the length of the base AB = 39, and the common distance of the ordinates A$m$, $mo$, $no$, $o$B, each = 9.75.



Here $\frac{1}{3}[(AD+BC)+4(mp+or)+2nq] \times 9.75 = \frac{1}{3}[(8.2+8.6)+4(7.4+10.2)+(9.2\times2)] \times 9.75 = \frac{1}{3}(16.8+70.4+18.4)\times9.75=(105.6\div3)\times9.75=343.2$, the area of the space ADCBA required.

*Ex.* 2. Required the area of an irregular space AD$qn$A bounded on one side by a curve line, and divided by three equidistant ordinates perpendicular to the base A$n$, the ordinates being AD=8, $mp$=6, and $nq$=10, the length of the base A$n$=14, and the common distance A$m$, $mn$, each equal 7.

Ans. 98.

*Ex.* 3. The abscissa of a parabola being 2, and the base or ordinate 12, required the area of the parabola.

Here, by taking three ordinates, of which the first and last are each nothing, the middle one being the abscissa=2, and the common distance=6; hence the area of the parabola=16 =Ans.

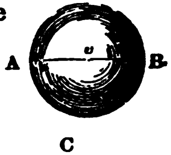
## MENSURATION OF SOLIDS.

### PROBLEM I.

### *To find the solidity of a sphere, spheroid, a spherical or an elliptical revoloid.*

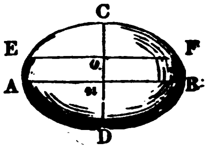
Rule.—Multiply a central conjugate section by the vertical axis, and take two-thirds of the product for the solidity.

Ex. 1. What is the solidity of a sphere, whose diameter is 10 feet?



$31.4159\times\frac{1}{2}=78.5397$ = to a central section; hence, $78.5397\times10\times\frac{2}{3}=523.931$ cubic feet the solidity.

Ex. 2. What is the solidity of a prolate spheroid, ACBD, whose vertical or fixed axis, AB, is 10, and its revolving axis, CD, is 5?

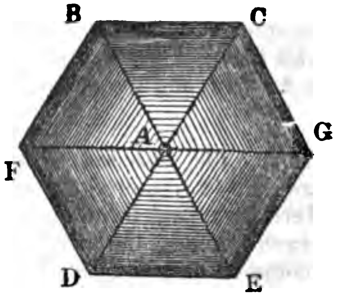


$3.14159\times5\times1\frac{1}{4}=19,63494$ = a central conjugate section.

Hence, $19.63494\times10\times\frac{2}{3}=130.9329$, the solidity required.

Ex 3. Required the solidity of a spherical hexagonal revoloid, BCGEDF, whose vertical axis is ten feet.

By referring to the table of Polygons. (*Mensuration El. Geom.*) we find the area of a hexagon, circumscribed about a circle whose diameter is 10, is 17,320508; hence, $17,320508\times10\times\frac{2}{3}=115,47005$, the solidity required.



Ex. 4. Required the solidity of an elliptical rectangular revoloid, whose vertical axis is 48 inches, and conjugate axis is 36 inches.

$$36\times36\times48\times\frac{2}{3}=41472 \text{ cubic inches.}$$

Ex. 5. What is the solidity of an elliptical rectangular revoloid, whose vertical axis is 36 inches, and whose conjugate is 48 inches? Ans. 55296 cubic inches.

Ex. 6. What is the solidity of an oblate spheroid, whose revolving axis = 48, and whose conjugate or fixed axis = 36 inches? Ans. 43429.4784 cubic inches.

Ex. 7. Required to find the solid content of the earth, supposing its circumference to be 25000 miles.

Ans. 263859375000 cubic miles.

Ex. 8. What is the solid content of a sphere, whose diameter AB = 25 feet?

Ans. 8181.25 cubic feet.

Ex. 9. Required the solidity of a sphere, whose circumference is 18.6 feet. Ans. 108.665413272 cubic feet.

### PROBLEM. II.

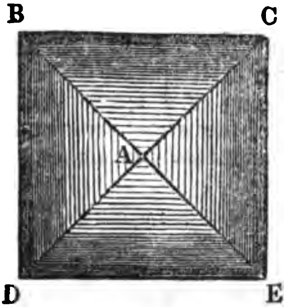
*To find the surface of a sphere or of a spherical revoloid.*

Rule.—Multiply the perimeter of its central conjugate section by the vertical axis, and the product is the whole surface.

Ex. 1. What is the surface of a sphere whose diameter is 10? Ans. 31.4159×10=314.159, the surface required.

Ex. 2. Required the surface of a rectangular spherical revoloid, BCED, whose vertical axis is 10 feet.

10×4×10=400 square feet, the surface required.



Ex. 3. Required the surface of a ball, whose diameter AB = 1 inch. Ans. 3.1416 square inches.

Ex. 4. How many square inches will cover a globe of 12 inches in diameter? Ans. 452.3904 square inches.

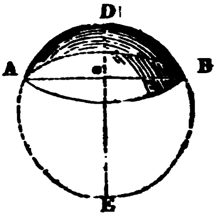
Ex. 5. Required the superficies of the terraqueous globe, supposing the diameter AB = 7958 miles. And if only one-fourth part of its surface be dry land, and two acres sufficient to produce food for one person; how many persons can live on the earth at one time.

Ans. { 198956786.5824 sq. miles, the surface of the globe
49739196.6456 sq. miles, dry land.
15916542927 persons can live on the earth. }

### PROBLEM III.

*To find the solidity of any segment or zone of a sphere.*

Rule.—To half the sum of the areas of the two bases multiplied by the altitude, add the solidity of a sphere whose diameter is equal to the altitude of the segment or zone.

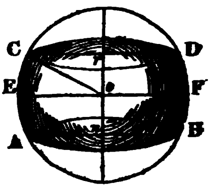


Ex. 1. What is the solidity of a spherical segment ABD, whose base is 10 and whose height *o*D is 2?

$$10\times2=20.$$

and $\frac{1}{6}\pi D^3=\frac{1}{6}\times3.14159\times2\times2\times2=4.18876.$

hence $20+4.18876=24,18876$ = the solidity of the segment.



Ex. 2. What is the solid content of a zone EFDCE, whose height *or* = 30 inches, the greater diameter EF = 60 inches, and the less diameter AB = 40 inches? Ans. 75398.4 cubic inches.

Ex. 4. Required the solidity of the middle zone of a sphere ABDCA, the diameter of the whole sphere EF = 80 inches, the height *nr*=64 inches.

Ans. 233070.5408 cubic inches.

### PROBLEM IV.

*To find the convex surface of any segment or zone of a sphere, or spherical revoloid.*

Rule.—Multiply the perimeter of a middle section of the whole sphere or revoloid, perpendicular to the vertical axis, by the height of the segment or zone.

*Scholium.* This is the same as the rule given in the Elements of Geometry for a spherical segment or zone, viz: its convex surface is there said to be equal to the height of the segment or zone, multiplied by the circumference of the sphere. The same rules as there given for segments and sectors of a sphere, will answer also for segments and sectors of right revoloids.

Ex. 1. What is the convex surface of a segment of a right revoloid, whose height is 2 feet, the perimeter of a central conjugate section of the whole revoloid being 40 feet?

$40\times2=80$, the surface required.

Ex. 2. Required the convex surface of a zone of a rectangular right revoloid whose height is 6 feet, the whole altitude of the revoloid being $10\frac{1}{2}$ feet. Ans. 252 feet.

Ex. 3. Required the convex surface of a segment of a hexagonal right revoloid, whose height is $5\frac{1}{2}$ feet, the axis of the revoloid being 10 feet. Ans. 190.5255825 square feet.

### PROBLEM V.

### *To find the solidity of a sector of a spherical or right revoloid.*

Rule.—Multiply its convex surface by one-third the semi-axis of the revoloid.

Ex. What is the solidity of a revoloidal sector, whose convex surface is 10 square feet, the axis of the revoloid being 10 feet?

$$10\times\frac{1}{3}\times5=16\frac{2}{3}, \text{ the solidity.}$$

### PROBLEM VI.

### *To find the solidity of a segment or zone of a spherical revoloid.*

Rule.—Find the solidity of the revoloidal sector having the same convex surface; find also, the solidity of the pyramid having the same base as the segment, and whose vertice is in the centre of the revoloid: subtract the solidity of the pyramid from that of the sector, which will give the solidity of the segment, if the segment is less than a semi-revoloid; and add the solidity of the pyramid to that of the sector, if the segment be greater than a semi-revoloid.

Ex. 1. What is the solidity of a segment of a rectangular revoloid whose convex surface is 40 square feet, the axis of the revoloid being 10 feet?

Here, the sector will be found $= 40\times\frac{1}{3}=66\frac{2}{3}$ solid feet, and the height of the segment will be found $= 1$ foot; hence, $(5^2-4^2)=36 =$ the base of the segment; and $36\times4\times\frac{1}{3}= 48$ $=$ the solidity of the pyramid.

Therefore, $66\frac{2}{3}-48=18\frac{2}{3}$ cubic feet the solidity required.

Ex. 2. Required the solidity of the segment of an octagonal revoloid, whose convex surface is 100 feet, the axis of the revoloid being 10 feet.

### PROBLEM VII.

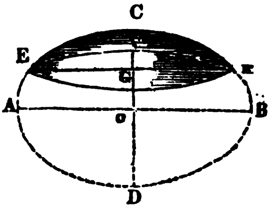
*To find the solidity of a segment of a spheroid, made by a plane parallel to either axis.*

*Scholium.* Since the segment of a spheroid is the segment of a sphere, expanded or contracted in the ratio of the major and minor axes ; hence, we have the following.

Rule. Find the solidity of a corresponding segment of the same altitude, from a sphere described on the same axis as that of the segment ; then. as this axis is to its conjugate, so is the spherical segment to the spheroidal segment, if the segments base is parallel to the fixed axis. Or as the square of this axis is to the square of its conjugate, so is the spherical segment to the spheroidal segment, if the base is circular or parallel to the revolving axis.

Note. The same will also apply to the segment of an elliptical revoloid compared with a corresponding segment of a spherical or right revoloid.

*Ex.* 1. In the prolate spheroid ACBD the fixed axis AB=50. the revolving axis CD=30, required the solidity of the segment EFCE, its height EG=6, the base being parallel to the fixed axis AB.



The solidity of a spherical segment, whose height is CG, the diameter being CD=1470.2688.

Hence, by the rule,

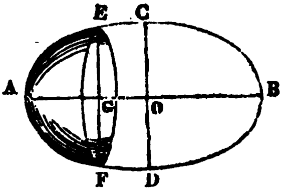
CD : AB : : 1470.2688 : EFCE;

or 30 : 50 : : 1470.2688 : 2450.448 the solidity of the segment EFCE.

*Ex.* 2. In an *oblate* spheroid, whose revolving axis AB=50, the fixed axis CD=30 ; required the solid content of the segment EFCE, whose height = 5, its base being perpendicular to the revolving axis. Ans. 1099.56.

*Ex.* 3. Required the solidity of the segment EFCE of the *prolate* spheroid ACBDA, the fixed axis DB=48, the revolving axis CD=38. and the height of the segment CG=16, the base being perpendicular to the revolving axis.

Ans. 13883.8878.



*Ex.* 4. Required the solidity of the segment EAFE of a *prolate* spheroid, the height AG=5 inches, its base being parallel to the revolving axis, which is 30 inches, its fixed axis being 50 inches.

The solidity of a spherical segment, whose altitude is AG of a sphere, whose axis is AB, is 1832.6 cubic inches.

$AB^2 : CD^2 :: 1832.6 : EAFE$;

or $2500 : 900 :: 1832.6 : 659.736$ the solidity of the segment EAFE.

*Ex.* 5. Required the solid content of the segment of the *prolate* spheroid EAFE, its base being parallel to the revolving axis; the height AG=1, the fixed axis AB—10, and the revolving axis CD=6. Ans. 5.2778'.

*Ex.* 6. The fixed axis CD of an *oblate* spheroid being 30, the revolving axis AB=50, and the height of the segment=6, its base being parallel to the revolving axis; required the solidity of the spheroidal segment. Ans. 4084.08.

### PROBLEM VIII.

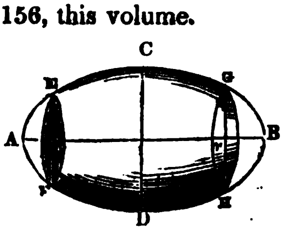
### *To find the solid content of the middle frustum of a spheroid.*

#### CASE I.

#### *When the ends are circular, or parallel to the revolving axis.*

RULE. To twice the square of the middle diameter, add the square of the diameter of one end; multiply this sum by the length of the frustum, and the product again by .2618 (*which is one-third of* .7854,) for the solidity of the middle frustum.

*Scholium.* This, and the following rule is derived from the principles contained in scholium page 156, this volume.



*Ex.* 1. Required the solidity of the middle frustum FCGHDFE of a *prolate* spheroid, the middle diameter CD=30, the diameter of each circular end EF or GH=18, and the length $cr=40$.

Here $(2CD^2+GH^2) \times cr \times .2618 = [(30^2 \times 2)+18^2] \times 40 \times .2618 = (1800+324) \times 40 \times .2618 = 22242.528$ the solidity of the middle frustum ECGHDFE required.

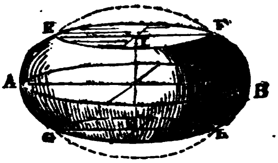
*Ex.* 2. What is the solidity of the middle frustum ECGHDFE of an *oblate* spheroid, having the less diameters of the circular ends EF and GH, each equal 40 ; the middle or greater diameter CD=50, and the length $cr=18$ ?

Ans. 31101.84.

### CASE II.

*When the ends are elliptical, or perpendicular to the revolving axis.*

Rule. To twice the product of the major and minor axes of the middle section, add the product of the major and minor axis of one end ; then multiply this sum by the length of the frustum, and the product again by .2618, for the solid content of the middle frustum.



*Ex.* 1. In the middle frustum EFHGE of an *oblate* spheroid, the major and minor axes of the middle or greater elliptic section AB are 50 and 30, and the major and minor axis at one end EF are 40 and 24, the height IK=9 ; required the solid content of the middle frustum.

Here $(50\times30\times2)+(40\times24)\times9\times.2618=(300+960)\times2.3562=9330.552$, the solidity of the frustum EFHGE required.

*Ex.* 2. In the middle frustum EFHGE of an *oblate* spheroid, the two axes of the middle ellipse are 50 and 30, and those of each end are 30 and 18, the height of the frustum IK = 40 ; required the solid content of the frustum EFHGE.

Ans. 37070.88.

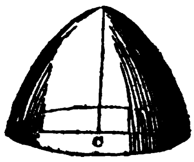
### PROBLEM IX.

*To find the solidity of a paraboloid or a vertical parabolic revoloid.*

Rule. Multiply the area of the base by half the height.

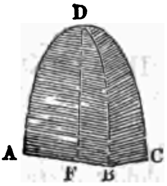
*Ex.* 1. If the diameter of the base of a paraboloid be 12 feet, and height 22 feet, what is the solidity ?

Ans. 1243.



*Ex.* 2. If the sides AB, CB of the rectangular base of a parabolic semi-revoloid or pyramoid ABCD are each = 10 inches, and the altitude FD=18 inches, required its solidity.

Ans. 1800 cubic inches.



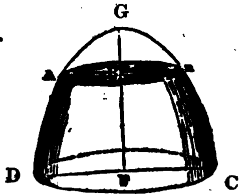
#### PROBLEM X.

*To find the solidity of the frustum of a parabolic conoid, or or paraboloid, or of a vertical parabolic revoloid.*

Rule. Multiply the sum of the areas of the two ends by half their distance.

*Ex.* 1. What is the solid content of the frustum of a paraboloid, the greater diameter DC = 30, the least diameter AB=24, and the altitude EF=9?

Ans. 5216.6268.



*Ex.* 2. What is the content in *wine gallons* of a cask in the form of two equal frustums of a paraboloid; the length=2EF =40 inches, the bung diameter DC=32 inches, and the head diameter AB=24 inches; the gallon containing 231 cubic inches?

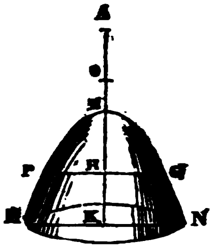
Ans. 108.768 gals.

#### PROBLEM XI.

*To find the solidity of a hyperbolic conoid, or otherwise called a hyperboloid.*

Rule. To the square of the radius of the base, add the square of the diameter in the middle, between the base and top; multiply this sum by the altitude, and the product again by .5236, for the solidity of the hyperboloid. (Art. 22, Chap. II, B. V.)

*Ex.* 1. What is the solidity of an hyperboloid MBNM, whose altitude KB=10, the radius of its base MK=12, and the middle diameter PG=$6\sqrt{7}$?



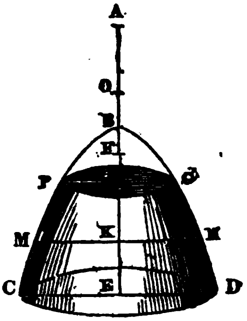
Here $(MK^2 + PG^2) \times KB \times .5236 = [12^2 + (6\sqrt{7})^2] \times 10 \times 5.236 = [144 + (6^2 \times 7)] \times .5236 = (144 + 252) \times 5.236 = 2073.456$, the solidity of the hyperboloid MBNM required.

*Ex.* 2. Required the solidity of the hyperboloid MBNM, whose altitude KB=50, the radius of its base MK=52, and the middle diameter PG=68. Ans. 191847.04.

###### PROBLEM, XII.

*To find the solidity of the frustum of a hyperbolic conoid, or hyperboloid.*

Rule. To four times the square of the middle diameter, add the sum of the squares of the greatest and least diameters; then multiply this sum by the altitude of the frustum, and that product again by .1309, (*it being the sixth part of* .7854,) for the solidity.



*Ex.* 1. Required the solidity of the frustum of the hyperbolic conoid PGDCP, the height EH=7, the greatest diameter CD=48, the middle diameter MN=38, and the least diameter PG=27.

Here $(4MN^2 + CD^2 + PG^2) \times EH \times .1309 = [(38^2 \times 4) + 48^2 + 27^2] \times .1309 = (5776 + 2304 + 729) \times 7 \times .1309 = 8809 \times 7 \times .1309 = 8071.6867$, the solid content of the frustum PGDCP required.

*Ex.* 2. Required the solidity of the frustum of a hyperbolic conoid PGDCP, whose greatest diameter CD = 10, the least diameter PG=6, the diameter MN=$8\frac{1}{2}$, and the altitude EH=12. Ans. 667.59.

*Ex.* 3. A cask, in the form of two equal frustums of a hyperbolic conoid, having its bung diameter CD=32 inches, its head diameter PG = 24 inches, and the diameter in the middle, between the bung and head MN=$\frac{8}{5}\sqrt{310}$, the length of the cask 2 EH = 40 inches; required the content in *wine gallons.*

Ans. 24998.7584 cubic inches=108.219, &c. gallons.

### PROBLEM XIII.

*To find, by a general rule, the solidity of any solid, frustum, or segment, produced by the revolution of any conic section, or of any revoloid circumscribing such solids.*

General Rule. To the sum of the ends, add four times a section equidistant therefrom, and multiply this sum by one-sixth of the length.

*Scholium.* This rule is true for any solid or segment, which is generated by any multiple or power of a series of numbers in arithmetical progression. (See Book V, Chap. 1 & 2.)

*Ex.* 1. What is the solidity of a sphere, whose diameter is 2? The area of its central section, or of its great circle is 3,14159.

Hence $3,14159\times4\times\frac{4}{6}=4,18878$ the solidity.

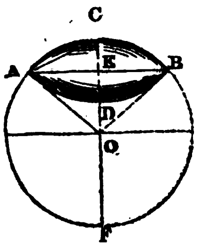
*Ex.* 2. What is the solidity of a zone of a spheroid, whose two bases are 10 and 5 square inches, and whose central section parallel to the bases is 9 square inches, the height of the zone being 18 inches? Ans. 153 cubic inches.

### PROBLEM XIV.

*To find the solidity of a circular spindle, produced by the revolution of a circular segment about its base or chord as an axis.*

Rule. From $\frac{1}{3}$ of the cube of half the axis, subtract the product of the central distance into half the revolving circular segment, and multiply the remainder by four times 3.14159.

If $a$=the area of the revolving circular segment,
$l$=half the length or axis of the spindle,
$c$=the distance of the axis from the centre of the circle to which the revolving segment belongs;
The solidity$=(\frac{1}{3}l^3-\frac{1}{2}ac)\times4\times3.14159$.



*Ex.* Let a circular spindle ACBD be produced by the revolution of the segment ABC, about AB. If the axis AB be 140, and CP half the middle diameter of the spindle be 38.4; what is the solidity?

| | |
|---|---|
| The area of the revolving segment is | 3791 |
| The central distance OE | 44.6 |
| The solidity of the spindle | 374402 |

### PROBLEM XV.

*To find the surface of a circular spindle.*

Rule. Subtract the product of the revolving arc, multiplied by the central distance, from the product of the length of the spindle into the radius, and multiply the remainder by twice 3.1416, and this product will be the surface.

*Ex.* Required the surface of a circular spindle ACBDA, whose length AB = 40, and middle diameter CD = 30, the length of the arc ACB being $53\frac{1}{3}$.

The height of the revolving segment=15, the radius of the circle=$20\frac{5}{6}$, the central distance OE=$5\frac{5}{6}$.

$(AB\times OC—ACB\times OE)\times 3,1416\times 2$
$=(40\times 20\frac{5}{6}—53\frac{1}{3}\times 5\frac{5}{6})\ 6.2832=3281.22\frac{2}{3}$, the surface of the spindle ACBDA required.

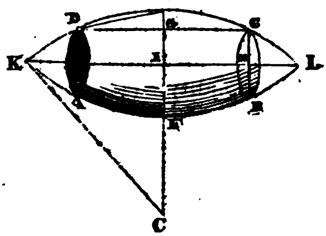
### PROBLEM XVI.

*To find the solidity of the middle frustum of a circular spindle.*

Rule. From the square of half the axis of the whole spindle subtract $\frac{1}{3}$ of the square of half the length of the frustum; multiply the remainder by this half length; from the product subtract the product of the revolving area into the central distance; and multiply the remainder by twice 3.14159.

If L=half the length or axis of the whole spindle,
$l$=half the length of the middle frustum,
$c$=the distance of the axis from the centre of the circle,
$a$=the area of the figure which, by revoling, produces the frustum;
The solidity$=((L^2—\frac{1}{3}l^2)\times l—ac)\times 2\times 3.14159$.

*Ex.* If the diameter of each end of a frustum of a circular spindle be 21.6, the middle diameter 60, and the length 70; what is the solidity?



| | |
|---|---|
| The length of the whole spindle is | 79.75 |
| The central distance | 11.5 |
| The revoling area | 1703.8 |
| The solidity | 136751.5 |

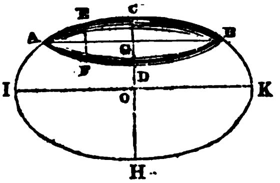
*Scholium.* The middle frustum of a circular spindle may be resolved into a cylinder, whose two bases are AD, CB, and a ring described by the revolution of the segment CDE about the axis KL; this ring may also be resolved into a cylindric segment, whose base is the segment DCE, and whose altitude is =the inner diameter of the ring, and a circular spindle formed by revolving the segment DCE about its chord DC. (Prop. XI, Cor. 3, B. III.) Hence its content may be calculated accordingly.

### PROBLEM XVII.

*To find the solidity of an elliptic spindle.*

Rule. First find the solidity of a circular spindle, generated from a segment, whose height CG, is the same as that of the elliptical segment, generating the elliptic spindle, the radius of the circle being CO.

Then, as the length of the circular spindle is to that of the elliptic spindle, so is the solidity of the circular spindle to that of the elliptic spindle.



*Ex.* If half the middle diameter CD of an elliptic spindle is 38.4, and its axis AB=200, its central distance OG being 44.6, what is its solidity ?

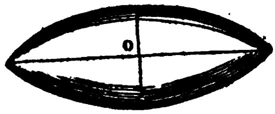
The solidity of a circular spindle having the same middle diameter, and the same central distance, we have found (Prob. XIV,)=374402, but its length is 140, therefore by the rule.

140 : 200 : 374402 : 534860 the solidity required.

### PROBLEM XVIII.

*To find the solidity of a parabolic spindle, produced by the revolution of a parabola about a double ordinate or base.*

Rule. Multiply the square of the middle diameter by $\frac{8}{15}$ of the axis, and the product by .7854.



*Ex.* If the axis of a parabolic spindle be 30, and the middle diameter 17, what is the solidity ?

Ans. 3631.7

### PROBLEM XIX.

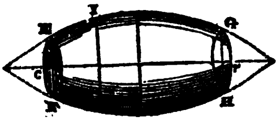
*To find the solidity of the middle frustum of a parabolic spindle.*

Rule. Add together the square of the end diameter, and twice the square of the middle diameter; from the sum subtract $\frac{2}{5}$ of the square of the difference of the diameters, and multiply the remainder by $\frac{1}{3}$ of the length, and the product by .7854.

If D and $d$=the two diameters, and $l$=the length;
The solidity$=(2D^2+d^2-\frac{2}{5}(D-d)^2)\times\frac{1}{3}l\times.7854$.

*Ex.* If the end diameters of a frustum of a parabolic spindle be each 12 inches, the middle diameter 16, and the length 30; what is the solidity?

Ans. 5102 inches.



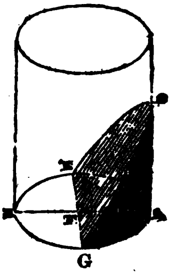
### PROBLEM XX.

*To find the convex surface of a cylindric ungula.*

Rule. From the product of the diameter and sine, subtract the product of the arc and cosine, and multiply the difference by the altitude divided by the versed sine.

Let $h$=the altitude AD,
$v$=the versed sine AF,
$d$=the diameter AB
$a$=the arc EAG,
$s$=the right sine FG,
$c$=the cosine of the half arc.

Then$\frac{ds-ac}{v}\times h$=the convex surface.



G

*Scholium* 1. When F is the centre of the base; then $v=s=d, c=o$; and then the rule becomes $dh$, viz., the surface is=the product of the diameter into the height.

2. When AF excedes $\frac{1}{2}$AB, then $ac$ must be added, and the expression becomes$\frac{ds+ac}{v}\times h$=the surface.

*Ex.* 1. What is the curve surface of an ungula EGDA, whose base is half the base of the cylinder and height, AD=10, the radius FG=10? Ans. 100.

*Ex.* 2. Given the diameter AB = 100 the height AD=140, and the versed sine AF=10, required the curve surface.

Ans. 5962,738.

*Scholium.* The same considerations will apply to cylindric ungulas, as for circular spindles, taking GE as the axis of the spindle, and AD the circumference of a middle section. (Prop. IV, Cor. 1, B. III.)

## PROBLEM XXI.

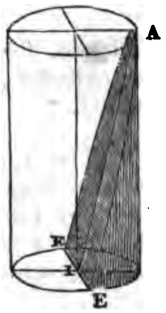
### *To find the solidity of a cylindric ungula.**

### CASE I.

*When the base of the ungula is = half the base of the cylinder*

RULE. Multiply the square of the radius of the base by the altitude of the ungula, and take $\frac{2}{3}$ the product for the solidity.

*Scholium.* This rule is absolute, without reference to the circle's quadrature, (Prop. VI, Cor. 5, B. III,) ($s=\frac{2}{3}r^2h$, Formula 6, Page 92.)



*Ex.* What is the solidity of the cylindric ungula AEFC, whose base EFC is half that of the cylinder, the diameter EF being 6, and the altitude CA of the ungula being 16, $3^2 \times 16 \times \frac{2}{3} = 96$ the solidity required.

### CASE II.

*When the base of the ungula is greater or less than half that of the cylinder.*

Subtract the product of the area of the base by the difference between the radius and the versed sine or height, from *one-twelfth* of the cube of the chord of the base, if the versed sine be less less than the radius, *otherwise add this product,* multiplying this result by the altitude of the ungula, and *divide this product by the versed* sine.

If $a$=the area FEC of the segment forming the base of the ungula, $r$=the radius, FI $v$=the versed sine CI, $c$ = the chord EF, and $h$=the altitude AC.

Then will the solidity of any cylindric ungula=

$$\left(\tfrac{1}{12}c \pm (r \backsim v)\, a\right)\frac{h}{v}$$

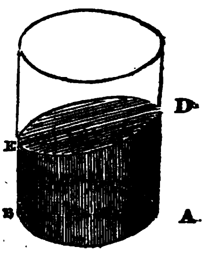
*Ex.* Given the diameter HC 50 inches, the altitude AC of the ungula=120 inches the versed sine IF=10 inches, required the solidity of the ungula.

The chord EF will be found=40 inches. The area of the base 279.56.

Whence, by the rule, AF being less than $\frac{1}{2}$HC, we have

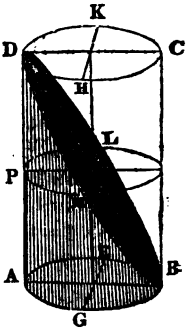
$$\left(\tfrac{1}{12}c^3-(r-v)a\right)\frac{h}{v}=136709\tfrac{1}{5}$$ cubic inches, the solidity of the ungula EFCA.

*Scholium.* When the section passes obliquely through the opposite sides of the cylinder, the content of the ungula may be found by multiplying the sum of the greatest and least heights of the ungula by the area of the base, and its surface may be found by multiplying ½ the sum of the greatest and least heights by the perimeter of the base.



Hence, the ungula ABD is equal to half the cylinder ABCD, both in its surface and solidity.

2. The complement LMHKD, DKHCB of any ungula LMPD, BGAED, may be found by subtracting the solidity of the ungula from a portion of the cylinder of equal base and altitude.



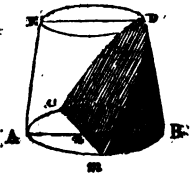
PROBLEM XXII.

*To find the solidity of a conical ungula. cut from the cone, or frustum, by a plane parallel to the side of the cone.*

Rule. Multiply the area of the base by the diameter of the base of the frustum, and divide the product by the difference of the diameters of the two bases ; from this quotient subtract four-thirds of the product of the less diameter by the square root of the product of the less diameter, and difference of the diameters. Multiply the remainder by one-third of the height, and the product will be the content.

Let $a$=the area of the base $cm$B,
D=AB the diameter of the base of the frustum,
$d$=ED the diameter at the top,
$h$=the height.
Then will the solidity of the ungula=

$$\left(\frac{aD}{D-d}-\frac{4}{3}d\times\sqrt{d(D-d)}\right)\frac{1}{3}h$$



*Ex.* If the diameter AB=30 inches, the diameter ED=19.2 inches, and the height $od$ = 18 inches, what is the content of the ungula ?
Ans. 1606.41.

*Scholium.* Other formulæ may be found in the general scholia, at pages 191, &c., for cylindric and conic ungulas; it is unnecessary to extend the problems here.

### PROBLEM XXIII.

### *To find the solidity of a cylindrical ring.*

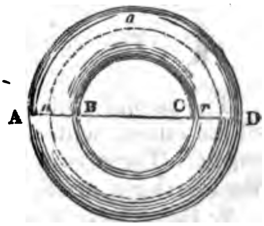
Rule. Multiply half the sum of the inner and outer circumferences by the area of a section of the ring.

*Scholium.* This rule answers for all rings, the virtual centres of whose sections are in the centre of magnitudes of such sections.

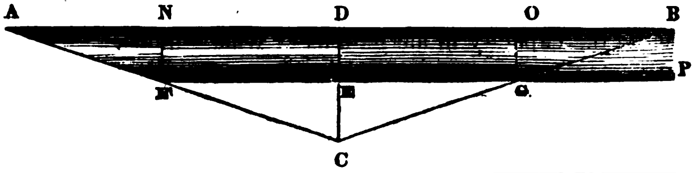
For all other rings see Prop. XI. and Corollaries B. III.

What is the solidity of the cylindric ring AD, whose inner diameter BC is 10 inches, and whose outer diameter AD is 20 inches?

Ans. 925.436 inches.

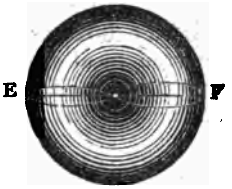


*Scholium.* This ring is equivalent to a segment ABGFA of a cylinder, whose section DE is equal to that of the ring, and whose length FG = the inner circumference, and AB = the outer circumference.

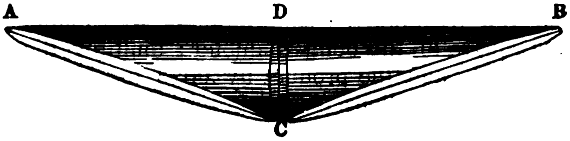


*Ex.* 2. What is the solidity of the cylindric ring EF, whose inner diameter is 0, and its outer diamer EF 10 inches?

Ans. 308,478 inches.



*Scholium.* This ring is equivalent to the segment ABC of a cylinder, whose length AB is = the outer circumference of the ring.



## GAUGING OF CASKS.

**Art. 1.** Gauging of casks is a practical art; and since casks are not commonly constructed in exact conformity with any regular mathematical figure, the subject does not admit of being treated in a very scientific manner; by most writers on the subject, however, they are considered as nearly coinciding with one of the following forms:

$$\begin{array}{l}1.\\2.\end{array}\left.\right\}\text{The middle frustum}\left\{\begin{array}{l}\text{of a spheroid,}\\\text{of a parabolic spindle.}\end{array}\right.$$

$$\begin{array}{l}3.\\4.\end{array}\left.\right\}\text{The equal frustums}\left\{\begin{array}{l}\text{of a paraboloid,}\\\text{of a cone.}\end{array}\right.$$

The *second* of these varieties agrees more nearly than any of the others, with the forms of casks, as they are commonly made. The first is too much curved, the third too little, and the fourth not at all, from the head to the bung.

2. Rules have already been given, for finding the capacity of each of the four varieties of casks. As the dimensions are taken in *inches*, these rules will give the contents in cubic inches. To abridge the computation, and adapt it to the particular measures used in gauging, the factor .7854 is divided by 282 or 321; and the quotient is used instead of .7853, for finding the capacity in ale gallons or wine gallons.

$$\text{Now}\frac{.7854}{282}=002785, \text{ or } .0028 \text{ nearly};$$

$$\text{And } \frac{.7854}{231}=.0034$$

If then .0028 and 0034 be substituted for .7854, in the rules referred to above; the contents of the cask will be given in ale gallons and wine gallons. These numbers are to each other nearly as 9 to 11.

### PROBLEM I.

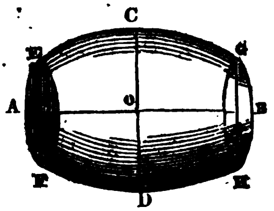
*To calculate the contents of a cask, in the form of the middle frustum of a spheroid, being a cask of the first variety.*

**Rule.** Add together the square of the head diameter, and twice the square of the bung diameter; multiply the sum by $\frac{1}{3}$ of the length, and the product by .0028 for ale gallons, or by .0034 for wine gallons.

If D and $d$ = the two diameter, and $l$ = the length;

The capacity in inches $= (2D^2+d^2)\times\frac{1}{3}l\times.7854$.

And by substituting .0028 or .0034 for .7854, we have the capacity in ale gallons or wine gallons.



*Ex.* What is the capacity of a cask of the first form, whose length AB is 30 inches, its head diameter EF 18, and its bung diameter CD 24?

Ans. 41.3 ale gallons,
or 50.2 wine gallons.

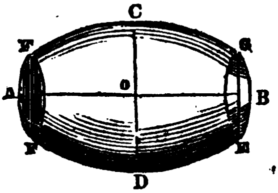
### PROBLEM II.

*To calculate the contents of a cask, in the form of the middle frustum of a parabolic spindle, being a cask of the second variety.*

Rule. Add together the square of the head diameter, and twice the square of the bung diameter, and from the sum subtract ⅖ of the square of the difference of the diameters; multiply the remainder by ⅓ of the length, and product by .0028 for ale gallons, or .0034 for wine gallons.

The capacity in inches $= (2D^2+d^2-\frac{2}{5}(D-d)^2)\times\frac{1}{3}l\times.7854$.

*Ex.* What is the capacity of a cask of the second form, whose length AB is 30 nches, its head diameter BF = 18, and its bung diameter CD=24?



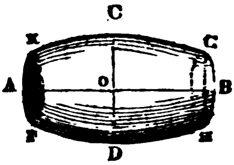
Answer 40.9 ale gallons,
or 49.7 wine gallons.

### PROBLEM III.

*To calculate the contents of a cask, in the form of two equal frustums of a paraboloid, being a cask of the third variety.*

Rule. Add together the square of the head diameter, and the square of the bung diameter; multiply the sum by half the length, and the product by .0028 for ale gallons, or .0034 for wine gallons.

The capacity in inches= $(D^2+d^2)\times\frac{1}{2}l\times.7854$.



*Ex.* What is the capacity of a cask of the third form, whose dimensions are, as before, 30, 18, and 24?

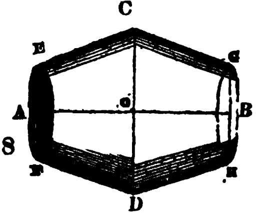
Ans. 37.8 ale gallons, or 45.9 wine gallons.

### PROBLEM IV.

*To calculate the contents of a cask, in the form of two equal frustums of a cone.*

RULE. Add together the square of the head diameter, the square of the bung diameter; and the product of the two diameters; multiply the sum by ⅓ of the length, and the product by .0028 for ale gallons, or .0034 for wine gallons.

The capacity in inches= $(D^2+d^2+Dd)\times\frac{1}{3}l\times.7854$.



*Ex.* What is the capacity of a cask of the fourth form, whose length AB is 30, and its diameters EF and CD = 18 and 24?

Ans. 37.3 ale gallons, or 45,3 wine gallons.

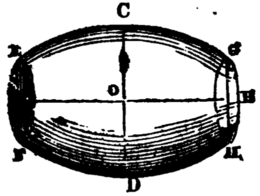
*Scholium.* In the preceding rules, it is supposed that the cask corresponds to the different varieties, whereas, it is seldom that a cask perfectly coincides with either; but for the greater certainty of the truth, when accuracy is required, the following rules, the demonstration of which will be found in Hutton's Mensuration, are to be preferred.

### PROBLEM V.

*To calculate the contents of any common cask from three dimensions.*

RULE. Add together
25 times the square of the head diameter,
39 times the square of the bung diameter, and
26 times the product of the two diameters.

Multiply the sum by the length, divide the product by 90, and multiply the quotient by .0028 for ale gallons, or .0034 for wine gallons.



The capacity in inches=

$(39D + 25d^2 + 26Dd)\times\frac{l}{90}\times.7854.$

*Ex.* What is the capacity of a cask whose length is 30 inches, the head diameter 18, and the bung diameter 24?

Ans. 39 ale gallons,
or 47½ wine gallons.

PROBLEM VI.

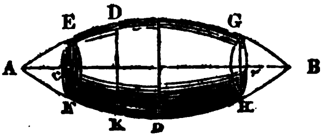
*To calculate the contents of a cask from four dimensions, the length, the head and bung diameters, and a diameter taken in the middle between the head and the bung.*

RULE. Add together the squares of the head diameter, of the bung diameter, and of double the middle diameter; multiply the sum by ⅙ of the length, and the product by .0028 for ale gallons, or .0034 for wine gallons.

If D = the bung diameter, $d$ = the head diameter, $m$ = the middle diameter, and $l$=the length;

The capacity in inches=

$(D^2+d^2+\overline{2m}^2)\times\frac{1}{6}l\times.7854.$



*Ex.* What is the capacity of a cask, whose length *cr* is 30 inches, the head diameter EF=18, the bung diameter CD = 24, and the middle diameter IK 22½?

Ans. 41 ale gallons,
or 49⅔ wine gallons.

*Scholium.* In making the calculations in gauging, according to the preceding rules, multiplications and divisions are frequently performed by means of a *Sliding Rule*, on which are placed a number of logarithmic lines, similar to those on Gunter's Scale.

Another instrument commonly used in gauging is the *Diagonal Rod.* By this, the capacity of a cask is very expeditiously found, from a single dimension, the distance from the bung to the intersectiou of the opposite stave with the head. The measure is taken by extending the rod through the cask, from the bung to the most distant part of the head. The number of gallons corresponding to the length of the line thus found, is marked on the rod. The *logarithmic* lines on the gauging rod are to be usad in the same manner, as on the sliding rule.

### ULLAGE OF CASKS.

Art. 2. When a cask is *partly* filled, the whole capacity is divided, by the surface of the liquor, into two portions; the *least* of which, whether full or empty, is called the *ullage*. In finding the ullage, the cask is supposed to be in one of two positions; either *standing*, with its axis perpendicular to the horizon; or *lying*, with its axis parallel to the horizon. The rules for ullage which are *exact*, particularly those for lying casks, are too complicated for common use. The following are sufficiently near approximations. See Hutton's Mensuration.

### PROBLEM VII.

### *To calculate the ullage of a standing cask.*

Rule. Add together the squares of the diameter at the surface of the liquor, of the diameter of the nearest end, and of double the diameter in the middle between the other two; multiply the sum by $\frac{1}{6}$ of the distance between the surface and the nearest end, and the product by .0028 for ale gallons, or .0034 for wine gallons.

If $D$=the diameter of the surface of the liquor,
$d$=the diameter of the nearest end,
$m$=the middle diameter, and
$l$=the distance between the surface and the nearest end;

The ullage in inches$=(D^2+d^2+2m^2)\times\frac{1}{6}l\times.7854$.

*Ex.* If the diameter at the surface of the liquor, in a standing cask, be 32 inches, the diameter of the nearest end 24, the middle diameter 29, and the distance between the surface of the liquor and the nearest end 12; what is the ullage?

Ans. 27⅘ ale gallons, or 33¾ wine gallons.

### PROBLEM VIII.

### *To calculate the ullage of a lying cask.*

Rule. Divide the distance from the bung to the surface of the liquor, by the whole bung diameter, find the area of a circular segment, whose versed sine is the quotient in a circle, whose diameter is 1, and multiply it by the whole capacity of the cask, and the product by 1¼ for the part which is empty.

If the cask be not half full, divide the depth of the liquor by the whole bung diameter, and find the area of the segment, multiply, &c., for the contents of the part which is full.

*Ex.* If the whole capacity of a lying cask be 41 ale gallons, or 49⅔ wine gallons, the bung diameter 24 inches, and the distance from the bung to the surface of the liquor 6 inches, what is the ullage? Ans. 7¾ ale gallons, or 9½ wine gallons.

### OF THE
## SPECIFIC GRAVITY OF SOLIDS AND FLUIDS.

The specific gravities of bodies are their relative weights, contained under the same given magnitude as a cubic foot, or a cubic inch, &c.

*A table of the Specific Gravities of Bodies, and the weight of a cubic foot of each, in ounces, avoirdupois.*

| | |
|---|---|
| Platinum, | |
| Rolled, | 22666 |
| Hammered, | 20335 |
| Pure gold | |
| Hammered, | 19360 |
| Cast, | 19256 |
| Gold 22 car. fine cast, | 17484 |
| Mercury, | 13596 |
| Lead, | 11351 |
| Silver, cast, | 10474 |
| Copper, | 8788 |
| Soft steel, | |
| Hammered, | 7839 |
| Cast, | 7832 |
| Hard steel, | |
| Hammered, | 7817 |
| Cast, | 7815 |
| Bar iron, | 7787 |
| Tin, | 7290 |
| Cast iron, | 7208 |
| Zinc, | 6860 |
| Granite, | 3500 to 4000 |
| Flint glass, | 3329 |
| Common stone, | 2520 |
| Loom, | 2160 |
| Clay, | 2160 |
| Brick, | 2000 |
| Ivory, | 1825 |
| Sand, | 1520 |
| Coal, | 1250 |
| Sulphuric acid, | 1840 |
| Nitrous acid, | 1550 |
| Nitric acid, | 1217 |
| Human blood, | 1054 |
| Cow's milk, | 1031 |
| Box-wood, | 1030 |
| Sea-water, | 1028 |
| Vinegar, | 1026 |
| Tar, | 1015 |
| Common water, | 1000 |
| Red wine, | 990 |
| Linseed oil, | 932 |
| Proof spirits at 510, | 923 |
| Olive oil, | 913 |
| Alcohol, pure, | 792 |
| Æther, | 726 |
| Air, | 1¼ |

*Note.* The several sorts of wood are supposed to be dry. Also as a cubic foot of water weighs just 1000 ounces, avoirdupois, the numbers in this table express not only the specific gravities of the several bodies, but also the weight of a cubic foot of each, in avoirdupois ounces; and hence, by proportion, the weight of any other quantity, or the quantity of any other weight, may be known as in the following problems.

### PROBLEM I.

*To find the magnitude of any body from its weight.*

Rule.—As the tabular specific gravity of the body is to its weight in avoirdupois ounces, so is one cubic foot, or 1728 cubic inches, to its content in feet, or inches respectively.

Ex. 1. Required the solid content of an irregular block of common stone, which weighs 1 cwt. or 1792 ounces.

Here, as 2520 oz : 1792 ox : : 1728 cubic inches : to its solid content.

*Or,*

As 5 oz. : 256 oz : : 24 cubic inches : $1228\frac{4}{5}$ cubic inches, the solid content required.

Ex. 2. How many cubic feet are there in a ton weight of dry oak? Ans. $38\frac{138}{185}$ cubic feet.

Ex. 3. What is the solid content, and diameter of a cast iron ball, that weighs 42 pounds, its specific gravity being 7208?

Ans. { Solidity = 161.095 cubic inches.
Diameter = 6.75 inches.

### PROBLEM II.

*To find the weight of a body from its magnitude.*

Rule.—As one cubic foot, or 1728 cubic inches, is to the solid content of the body, so is its tabular specific gravity to the weight of the body.

Ex. 1. Required the weight of a block of marble, whose specific gravity being 2700, the length = 63 feet, the breadth and thickness each = 12 feet; this block being the dimensions of one of the stones in the walls of Balbeck.

Here, as 1 cubic foot : 63 × 12 × 12 (= 9072 cubic feet) : : 2700 oz. : $683\frac{7}{16}$ tons weight, almost equal to the burthen of an East India ship.

Ex. 2. What is the weight of a block of dry oak, which measures 10 feet long, 3 feet broad, and $2\frac{1}{2}$ feet deep?

Ans. $4335\frac{15}{16}$ pounds.

Ex. 3. What is the weight of a leaden ball, $4\frac{1}{4}$ inches in diameter, its specific gravity being 11351?

Ans. 16 lbs. 7 oz.

Ex. 4. Required the weight of a cast iron shell, 3 inches thick, its external diameter being 16 inches, and its specific gravity 7208.

Ans. { Solidity = 1621.0656 cubic inches.
Weight = 2 cwt. 3 qr. 9 lb. .5 oz.

### PROBLEM III.

*To find the specific gravity of a body heavier than water.*

Rule.—Weigh the body both *in water* and *out of water*, by a *hydrostatic balance*, and take the difference of these results, which will be the weight lost *in water*.

Then say, as the weight lost *in water*, is to the weight of the body *in air*, so is the specific gravity of water, to the specific gravity of the body.

Ex. 1. A piece of stone weighed ten pounds *in air;* but, *in water*, only 6¾ pounds; required the specific gravity.

Here, as 10 — 6¾ (= 3¼) : 10 : : 1000 : to the specific gravity of the body.

*Or,*

As 13 lbs. : 40 lbs. : : 1000 oz. : 3077 oz. = Ans.

Ex. 2. A piece of copper weighs 36 oz. *in air*, and only 31.904 oz. *in water;* required the specific gravity of copper.

Ans. 8788 ounces.

Eq. 2. Required the specific gravity of a piece of granite stone which weighs 7 lbs. *in air*, and 5 lbs. *in water*.

Ans. 3500 ounces.

### PROBLEM IV.

*To find the specific gravity of a body lighter than water.*

Rule.—Fasten to the lighter body, by a slender thread, another body heavier than water, so that the mass compounded of the two may sink together. Weigh the heavier body, and the compound mass, separately, both *in water* and *out of it*, then find how much each loses *in water*, by subtracting its weight *in water* from its weight *in air*.

Then say, as the difference of these remainders is to the weight of the lighter body *in air*, so is the specific gravity of water to the specific gravity of the lighter body.

Ex. 1. Suppose a piece of elm weighs 12 lbs. *in air*, and that a piece of metal, which weighs 18 lbs. *in air*, and 16 lbs. *in water*, is affixed to it, and that the compound weight is 6 lbs. *in water;* required the specific gravity of the elm.

Here 18 — 16=2 pounds, the metal lost *in water;*
and (18+15) — 6 = 33 — 6=27 pounds, the compound lost *in water*.

Then 27 — 2=25 pounds, the elm lost *in water*.

As 27 – 2 (=25 lbs.) : 15 lbs. : : 1000 oz. : to the specific gravity of the elm.

*Or,*

As 1 lb. : 3 lbs. : : 200 oz. : 600 oz. = Ans.

Ex. 1. A piece of ash weighs 20 lbs. *in air*, to which is affixed a piece of metal, which weighs 15 lbs. *in air*, and *in water*, $13\frac{1}{3}$ lbs.; and the compound, *in water*, weighs only $8\frac{1}{3}$ lbs; required the specific gravity of the ash.

Ans. 800 ounces.

Ex. 2. Suppose a piece of fir weighs 11 lbs. *in air*, and a piece of steel being affixed which weighed 16 lbs. *in air*, and *in water* 14 lbs.; and the compound *in water* weighs only 5 lbs.; what is the specific gravity of the fir?

Ans. 550 ounces.

Ex. 4. A piece of cork weighing 20 lbs. *in air*, had a piece of granite fixed to it, that weighed 120 lbs. *in air*, and 80 lbs. *in water*; the compound mass weighed $16\frac{2}{3}$ lbs. *in water*; what was the specific gravity of the cork?

Ans. 240 ounces.

## QUESTIONS FOR EXERCISE.

1. Having a rectangular marble slab, 58 inches by 27, I would have a square foot cut off parallel to the shorter edge; I would then have the like quantity divided from the remainder, parallel to the longer side; and this alternately repeated, till there shall not be the quantity of a foot left; what will be the dimensions of the remaining piece?

Ans. 20.7 inches by 6.086.

2. Given two sides of an obtuse angled triangle, which are 20 and 40 poles; required the third side, that the triangle may contain just an acre of land? Ans. 58.876 or 23.099.

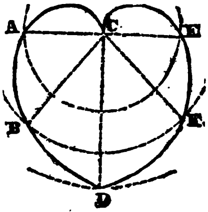
3. The ellipse in Grosvenor-square measures 840 links across the longest way, and 612 the shortest, within the rails; now the walls being 14 inches thick, what ground do they inclose, and what do they stand upon?

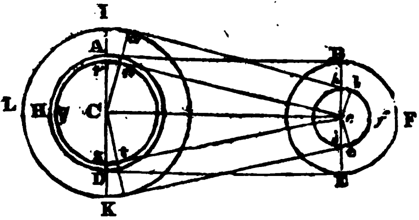
Ans. { inclose 4*a*. 0*r*. 6*p*.
stand on $1760\frac{1}{2}$ sq. feet.

4. What is the length of a chord, which cuts off one-third of the area, from a circle whose diameter is 289?

Ans. 278.6716.

5. What is the area of the heart CABDFEC, the axis CD = 10 inches. (See article on spirals, page 140.)
Ans. 104.7193 inches.





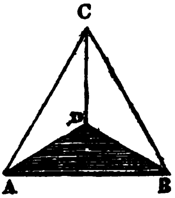
6. There are two pulleys AHD, BFE, the diameters AD, and BE are each = 20 inches, and the distance Cc is 4 feet; the pulley BFE is put in motion around its axis by a belt ABFEDH, passing round AHD. Now if the pulley ILK on the same axis with AHD, is 40 inches in diameter, what must be the diameter *hi* of a corresponding pulley *hfi*, around which the belt I*lfe*KL may pass, so as to be of the same length and tension as that of the belt ABFEDH, and what will be the ratio of the angular velocity of the two pulleys.

Draw C*a*, *cb* perpendicular to *al*, and from *c* draw *cn* perpendicular to C*a*; *cl* and *cn* will be parallel to *al*, and hence will be = *al*: with the radius C*n* describe a circle *nrqst*, and the tangent *nc* will be = the tangent *al*, $=\sqrt{(Cc^2-(IC-hc)^2)}$.

The arc *rn* is = arc I*a* — arc *hl*.

It is required from these data to find the diameter *ik*.

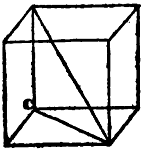
*Scholium.* In the solution of this problem, it will be necessary to express the arc *rn* in terms of its functions; the mode of conducting the solution will be left for the student.



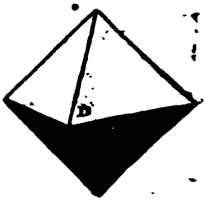
7. Required an expression for the superficies, and also for the solidity of a tetrædron ABCD, in terms of its linear edge, AB=A.

Ans. $A^2\sqrt{3}$ = the surface.
$\frac{1}{12}A^3\sqrt{2}$ = the solidity.

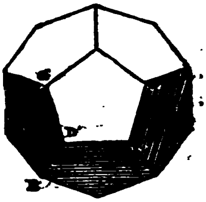
8. Required expressions for the surface and solidity of a regular hexædron or cube in terms of its edge. Ans. $6A^2$, = its surface.
$A^3$, = its solidity.



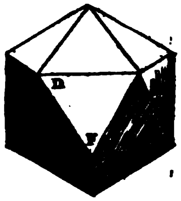
9. How will you express the surface and solidity of an Octædron in terms of its edge? Ans. $2A^2\sqrt{3}$ = the surface.
$\frac{1}{3}A^3\sqrt{2}$ = the solidity.



10. Let the surface and solidity of a dodecædron be expressed in terms of its edge.
Ans. $15A^2\sqrt{(1+\frac{2}{5}\sqrt{5})}$ = the surface.
$5A^3\frac{\sqrt{47+21\sqrt{5}}}{40}$ = the solidity



11. Express in terms of its edge, the surface and solidity of an Icosædron.
Ans. $5A^2\sqrt{3}$ = the surface.
$\frac{5}{6}A^3\frac{\sqrt{7+3\sqrt{5}}}{2}$ = the solidity.



12. The diameter of the Winchester bushel was 18½ inches, and its depth 8 inches: what must the diameter of a bushel be when its depth is 7½ inches? Ans. 19.1067.

13. Of what diameter must the bore of a cannon be, which is cast for a ball of 24 lbs. weight, so that the diameter of the bore may be 1-10 of an inch more than that of the ball, and supposing a 9 lb. ball to measure 4 inches in diameter?
Ans. 5.646 inches.

14. Suppose the ball on the top of St. Paul's church is 6 feet in diameter, what did the gilding of it cost, at 3½*d.* per square inch? Ans. £237, 10*s.* 1*d.*

15. What will the gilding of a right rectangular revoloid, whose axis is 4 feet, come to at 5 cents per square inch?
Ans. $720.

16. A silver cup, in form of the frustum of a cone, whose top diameter is 3 inches, its bottom diameter 4, and its altitude 6 inches, being filled with beer, a person drank out of it till he could see the middle of the bottom; it is required to find how much he drank?

Ans. 42.899844 cubic inches = .152127 ale gallons, or 1 gill and ½ nearly, the quantity required.

17. Two persons would divide between them, by a plane perpendicular to the base, a hay rick, in the form of a paraboloid, whose altitude is 40, and the diameter of its base 30 feet; it is required to find the difference between the solidities of the parts, supposing the altitude of the section to be 28 feet. Ans. 11265.75803, the difference required.

18. In the construction of a railroad, having contracted for the sum of $1000 to excavate a certain section, the area of whose conjugate sections in 11 different places, taken at equal distances of 3 rods each, including the ends, are as follows, viz: the first, 150 square feet, the second, 160. the third, 165, the fourth, 172, the fifth, 190, the sixth, 210, the seventh, 224, the eighth, 240, the ninth, 202, the tenth, 108, and the eleventh 0. After having disposed of the materials to be excavated at 10 cents per cubic yard, to be delivered on an adjoining section, I afterward received an offer to have the whole labor of excavation and delivery performed for 30 cents per yard; shall I gain or lose by my contract if I accept of the offer, and how much? Ans. I shall gain $355,64.

19. To determine the weight of a hollow spherical iron shell, 5 inches in diameter, the thickness of the metal being one inch? Ans. 11.79lb.

20. It is proposed to determine the proportional quantities of matter in the earth and moon; the density of the former being to that of the latter, as 10 to 7, and their diameters as 7930 to 2160. Ans. as 71 to 1 nearly.

21. What difference is there, in point of weight, between a block of marble containing 1 cubic foot and a half, and another of brass of the same dimensions, whose specific gravity is 8000? Ans. 496lb. 14oz.

22. What position in the line between the earth and moon, is their common centre of gravity; supposing the earth's diameter to be 7920 miles, and the moon's 2160; also the density of the former to that of the latter, as 99 to 68, or as 10 to 7 nearly, and their mean distance 30 of the earth's diameters?

Ans. 633.65 miles below the surface of the earth.

23. How deep will a cube of oak sink in common water; each side of the cube being 1 foot? Ans. $11\frac{1}{10}$ inches.

24. How deep will a globe of oak sink in water; the diameter being 1 foot? Ans. 9.9867 inches.

25. If a cube of wood, floating in common water, have three inches of it dry above the water, and $4\frac{8}{103}$ inches dry when in sea-water; it is proposed to determine the magnitude of the cube, and what sort of wood it is made of?

Ans. the wood is oak, and each side 40 inches.

26. Hiero, king of Sicily, ordered his jeweller to make him a crown, containing 63 ounces of gold. The workmen thought that substituting part silver was only a proper perquisite; but Hiero, suspecting that fraud had been practised, Archimides was appointed to examine it; who on putting it into a vessel of water, found it raised the fluid 8.2245 cubic inches; and having discovered that the inch of gold more critically weighed 10.36 ounces, he found by calculation what part of the king's gold had been changed. And you are desired to repeat the process. Ans. 28.8 ounces.

27. Supposing the cubic inch of common glass weigh 1.4921 ounces troy, the same of sea-water .59542, and of brandy .5368; then a seaman having a gallon of this liquor in a glass bottle, which weighs 3.84lb out of water, and, to conceal it from the officers of the customs, throws it overboard. It is proposed to determine, if it will sink, how much force will just buoy it up? Ans. 14.1496 ounces||

28. Suppose, by measurement, it be found that a man of war, with its ordinance, rigging, and appointments, sinks so deep as to displace 50000 cubic feet of fresh water; what is the whole weight of the vessel? Ans. $1395\frac{1}{11}$ tons.

29. It is required to determine what would be the height of the atmosphere, if it were every where of the same density as at the surface of the earth, when the quicksilver in the barometer stands at 39 inches; and also, what would be the height of a water barometer at the same time?

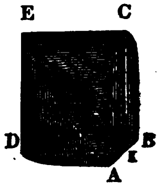
Ans. height of the air $29166\frac{2}{3}$ feet, or 5.5240 miles,
height of water 35 feet.

30. If the inner axis of a hollow globe of copper, exhausted of air, be 100 feet; what thickness must it be of, that it may just float in the air? Ans. .02624 of an inch thick.

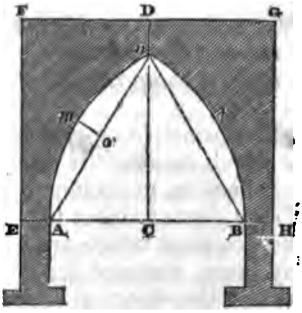
31. If a spherical baloon of copper, of $\frac{1}{100}$ of an inch thick, have its cavity of 100 feet diameter, and be filled with inflammable air, of $\frac{1}{100}$ of the gravity of common air, what weight will just balance it, and prevent it from rising up into the atmosphere? Ans. 21785lb.

32. Construct a quantity $\frac{a\underline{x}^{2}\overline{z}}{\underline{u}}$ and show its value. (See Book IV., Chap. II.)

33. Express in a series of variables, the symmetrical cylindrical ungulas, DBAECAB. where, AB=2ID=$x$, and IC=$z$.



33. Required the solidity of the vacuity of a gothic roof, and also the solidity of the materials of the the roof; the span AB = 30 feet, the chords of each arc A*n* and B*n* =40 feet, the versed sine *mo* = 6 feet, the thickness of the pear EA or BH = 17 feet, at the spring of the arc, the thickness of the crown of the arch D*n*=4 feet of the roof 60 feet.



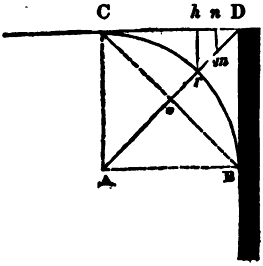
Ans. { 52896.89319 cubic feet, the solidity of the vacuity.
104854.11776 cubic feet, the solidity of the materials. }

34. Required the superficies of a dome in the form of a right hexagonal revoloid, each side of the base being 10 feet, and height 10 feet.

Ans. 519.61524 square feet.

35. The circumference of the base of a circular dome is 150 feet, and its height 23.873 feet; required the superficies.
Ans. 3581.1 square feet.

36. If the height BC of a saloon be 3.2 feet, the B*o*D, of its front 4.5 feet, the distance *or*, of its middle part, from the arc 9 inches, and the mean circumference at *m*=50 feet; required the solidity of the saloon.
Ans. 138.26489 cubic feet.



37. What is the whole surface of a saloon round a rectangular room, the mean compass at *r*=67.3137 feet, the girt D*r*B 3.1416 feet, and the ceiling measures 16 feet long, and 12 feet broad?

Ans. 403.4727 square feet.

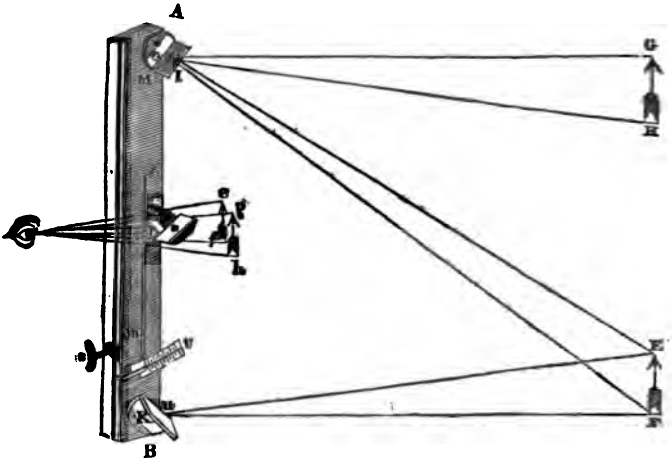
38. Required the concave surface of a circular arch, raised on a rectangular base, whose sides are 27 feet 3 inches by 20 feet 4 inches.

Ans. $632.5415\frac{1}{3}$ square feet.

## *Description of an instrument constructed by the author, for measuring distances and heights, by a single observation, and without changing the position, or measuring any base line.*

The principle on which this instrument is constructed, and by which the result is produced, consists in arranging mirrors, or reflectors, in such manner as to convey two distinct images of any distant object to the eye of the observer, as seen from two positions which are indicated by two mirrors, placed at any given distance from each other on the instrument, and causing the object to appear in two positions at the same time, and then measuring the apparent angle under which the two images appear.

For this purpose the following diagram represents one form of the construction of the instrument.



AB, represents the stock or base; at the extremities of which let two mirrors, I, *u*, be placed in the manner of the index glass to a common quadrant, and set perpendicular to the plane of the instrument, and at an angle of 45° with its axis or one of its edges; these we will call the object glasses, one of which, viz: I, is placed on the centre of motion of an index, M*m*, which is moveable about a centre at *e*, and with it the mirror, which for distinction is called also the index glass. There are two other reflectors, *n*, *i*. placed near the middle of the stock of the instrument, one above the other, their edges being in contact, the planes of which cross each other at right angles; one of these reflectors is parallel to one of the object glasses *n*, the other to the index glass, I.

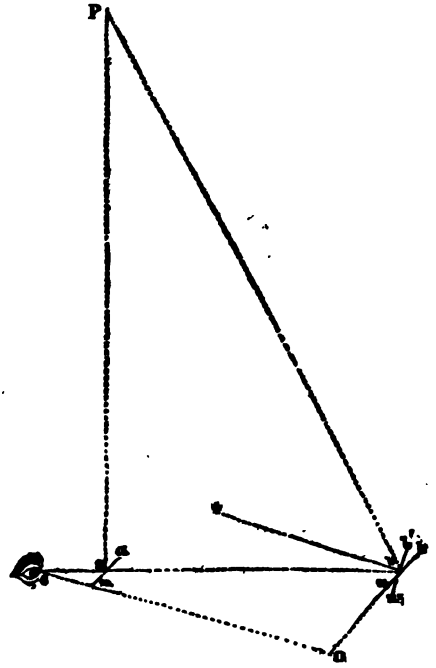
In using this instrument it must be placed or held so that its axis shall be perpendicular to a line from the object EF, to the fixed object glass *u*, and images of the object will be formed in the two object glasses I, K, and reflected into the two glasses *n* and *i*, and hence, to the eye of the observer, forming two distinct images, *ef*, *ph*, one of which appears higher than the other; and the angle under which the two objects appear, varies according to the distance of the object. In order to measure the angle contained by the apparent positions as indicated by the two images, the index is moved, and with it the index glass *i*, till the object shall appear in the same position in both glasses, so that there would appear to be but one image formed in both, or till the images formed in both would appear identical. Now if a *vernier scale*, *v*, is attached to the index and graduated, it will indicate the apparent angle under which the two images appear, or the angle under which the distance between the two object glasses would appear if placed at the distance of the object. But by the law of reflection, the index would be moved only through half the angular distance of the two images in order to produce an apparent coincidence; and hence the scale should be graduated with double the ordinary divisions, for the angle would be indicated by twice the angular motion of the index. In order that the index may be adjusted with accuracy, a tangent screw *s*, is provided, by which it can be adjusted with any precision required. When the observation and adjustment is completed, we have a right angled triangle whose base is the distance of the two mirrors, and whose altitude is the distance of the fixed object glass to the object, and since we have one of the acute angles, the other side or distance of the object becomes known.

When the object EF, is at any considerable distance, the angle *u*FI, or *u*EI, becomes smaller, and the object will appear to come to the mirror I, from the position GH; so that HI and F*u* produced, shall make an angle with each other equal to the apparent distance of the two images. But when the object is at an infinite distance, then the lines F*u*, HI, become parallel, and the two images coincide, so that but one image appears to the observer; and hence, in this case, the distance cannot be measured.

In order to save the trouble of calculation in each case, a table may be constructed to accompany the instrument, which shall contain the distance corresponding to any given angle pointed out on the scale, or, the scale itself may be graduated to specific distances, which may be read off instead of the angles.

Another construction of the instrument, using but two glasses is as follows:

Let two mirrors, *a* and *b*, be placed parallel to each other at the opposite extremities of the instrument, making an angle of 45° with its axis as before, and let the mirror *b*, be moveable about a centre by means of an index as described above.



When any distance is to be measured by this instrument, it is placed so that a line from the object P, to the mirror *a*, shall be perpendicular to the axis *rm*, of the instrument, the eye of the observer being at any point *e*, in the axis produced; and the image of the object seen in the mirror *a*, will, by the law of reflection, appear in the direction of this axis; and the image P, in the other mirror, if seen at all, will appear at *o*, not coinciding with *m*. But by turning the mirror *b* about its axis, the image will advance toward *v*, in the line of the axis of the instrument, and will ultimately coincide with it; when the two images will be seen at *e*, in the same line, *ev*.

It may be easily shown that the angular motion of the mirror *b*, necessary to bring it into the position *b' n'* so that the two images shall appear to coincide, will be half the angle at P, hence the distance is determined as before.

In this form of construction, I place the mirror in a wooden case or tube, leaving openings in its side in front of each mirror, and a small hole at the end for the eye of the observer; the index and scale is on the out-side of the case: the mirror *a*, must occupy but half a section of the tube, so that the mirror *b* may be seen over the edge of the mirror *a*; and the object is attained by bringing the images in both mirrors to coincide, or so as to appear as one image.

In order to determine the powers of this instrument, it is only necessary to observe that the distance *m*P, is the cotangent of the angle P to the radius *mr*, and that when *mr* is given, the value of *m*P may be calculated for any assumed value of the angle P, which is half the angle measured by the index and scale. Assuming the distance between the mirrors to be five feet, the angular motion of the mirror *b* from its position parallel to *a*, the zero of the scale, will be for 1000 feet nearly 8′35″, and for 1100 feet 7′50″, a difference of 45″ or three-fourths of a minute for a difference of 100 feet, or ten per cent of the first distance.

If we assume that by the divisions on the scale attached to the index, the motion of the mirror may be correctly found to half minutes, then the distance between the mirrors being taken at five feet, a change of half a minute would correspond at 10 feet to .007 of a foot, at 100 feet to .62 of a foot, at 1000 feet to 63 feet, and at 10000 feet or 1.9 mile the whole angle is but 51″, and considerable variations would entirely escape detection; but by the application of a telescope to the instrument in making the observations, its powers and accuracy may be considerably extended.

After having measured the distance to any object, as a house, or a tree, its altitude may be easily found by moving the index so that the top of the object in one mirror shall coincide with the bottom in the other, when the angle indicated on the scale, less the angle first found, corresponding to the to the distance, is the angle under which the object appears; whence having the angle and distance of the object, its altitude becomes known.

The following investigation of the powers of the instrument, showing its limits of practical accuracy, is taken from a report furnished by a committee of the Franklin Institute of Pennsylvania, to whom the two instruments designated above were submitted by the author in 1833. (Published in Vol. XI., No. 3, Journal of the Franklin Institute.)

Call the variation in the angle P, $y$, the distance $m$ P $a$, and $mr$, $b$. Suppose the angle P, to become P $- y$, and that then $a+x$, $x$ denoting the increase of length of $a$, corresponding to a decrease, $y$, of the angle P.

By trigonometry,

$$\tan. P = \frac{b}{a} \text{ and}$$

$$\tan. (P - y) = \frac{b}{a+x}; \text{ but}$$

$$\tan. (P - y = \frac{\tan. P - \tan. y}{1 + \tan. P \times \tan. y},$$ or by substituting for tan. P and tan. (P $- y$) their values found above.

$$\frac{b}{a+x} = \frac{\frac{b}{a} - \tan. y}{1 + \frac{b}{a} \div \tan. y} \text{ or}$$

$$\frac{b}{a+x} = \frac{b - a \div \tan. y}{a + b \div \tan. y}, \text{ whence}$$

$$x = \frac{(a^2+b^2) \div \tan. y}{b - a \times \tan. y}, \text{ or}$$

$$x = \frac{\frac{a^2+b^2}{b}}{\frac{b}{\tan. y} - a}$$

If, as assumed above, $b = 5$, and $y = 1$, the general equation becomes

$$x = \frac{25+a^2}{17241.4 - a}$$

$$\text{When } a = 10, x = \frac{125}{17231.4} = .007.$$

$$\text{For } a = 100, x = \frac{10.025}{17141.4} = .62$$

For $a = 1000$, $x = 61.6$, and for $a = 10{,}000$, $x = 13{,}809$; which is greater than the distance $a$.

By assuming a limit to the accuracy required, calculation will show how the instrument may be adapted to this limit when possible. For example, let the greatest inaccuracy allowed be one foot in 100, then $b$ and $y$ must be so adjusted that at the greatest distance for which the instrument is to be used $x = \frac{a}{100}$. Calling this value of $a$, $a'$, we shall have,

$$.01\,a' = \frac{a'^2+b^2}{\frac{b}{\tan. y} - a'}, \text{ or}$$

$$b^2 - \frac{.01\,a'}{\tan. y} \times b = -1.01\,a'^2,$$ an equation which must exist in order that the required accuracy may be attainable. To examine by it the instrument already supposed, let us ascertain whether at 1000 feet, as the greatest distance at which it is to be used, the accuracy will come within the limit of one foot variation, in 100. In this case $a' = 1000$, $b$, as before, $= 5$, and $\tan. y = \tan. 1'$. Whence $b^2 = 25$, $\frac{b}{\tan. y} = 17241.4$, $01a' = 10$, and $1.01\,a'^2 = 1.010.000$. Substituting these values in the equation above, it requires

$$25 - 172,414 = -1,010,000,$$ the equation is not fulfilled, and the instrument does not come up to the requirement. It would be easy to determine values of $y$ and $b$ required for all possible degrees of accuracy, and thus by the possibility of making the half divisions accurate, and by the length which convenience might limit, to ascertain whether the instrument could be constructed to give the required degree of accuracy.

The investigation may be made more general, thus; let $\frac{a}{b}$ express the required limit of accuracy at the greatest distance for which the instrument is to be used, then at that distance $x = \frac{a}{n}$, or calling, as before, the value of $a$, $a'$,

$$\frac{a}{n} = \frac{a'^2+b^2}{\frac{b}{\tan. y} - a} \text{ whence}$$

$$b^2 - \frac{a'}{n \tan. y} b = -\frac{a'^2}{n} - a'^2 - a'^2 \left(1 + \frac{1}{n}\right)$$

$$b = \frac{a'}{2n \tan. y} \pm \sqrt{\frac{a'^2}{4\,n^2 \tan.^2 y} - a'^2 \left(1 + \frac{1}{n}\right)}$$

$$b = a' \frac{\left(1 \pm \sqrt{1 - 4\,n^2 \tan.^2 y \left(1 + \frac{1}{n}\right)}\right)}{2n \tan. y}$$

This equation is possible when $4n \tan.^2 y\,(n+1) < 1$.

# NOTES.

## BOOK I.

Having in the preceding volume treated of the properties of the *Parabola*, *Ellipse*, and *Hyperbola*, we show in this, that these curves are the sections of the cone, and are each formed by a plane passing through a cone, according to certain conditions. Of these sections the quadrature of the parabola is easily attained, by the principles embraced in Prop. IV., in relation to the sectional divisions of a prism. Props. VI. and VII., show the application of the principles* to the quadrature of the parabola.

The quadrature of the *Ellipse*, though not so accurately determined as that of the parabola, is nevertheless easily expressed in terms of the circle's quadrature.

The *Hyperbola* is of more difficult determination, in relation to its quadrature, than the other conic sections, but, is nevertheless susceptible of being approximately determined to any extent required.

## BOOK II.

### ON SOLID SECTIONS OR SEGMENTS.

Since the term section, though originally applied only to surfaces, made by cutting a solid, is extensively used in the arts, as expressing a definite portion of a solid, I have taken the liberty of making solid sections, as synonimous with segments of solids. This book, consists mostly of the comparison of cylindric, and conical ungulas.

## BOOK III.

### OF REVOLOIDS.

Revoloids are a class of bodies not usually treated of in works on Geometry ; but, from the important considerations connected with their organization, and from the relations which they bear, both to rectilineal and curvilineal solids, it is of the highest importance to Geometry, that their properties should be discussed, and that they should receive a conspicuous place among geometrical solids ; more especially, as they are almost the only curvilinear solids, that are absolutely cubable. For, we have shown, in the progress of the work, that, not only the right or spherical revoloid, and the elliptical revoloid, Props. VI. and VII. are cubable, is absolute terms, but also *Parabolic* and *Hyperbolic Revoloids* ; moreover, we have shown (Prop. III.,) the quadrature of the surface of a right revoloid, without regard to the circle's quadrature, although it is bounded by cylindric surfaces.

Some important principles are derived from the subject of mathematical transformation, as in Prop. IV., V., XI., and Corollaries.

---

* In Prop. VI, the parallel lines *k*K &c., should be parallel to the axis EF of the parabola, instead of the position as there expressed, otherwise, the argument is not correct ; but since the principle to be established, is not affected by the error in that diagram and argument, it is deemed best not to alter the demonstration in this edition. The Scholium to Prop. VII, may be corrected by making AB or CD the axis of the parabola, instead of AC or BD. In all references to these propositions in the succeeding parts of the work, it will be perceived that the conditions of their application is expressed.

## BOOK IV.

The quadrature of the revoloidal surface treated of in Book III., furnishes us with data for the quadrature of the circle, through the medium of the revoloidal curve; this subject is amply discussed in this book; through the properties of this curve, we are also enabled to deduce some important trignometrical functions. In Prop. III., it is shown that the revoloidal curve may be by transformation derived from an elliptical curve.

Expressions are obtained in Props. IX. and XII., for the length of the circle's circumference; it is there shown, that these approximations may be extended indefinitely, so that the circumference may be obtained to any degree of exactness required.

By applying the principles of the parabola to those of the revoloidal curve in Prop. XVII., a remarkable approximation is obtained, so that if the sine and cosine of a small arc, and sine of half the given arc is obtained, the arc itself may be expressed in terms of those functions; and it is shown that these functions may be so taken, that the arc shall be truly expressed to the same number of decimal places, that those functions are truly expressed; in pursuance of this, will be found, an example in Mensuration, page 194, where the arc is correctly calculated to 20 decimal places, by a simple process, having the sines and cosine given, as data, to 21 decimal places. Other modes of approximation, for the arc of the circumference may be pointed out, depending on the same principles, derived both from the quadrature of the revoloidal surface, and cubature of the reveloid; but, by pursuing the course pointed out, page 125, the arc of the circumference may be found, with certainty, by this process, to any number of decimal places we have patience to pursue it.

It may be here remarked that, by some computist, the circumference of a circle whose diameter is 1, has been recently developed to 154 decimal places, which is as follows:—

3.1415926535897932384626433832795028841971693993751058209749445
923078164062862089986280348253421170679821480865132723066470 9
38446460955051822317253594081284802.

It is therefore folly to attempt its farther developement, considering tha no practical or speculative advantage would accrue therefrom.

Props. XVIII and XIX, prepare us for the construction of a curve, described in Prop. XX., termed the curve of the circle's quadrature, the properties of which are, that a line drawn from any point in this curve, perpendicular to its conjugate diameter, will be equal to the arc of the inscribed circle cut off by a secant drawn from the centre of the circle to this point, and if another line be drawn from the same point in the curve, to the extremity of the conjugate diameter, the area of the space, intercepted by the two lines without the circle, will be equal to the area of the segment of the circle cut off by the latter line. From these properties, we are enabled to deduce an important theorem, in relation to segments of the circle, viz., that the area of any segment of a circle is equal to the difference between the arc of the segment, and its sine, multiplied by half the radius.

## BOOK V.

In this book, all geometrical magnitudes are discussed from their principles of organization, from elementary magnitudes, without referring them to any specific forms or relations. And after introducing, and explaining the principles of the production of geometrical magnitudes, from variable elements, in Chap. I., in order to render this science subject to analytical, and algebraic consideration, a peculiar notation has been introduced as the subect of the second Chapter; and the mode of application, of this notation, o such subjects is there explained.

By this mode of conducting geometrical investigations, and by this notation, we are enabled in a manner, somewhat more obvious than that of the calculus, to arrive at the same results as are obtained by that science.

And although this forms but an introduction to the subject, yet it furnishes evidence of its adaptation to the investigation of the properties of all geometrical magnitudes; and may, perhaps, be rendered of equal universality with the calculus, with which it is most intimately allied.

Here instead of the infinitely small momentary increments of variable magnitudes, or instead of the differentials of variables, this notation recognizes only the conditions of the variables themselves, in their associated capacity, which, from the principles of the science, may be integrated, as certainly, and with more obvious rationality, than those performed by the calculus from their differentials.

By this notation, we are enabled to get a positive expression for the circle's quarature, in known functions of the diameter; which, since all Geometricians are satisfied of the incommensurability of the circumference in direct terms of the diameter, should be received as the quadrature itself. For, from this expression, means may be devised of developing, decimally, the quadrature to any desirable extent. And if it had not been previously shown, as it has been done by Legendre and others, that the circumference, and consequently the quadrature of the circle, could not be expressed numerically in terms of the diameter, its impossibility might very obviously be proved here; and for the same reason that a surd quantity cannot be thus numerically expressed, since the quadrature of the circle consists of the production of a series of surd quantities, drawn into a function of the diameter.

Chapter III. is an introduction to the differential and integral calculus, designed to show the first principles of that science, and to show its connection with that of Chapter II., both of which are evidently in their essential particulars, based on the principles contained in Chapter I.

Chapter IV. is devoted to the application of the principles previously discussed, to determine the position of the virtual centre, or centre of gravity of geometrical magnitudes. This has universally been denominated by all authors, hitherto as the *centre* of *gravity*. I doubt not, I shall have the approbation of most Mathematicians in discarding that term, and supplying in its place that of the *virtual centre*. The term *centre* of *gravity*, though perfectly proper in works on Mechanics and Natural Philosophy, is highly incongruous in a work on pure mathematics, where the physical properties of bodies is not a subject of investigation, and perhaps not intelligibly understood.

## MENSURATION.

Such subjects in the mensuration of surfaces and solids, as could not consistently be introduced into the elementary part of the work, is introduced here; most of the subjects embraced in this, have been discussed in the geometrical part of the work, and the principles demonstrated; where this is not the case, a reference has been made to the author, where such demonstration may be found; on this subject free use has been made of Hatton's Mensuration; limited quotations have also been made from other authors.

The article on Guaging is mostly taken from Day's Mathematics, though originally derived from Hutton.

# TABLE OF NATURAL SINES.

In the following table of Natural Sines, the sines are exhibited to every degree and minute of the quadrant, and so arranged, that the degrees corresponding to the sines will be found on the top of the page in the same column with the sine; the minutes in the left hand column opposite their sines: and the degrees answering to the cosine will be found in the same manner, at the bottom, with their minutes in the right hand column.

The sine or cosine of an arc greater than 90°, is the same as the sine or cosine of its supplement; or the same as those of the difference of 180° and the given arc.

The sines and cosines of any arcs greater than 180°, are the same as those for arcs less than 180°, but with contrary signs, being by trigonometry considered negative; they may be found in the table as above, by deducting 180° from the given arcs.

If the sine or cosine is required to degrees, minutes, and seconds.

Take out the sines or cosines answering to the next less, and the next greater arc expressed in the table; multiply their difference by the given number of seconds and divide the product by 60; then the quotient added to the sine of the next less arc will be the sine or cosine required.

*Example.* Required the sine of 32° 21′ 45″, or its supplement 147° 38′ 15″.

The sine of 32° 21′ is, .535090
The sine of 32° 22′ is, .535335

The difference is, .000245.

Hence, $245 \times 45 \div 60$

$= .000245 \times \frac{3}{4} = .000184$, the quantity to be added to .535090.

Therefore, .535090
+.000184

The sine required. = .535274.

If the arc of a given sine or cosine is required in degrees, minutes, and seconds:

Take out the arcs answering to the next less and the next greater sine and cosine, and multiply their difference by 60, and that product by the difference of the next less, and the given sine or cosine divided by the difference between the next less and next greater sine, and add or subtract as before for the arc of the sine or cosine.

*Example.* Required the degrees, minutes, and seconds corresponding to the sine .495994.

The sine next less than that given is, .495964
The next greater sine is, .496217

The difference, .000253.

The difference of the next less and the given sine is, .000030.

The arc corresponding to the next less sine is 29° 44′.

Hence, $000030 \times 60 \div 000253 = 7''$ = the number of seconds to add to 29° 44′.

tTherefore, e required arc is 29° 44′ 7″.

# NATURAL SINES.

| M | 0° | 1° | 2° | 3° | 4° | 5° | 6° | 7° | 8° | 9° | M |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 000000 | 017452 | 034899 | 052336 | 069756 | 087156 | 104528 | 121869 | 139173 | 156434 | 60 |
| 1 | 000291 | 017743 | 035190 | 052626 | 070047 | 087446 | 104818 | 122158 | 139461 | 156722 | 59 |
| 2 | 000582 | 018034 | 035481 | 052917 | 070337 | 087735 | 105107 | 122447 | 139749 | 157009 | 58 |
| 3 | 000873 | 018325 | 035772 | 053207 | 070627 | 088025 | 105396 | 122735 | 140037 | 157296 | 57 |
| 4 | 001164 | 018616 | 036062 | 053498 | 070917 | 088315 | 105686 | 123024 | 140325 | 157584 | 56 |
| 5 | 001454 | 018907 | 036353 | 053788 | 071207 | 088605 | 105975 | 123313 | 140613 | 157871 | 55 |
| 6 | 001745 | 019197 | 036644 | 054079 | 071497 | 088894 | 106264 | 123601 | 140901 | 158158 | 54 |
| 7 | 002036 | 019488 | 036934 | 054369 | 071788 | 089184 | 106553 | 123890 | 141189 | 158445 | 53 |
| 8 | 002327 | 019779 | 037225 | 054660 | 072078 | 089474 | 106843 | 124179 | 141477 | 158732 | 52 |
| 9 | 002618 | 020070 | 037516 | 054950 | 072368 | 089763 | 107132 | 124467 | 141765 | 159020 | 51 |
| 10 | 002909 | 020361 | 037806 | 055241 | 072658 | 090053 | 107421 | 124756 | 142053 | 159307 | 50 |
| 11 | 003200 | 020652 | 038097 | 055531 | 072948 | 090343 | 107710 | 125045 | 142341 | 159594 | 49 |
| 12 | 003491 | 020942 | 038388 | 055822 | 073238 | 090633 | 107999 | 125333 | 142629 | 159881 | 48 |
| 13 | 003782 | 021233 | 038678 | 056112 | 073528 | 090922 | 108289 | 125622 | 142917 | 160168 | 47 |
| 14 | 004072 | 021524 | 038969 | 056402 | 073818 | 091212 | 108578 | 125910 | 143205 | 160455 | 46 |
| 15 | 004363 | 021815 | 039260 | 056693 | 074108 | 091502 | 108867 | 126199 | 143403 | 160743 | 45 |
| 16 | 004654 | 022106 | 039550 | 056983 | 074399 | 091791 | 109156 | 126488 | 143780 | 61030 | 44 |
| 17 | 004945 | 022397 | 039841 | 057274 | 074689 | 092081 | 109445 | 126776 | 144068 | 161317 | 43 |
| 18 | 005236 | 022687 | 040132 | 057564 | 074979 | 092371 | 109734 | 127065 | 144356 | 161604 | 42 |
| 19 | 005527 | 022978 | 040422 | 057854 | 075269 | 092660 | 110023 | 127353 | 144644 | 161891 | 41 |
| 20 | 005818 | 023269 | 040713 | 058145 | 075559 | 092950 | 110313 | 127642 | 144932 | 162178 | 40 |
| 21 | 006109 | 023560 | 041004 | 058435 | 075849 | 093239 | 110602 | 127930 | 145220 | 162465 | 39 |
| 22 | 006399 | 023851 | 041294 | 058726 | 076139 | 093529 | 110891 | 128219 | 145507 | 162752 | 38 |
| 23 | 006690 | 024141 | 041585 | 059016 | 076429 | 093819 | 111180 | 128507 | 145795 | 163039 | 37 |
| 24 | 006981 | 024432 | 041876 | 059306 | 076719 | 094108 | 111469 | 128796 | 146083 | 163326 | 36 |
| 25 | 007272 | 024723 | 042166 | 059597 | 077009 | 094398 | 111758 | 129084 | 146371 | 163613 | 35 |
| 26 | 007563 | 025014 | 042457 | 059887 | 077299 | 094687 | 112047 | 129373 | 146659 | 163900 | 34 |
| 27 | 007854 | 025305 | 042748 | 060177 | 077589 | 094977 | 112336 | 129661 | 146946 | 164187 | 33 |
| 28 | 008145 | 025595 | 043038 | 060468 | 077879 | 095267 | 112625 | 129949 | 147234 | 164474 | 32 |
| 29 | 008436 | 025886 | 043329 | 060758 | 078169 | 095556 | 112914 | 130238 | 147522 | 164761 | 31 |
| 30 | 008727 | 026177 | 043619 | 061049 | 078459 | 095846 | 113203 | 130526 | 147809 | 165048 | 30 |
| 31 | 009017 | 026468 | 043910 | 061339 | 078749 | 096135 | 113492 | 130815 | 148097 | 165334 | 29 |
| 32 | 009308 | 026759 | 044201 | 061629 | 079039 | 096425 | 113781 | 131103 | 148385 | 165621 | 28 |
| 33 | 009599 | 027049 | 044491 | 061920 | 079329 | 096714 | 114070 | 131391 | 148672 | 165908 | 27 |
| 34 | 009890 | 027340 | 044782 | 062210 | 079619 | 097004 | 114359 | 131680 | 148960 | 166195 | 26 |
| 35 | 010181 | 027631 | 045072 | 062500 | 079909 | 097293 | 114648 | 131968 | 149248 | 166482 | 25 |
| 36 | 010472 | 027922 | 045363 | 062791 | 080199 | 097583 | 114937 | 132256 | 149535 | 166769 | 24 |
| 37 | 010763 | 028212 | 045654 | 063081 | 080489 | 097872 | 115226 | 132545 | 149823 | 167056 | 23 |
| 38 | 011054 | 028503 | 045944 | 063371 | 080779 | 098162 | 115515 | 132833 | 150111 | 167342 | 22 |
| 39 | 011344 | 028794 | 046235 | 063661 | 081069 | 098451 | 115804 | 133121 | 150398 | 167629 | 21 |
| 40 | 011635 | 029085 | 046525 | 063952 | 081359 | 098741 | 116093 | 133410 | 150686 | 167916 | 20 |
| 41 | 011926 | 029375 | 046816 | 064242 | 081649 | 099030 | 116382 | 133698 | 150973 | 168203 | 19 |
| 42 | 012217 | 029666 | 047106 | 064532 | 081939 | 099320 | 116671 | 133986 | 151261 | 168489 | 18 |
| 43 | 012508 | 029957 | 047397 | 064823 | 082228 | 099609 | 116960 | 134274 | 151548 | 168776 | 17 |
| 44 | 012799 | 030248 | 047688 | 065113 | 082518 | 099899 | 117249 | 134563 | 151836 | 169063 | 16 |
| 45 | 013090 | 030539 | 047978 | 065403 | 082808 | 100188 | 117537 | 134851 | 152123 | 169350 | 15 |
| 46 | 013380 | 030829 | 048269 | 065693 | 083098 | 100477 | 117826 | 135139 | 152411 | 169636 | 14 |
| 47 | 013671 | 031120 | 048559 | 065984 | 083388 | 100767 | 118115 | 135427 | 152698 | 169923 | 13 |
| 48 | 013962 | 031411 | 048850 | 066274 | 083678 | 101056 | 118404 | 135716 | 152986 | 170209 | 12 |
| 49 | 014253 | 031702 | 049140 | 066564 | 083968 | 101346 | 118693 | 136004 | 153273 | 170496 | 11 |
| 50 | 014544 | 031992 | 049431 | 066854 | 084258 | 101635 | 118982 | 136292 | 153561 | 170783 | 10 |
| 51 | 014835 | 032283 | 049721 | 067145 | 084547 | 101924 | 119270 | 136580 | 153848 | 171069 | 9 |
| 52 | 015126 | 032574 | 050012 | 067435 | 084837 | 102214 | 119559 | 136868 | 154136 | 171356 | 8 |
| 53 | 015416 | 032864 | 050302 | 067725 | 085127 | 102503 | 119848 | 137156 | 154423 | 171643 | 7 |
| 54 | 015707 | 033155 | 050593 | 068015 | 085417 | 102793 | 120137 | 137445 | 154710 | 171929 | 6 |
| 55 | 015998 | 033446 | 050883 | 068306 | 085707 | 103082 | 120426 | 137733 | 154998 | 172316 | 5 |
| 56 | 016289 | 033737 | 051174 | 068596 | 085997 | 103371 | 120714 | 138021 | 155285 | 172502 | 4 |
| 57 | 016580 | 034027 | 051464 | 068886 | 086286 | 103661 | 121003 | 138309 | 155572 | 172789 | 3 |
| 58 | 016871 | 034318 | 051755 | 069176 | 086576 | 103950 | 121292 | 138597 | 55860 | 173075 | 2 |
| 59 | 017162 | 034609 | 052045 | 069466 | 086866 | 104239 | 121581 | 138885 | 56147 | 173362 | 1 |
| 60 | 017452 | 034899 | 052336 | 069756 | 087156 | 104528 | 121869 | 139173 | 56434 | 173648 | 0 |
| M | 89° | 88° | 87° | 86° | 85° | 84° | 83° | 82° | 81° | 80° | M |

Natural Co-sines.

# NATURAL SINES.

| M | 10° | 11° | 12° | 13° | 14° | 15° | 16° | 17° | 18° | 19° | M |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 173648 | 190809 | 207912 | 224951 | 241922 | 258819 | 275637 | 292372 | 309017 | 325568 | 60 |
| 1 | 173935 | 191095 | 208196 | 225234 | 242204 | 259100 | 275917 | 292650 | 309294 | 325843 | 59 |
| 2 | 174221 | 191380 | 208480 | 225518 | 242486 | 259381 | 276197 | 292928 | 309570 | 326118 | 58 |
| 3 | 174508 | 191666 | 208765 | 225801 | 242769 | 259662 | 276476 | 293206 | 309847 | 326393 | 57 |
| 4 | 174794 | 191951 | 209050 | 226085 | 243051 | 259943 | 276756 | 293484 | 310123 | 326668 | 56 |
| 5 | 175080 | 192237 | 209334 | 226368 | 243333 | 260224 | 277035 | 293762 | 310400 | 326943 | 55 |
| 6 | 175367 | 192522 | 209619 | 226651 | 243615 | 260505 | 277315 | 294040 | 310676 | 327218 | 54 |
| 7 | 175653 | 192807 | 209903 | 226935 | 243897 | 260785 | 277594 | 294318 | 310953 | 327493 | 53 |
| 8 | 175939 | 193093 | 210187 | 227218 | 244179 | 261066 | 277874 | 294596 | 311229 | 327768 | 52 |
| 9 | 176226 | 193378 | 210472 | 227501 | 244461 | 261347 | 278153 | 294874 | 311506 | 328042 | 51 |
| 10 | 176512 | 193664 | 210756 | 227784 | 244743 | 261628 | 278432 | 295152 | 311782 | 328317 | 50 |
| 11 | 176798 | 193949 | 211040 | 228068 | 245025 | 261908 | 278712 | 295430 | 312059 | 328592 | 49 |
| 12 | 177085 | 194234 | 211325 | 228351 | 245307 | 262189 | 278991 | 295708 | 312335 | 328867 | 48 |
| 13 | 177371 | 194520 | 211609 | 228634 | 245589 | 262470 | 279270 | 295986 | 312611 | 329141 | 47 |
| 14 | 177657 | 194805 | 211893 | 228917 | 245871 | 262751 | 279550 | 296264 | 312888 | 329416 | 46 |
| 15 | 177944 | 195090 | 212178 | 229200 | 246153 | 263031 | 279829 | 296542 | 313064 | 329691 | 45 |
| 16 | 178230 | 195376 | 212462 | 229484 | 246435 | 263302 | 280108 | 296819 | 313440 | 329965 | 44 |
| 17 | 178516 | 195661 | 212746 | 229767 | 246717 | 263592 | 280388 | 297097 | 313716 | 330240 | 43 |
| 18 | 178802 | 195946 | 213030 | 230050 | 246999 | 263873 | 280667 | 297375 | 313992 | 330514 | 42 |
| 19 | 179088 | 196231 | 213315 | 230333 | 247281 | 264154 | 280946 | 297653 | 314269 | 330789 | 41 |
| 20 | 179375 | 196517 | 213599 | 230616 | 247563 | 264434 | 281225 | 297930 | 314545 | 331063 | 40 |
| 21 | 179661 | 196802 | 213 83 | 230899 | 247845 | 264715 | 281504 | 298208 | 314821 | 331338 | 39 |
| 22 | 179947 | 197087 | 214167 | 231182 | 248126 | 264095 | 281783 | 298486 | 315097 | 331612 | 38 |
| 23 | 180233 | 197372 | 214451 | 231465 | 248408 | 265276 | 282062 | 298763 | 315373 | 331887 | 37 |
| 24 | 180519 | 197657 | 214735 | 231748 | 248690 | 265556 | 282341 | 299041 | 315649 | 332161 | 36 |
| 25 | 180805 | 197942 | 215019 | 232031 | 248972 | 265837 | 282620 | 299318 | 315925 | 332435 | 35 |
| 26 | 181091 | 198228 | 215303 | 232314 | 249253 | 266117 | 282900 | 299596 | 316201 | 332710 | 34 |
| 27 | 181377 | 198513 | 215588 | 232597 | 249535 | 266397 | 283179 | 299873 | 316477 | 332984 | 33 |
| 28 | 181663 | 198798 | 215872 | 232880 | 249817 | 266678 | 283457 | 300151 | 316753 | 333258 | 32 |
| 29 | 181950 | 199083 | 216156 | 233163 | 250098 | 266958 | 283736 | 300428 | 317029 | 333533 | 31 |
| 30 | 182236 | 199368 | 216440 | 233445 | 250380 | 267238 | 284015 | 300706 | 317305 | 333807 | 30 |
| 31 | 182522 | 199653 | 216724 | 233728 | 250662 | 267519 | 284294 | 300983 | 317580 | 334081 | 29 |
| 32 | 182808 | 199938 | 217008 | 234011 | 250943 | 267799 | 284573 | 301261 | 317856 | 334355 | 28 |
| 33 | 183094 | 200223 | 217292 | 234294 | 251225 | 268079 | 284852 | 301538 | 318132 | 334629 | 27 |
| 34 | 183379 | 200508 | 217575 | 234577 | 251506 | 268359 | 285131 | 301815 | 318408 | 334903 | 26 |
| 35 | 183665 | 200793 | 217859 | 234859 | 251788 | 268640 | 285410 | 302093 | 318684 | 335178 | 25 |
| 36 | 183951 | 201078 | 218143 | 235142 | 252069 | 268920 | 285688 | 302370 | 318959 | 335452 | 24 |
| 37 | 184237 | 201363 | 218427 | 235425 | 252351 | 269200 | 285967 | 302647 | 319225 | 335726 | 23 |
| 38 | 184523 | 201648 | 218711 | 235708 | 252632 | 269480 | 286246 | 302924 | 319511 | 336000 | 22 |
| 39 | 184809 | 201933 | 218995 | 235990 | 252914 | 269760 | 286525 | 303202 | 319786 | 336274 | 21 |
| 40 | 185095 | 202218 | 219279 | 236273 | 253195 | 270040 | 286803 | 303479 | 320062 | 336547 | 20 |
| 41 | 185380 | 202502 | 219562 | 236556 | 253477 | 270320 | 287082 | 303756 | 320337 | 336821 | 19 |
| 42 | 185667 | 202787 | 219846 | 236838 | 253758 | 270600 | 287361 | 304033 | 320613 | 337095 | 18 |
| 43 | 185952 | 203072 | 220130 | 237121 | 254039 | 270880 | 287639 | 304310 | 320889 | 337369 | 17 |
| 44 | 186238 | 203357 | 220414 | 237403 | 254321 | 271160 | 287918 | 304587 | 321164 | 337643 | 16 |
| 45 | 186524 | 203642 | 220697 | 237686 | 254602 | 271440 | 288196 | 304864 | 321439 | 337917 | 15 |
| 46 | 186810 | 203927 | 220981 | 237968 | 254883 | 271720 | 288475 | 305141 | 321715 | 338190 | 14 |
| 47 | 187096 | 204211 | 221265 | 238251 | 255165 | 272000 | 288753 | 305418 | 321990 | 338464 | 13 |
| 48 | 187381 | 204496 | 221548 | 238533 | 255446 | 272280 | 289032 | 305695 | 322266 | 338738 | 12 |
| 49 | 187667 | 204781 | 221832 | 238816 | 255727 | 272560 | 289310 | 305972 | 322541 | 339012 | 11 |
| 50 | 187953 | 205065 | 222116 | 239098 | 256008 | 272840 | 289589 | 306249 | 322816 | 339285 | 10 |
| 51 | 188238 | 205350 | 222399 | 239381 | 256289 | 273120 | 289867 | 306526 | 323092 | 339559 | 9 |
| 52 | 188524 | 205635 | 222683 | 239663 | 256571 | 273400 | 290145 | 306803 | 323367 | 339832 | 8 |
| 53 | 188810 | 205920 | 222967 | 239946 | 256852 | 273679 | 290424 | 307080 | 323642 | 340106 | 7 |
| 54 | 189095 | 206204 | 223250 | 240228 | 257133 | 273959 | 290702 | 307357 | 323917 | 340380 | 6 |
| 55 | 189381 | 206489 | 223534 | 240510 | 257414 | 274239 | 290981 | 307633 | 324193 | 340653 | 5 |
| 56 | 189667 | 206773 | 223817 | 240793 | 257695 | 274519 | 291259 | 307910 | 324468 | 340927 | 4 |
| 57 | 189952 | 207058 | 224101 | 241075 | 257976 | 274798 | 291537 | 308187 | 324743 | 341200 | 3 |
| 58 | 190238 | 207343 | 224384 | 241357 | 258257 | 275078 | 291815 | 308464 | 325018 | 341473 | 2 |
| 59 | 190523 | 207627 | 224668 | 241640 | 258538 | 275358 | 292094 | 308740 | 325293 | 341747 | 1 |
| 60 | 190809 | 207912 | 224951 | 241922 | 258819 | 275637 | 292372 | 309017 | 325568 | 342020 | 0 |
| M | 79° | 78° | 77° | 76° | 75° | 74° | 73° | 72° | 71° | 70° | M |

Natural Co-sines.

# NATURAL SINES.

| M | 20° | 21° | 22° | 23° | 24° | 25° | 26° | 27° | 28° | 29° | M |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 342020 | 358368 | 374607 | 390731 | 406737 | 422618 | 438371 | 453990 | 469472 | 484810 | 60 |
| 1 | 342293 | 358640 | 374876 | 390990 | 407002 | 422882 | 438633 | 454250 | 469728 | 485064 | 59 |
| 2 | 342567 | 358911 | 375146 | 391267 | 407268 | 423145 | 438894 | 454509 | 469985 | 485318 | 58 |
| 3 | 342840 | 359183 | 375416 | 391534 | 407534 | 423409 | 439155 | 454768 | 470242 | 485573 | 57 |
| 4 | 343113 | 359454 | 375685 | 391802 | 407799 | 423673 | 439417 | 455027 | 470499 | 485827 | 56 |
| 5 | 343387 | 359725 | 375955 | 392070 | 408065 | 423936 | 439678 | 455286 | 470755 | 486081 | 55 |
| 6 | 343660 | 359997 | 376224 | 392337 | 408330 | 424199 | 439939 | 455545 | 471012 | 486335 | 54 |
| 7 | 343933 | 360268 | 376494 | 392605 | 408596 | 424463 | 440200 | 455804 | 471268 | 486590 | 53 |
| 8 | 344206 | 360540 | 376763 | 392872 | 408861 | 424726 | 440462 | 456063 | 471525 | 486844 | 52 |
| 9 | 344479 | 360811 | 377033 | 393140 | 409127 | 424990 | 440723 | 456322 | 471782 | 487098 | 51 |
| 10 | 344752 | 361082 | 377302 | 393407 | 409392 | 425253 | 440984 | 456580 | 472038 | 487352 | 50 |
| 11 | 345025 | 361353 | 377571 | 393675 | 409658 | 425516 | 441245 | 456839 | 472294 | 487606 | 49 |
| 12 | 345298 | 361625 | 377841 | 393942 | 409923 | 425779 | 441506 | 457098 | 472551 | 487860 | 48 |
| 13 | 345571 | 361896 | 378110 | 394209 | 410188 | 426042 | 441767 | 457357 | 472807 | 488114 | 47 |
| 14 | 345844 | 362167 | 378379 | 394477 | 410454 | 426306 | 442028 | 457615 | 473063 | 488367 | 46 |
| 15 | 346117 | 362438 | 378649 | 394744 | 410719 | 426569 | 442289 | 457874 | 473320 | 488621 | 45 |
| 16 | 346390 | 362709 | 378918 | 395011 | 410984 | 426832 | 442550 | 458133 | 473576 | 488875 | 44 |
| 17 | 346663 | 362980 | 379187 | 395278 | 411249 | 427095 | 442810 | 458391 | 473832 | 489129 | 43 |
| 18 | 346936 | 363251 | 379456 | 395546 | 411514 | 427358 | 443071 | 458650 | 474088 | 489382 | 42 |
| 19 | 347208 | 363522 | 379725 | 395813 | 411779 | 427621 | 443332 | 458908 | 474344 | 489636 | 41 |
| 20 | 347481 | 363793 | 379994 | 396080 | 412045 | 427884 | 443593 | 459166 | 474600 | 489890 | 40 |
| 21 | 347754 | 364064 | 380263 | 396347 | 412310 | 428147 | 443853 | 459425 | 474856 | 490143 | 39 |
| 22 | 348027 | 364335 | 380532 | 396614 | 412575 | 428410 | 444114 | 459683 | 475112 | 490397 | 38 |
| 23 | 348299 | 364606 | 380801 | 396881 | 412840 | 428672 | 444375 | 459942 | 475368 | 490650 | 37 |
| 24 | 348572 | 364877 | 381070 | 397148 | 413104 | 428935 | 444635 | 460200 | 475624 | 490904 | 36 |
| 25 | 348845 | 365148 | 381339 | 397415 | 413369 | 429198 | 444896 | 460458 | 475880 | 491157 | 35 |
| 26 | 349117 | 365418 | 381608 | 397682 | 413634 | 429461 | 445156 | 460716 | 476136 | 491411 | 34 |
| 27 | 349390 | 365689 | 381877 | 397949 | 413899 | 429723 | 445417 | 460974 | 476392 | 491664 | 33 |
| 28 | 349662 | 365960 | 382146 | 398215 | 414164 | 429986 | 445677 | 461232 | 476647 | 491917 | 32 |
| 29 | 349935 | 366231 | 382415 | 398482 | 414429 | 430249 | 445937 | 461491 | 476903 | 492170 | 31 |
| 30 | 350207 | 366501 | 382683 | 398749 | 414693 | 430511 | 446198 | 461749 | 477159 | 492424 | 30 |
| 31 | 350480 | 366772 | 382952 | 399016 | 414958 | 430774 | 446458 | 462007 | 477414 | 492677 | 29 |
| 32 | 350752 | 367042 | 383221 | 399283 | 415223 | 431036 | 446718 | 462265 | 477670 | 492930 | 28 |
| 33 | 351025 | 367313 | 383490 | 399549 | 415487 | 431299 | 446979 | 462523 | 477925 | 493183 | 27 |
| 34 | 351297 | 367584 | 383758 | 399816 | 415752 | 431561 | 447239 | 462780 | 478181 | 493436 | 26 |
| 35 | 351569 | 367854 | 384027 | 400082 | 416016 | 431823 | 447499 | 463038 | 478436 | 493689 | 25 |
| 36 | 351842 | 368125 | 384295 | 400349 | 416281 | 432086 | 447759 | 463296 | 478692 | 493942 | 24 |
| 37 | 352114 | 368395 | 384564 | 400606 | 416545 | 432348 | 448019 | 463554 | 478947 | 494195 | 23 |
| 38 | 352386 | 368665 | 384832 | 400882 | 416810 | 432610 | 448279 | 463812 | 479203 | 494448 | 22 |
| 39 | 352658 | 368936 | 385101 | 401149 | 417074 | 432873 | 448539 | 464069 | 479458 | 494700 | 21 |
| 40 | 352931 | 369206 | 385369 | 401415 | 417338 | 433135 | 448799 | 464327 | 479713 | 494953 | 20 |
| 41 | 353203 | 369476 | 385638 | 401681 | 417603 | 433397 | 449059 | 464584 | 479968 | 495206 | 19 |
| 42 | 353475 | 369747 | 385906 | 401948 | 417867 | 433659 | 449319 | 464842 | 480223 | 495459 | 18 |
| 43 | 353747 | 370017 | 386174 | 402214 | 418131 | 433921 | 449579 | 465100 | 480479 | 495711 | 17 |
| 44 | 354019 | 370287 | 386443 | 402480 | 418396 | 434183 | 449839 | 465357 | 480734 | 495964 | 16 |
| 45 | 354291 | 370557 | 386711 | 402747 | 418660 | 434445 | 450098 | 465615 | 480989 | 496217 | 15 |
| 46 | 354563 | 370828 | 386979 | 403013 | 418924 | 434707 | 450358 | 465872 | 481244 | 496469 | 14 |
| 47 | 354835 | 371098 | 387247 | 403279 | 419188 | 434969 | 450618 | 466129 | 481499 | 496722 | 13 |
| 48 | 355107 | 371368 | 387516 | 403545 | 419452 | 435231 | 450878 | 466387 | 481754 | 496974 | 12 |
| 49 | 355379 | 371638 | 387784 | 403811 | 419716 | 435493 | 451137 | 466644 | 482009 | 497226 | 11 |
| 50 | 355651 | 371908 | 388052 | 404078 | 419980 | 435755 | 451397 | 466901 | 482263 | 497479 | 10 |
| 51 | 355923 | 372078 | 388320 | 404344 | 420244 | 436017 | 451656 | 467158 | 482518 | 497731 | 9 |
| 52 | 356194 | 372448 | 388588 | 404610 | 420508 | 436278 | 451916 | 467416 | 482773 | 497983 | 8 |
| 53 | 356466 | 372718 | 388856 | 404876 | 420772 | 436540 | 452175 | 467673 | 483028 | 498236 | 7 |
| 54 | 356738 | 372988 | 389124 | 405142 | 421036 | 436802 | 452435 | 467930 | 483282 | 498488 | 6 |
| 55 | 357010 | 373258 | 389392 | 405408 | 421300 | 437063 | 452694 | 468187 | 483537 | 498740 | 5 |
| 56 | 357281 | 373528 | 389660 | 405673 | 421563 | 437325 | 452953 | 468444 | 483792 | 498992 | 4 |
| 57 | 357553 | 373797 | 389928 | 405939 | 421827 | 437587 | 453213 | 468701 | 484046 | 499244 | 3 |
| 58 | 357825 | 374067 | 390196 | 406205 | 422091 | 437848 | 453472 | 468958 | 484301 | 499496 | 2 |
| 59 | 358096 | 374337 | 390463 | 406470 | 422355 | 438110 | 453731 | 469215 | 484555 | 499748 | 1 |
| 60 | 358368 | 374607 | 390731 | 406737 | 422618 | 438371 | 453990 | 469472 | 484810 | 500000 | 0 |
| M | 69° | 68° | 67° | 66° | 65° | 64° | 63° | 62° | 61° | 60° | M |

Natural Co-sines.

| M | 30° | 31° | 32° | 33° | 34° | 35° | 36° | 37° | 38° | 39° | M |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 500000 | 515038 | 529919 | 544639 | 559193 | 573576 | 587785 | 601815 | 615661 | 629320 | 60 |
| 1 | 500252 | 515287 | 530166 | 544883 | 559434 | 573815 | 588021 | 602047 | 615891 | 629546 | 59 |
| 2 | 500504 | 515537 | 530413 | 545127 | 559675 | 574053 | 588256 | 602280 | 616120 | 629772 | 58 |
| 3 | 500756 | 515786 | 530659 | 545371 | 559916 | 574291 | 588491 | 602512 | 616349 | 629998 | 57 |
| 4 | 501007 | 516035 | 530906 | 545615 | 560157 | 574529 | 588726 | 602744 | 616578 | 630224 | 56 |
| 5 | 501259 | 516284 | 531152 | 545858 | 560398 | 574767 | 588961 | 602976 | 616807 | 630450 | 55 |
| 6 | 501511 | 516533 | 531399 | 546102 | 560639 | 575005 | 589196 | 603208 | 617036 | 630676 | 54 |
| 7 | 501762 | 516782 | 531645 | 546346 | 560880 | 575243 | 589431 | 603440 | 617265 | 630902 | 53 |
| 8 | 502014 | 517031 | 531891 | 546589 | 561121 | 575481 | 589666 | 603672 | 617494 | 631127 | 52 |
| 9 | 502266 | 517280 | 532138 | 546833 | 561361 | 575719 | 589901 | 603904 | 617722 | 631353 | 51 |
| 10 | 502517 | 517529 | 532384 | 547076 | 561602 | 575957 | 590136 | 604136 | 617951 | 631578 | 50 |
| 11 | 502769 | 517778 | 532630 | 547320 | 561843 | 576195 | 590371 | 604367 | 618180 | 631804 | 49 |
| 12 | 503020 | 518027 | 532876 | 547563 | 562083 | 576432 | 590606 | 604599 | 618408 | 632029 | 48 |
| 13 | 503271 | 518276 | 533122 | 547807 | 562324 | 576670 | 590840 | 604831 | 618637 | 632255 | 47 |
| 14 | 503523 | 518525 | 533368 | 548050 | 562564 | 576908 | 591075 | 605062 | 618865 | 632480 | 46 |
| 15 | 503774 | 518773 | 533615 | 548293 | 562805 | 577145 | 591310 | 605294 | 619094 | 632705 | 45 |
| 16 | 504025 | 519022 | 533867 | 548536 | 563045 | 577383 | 591544 | 605526 | 619322 | 632931 | 44 |
| 17 | 504276 | 519271 | 534106 | 548780 | 563286 | 577620 | 591779 | 605757 | 619551 | 633156 | 43 |
| 18 | 504528 | 519519 | 534352 | 549023 | 563526 | 577858 | 592013 | 605988 | 619779 | 633381 | 42 |
| 19 | 504779 | 519768 | 534598 | 549266 | 563766 | 578095 | 592248 | 606220 | 620007 | 633606 | 41 |
| 20 | 505030 | 520016 | 534844 | 549509 | 564007 | 578332 | 592482 | 606451 | 620235 | 633831 | 40 |
| 21 | 505281 | 520265 | 535090 | 549752 | 564247 | 578570 | 592716 | 606682 | 620464 | 634056 | 39 |
| 22 | 505532 | 520513 | 535335 | 549995 | 564487 | 578807 | 592951 | 606914 | 620692 | 634281 | 38 |
| 23 | 505783 | 520761 | 535581 | 550238 | 564727 | 579044 | 593185 | 607145 | 620920 | 634506 | 37 |
| 24 | 506034 | 521010 | 535827 | 550481 | 564967 | 579281 | 593419 | 607376 | 621148 | 634731 | 36 |
| 25 | 506285 | 521258 | 536072 | 550724 | 565207 | 579518 | 593653 | 607607 | 621376 | 634955 | 35 |
| 26 | 506535 | 521506 | 536318 | 550966 | 565447 | 579755 | 593887 | 607838 | 621604 | 635180 | 34 |
| 27 | 506786 | 521754 | 536563 | 551209 | 565687 | 579992 | 594121 | 608069 | 621831 | 635405 | 33 |
| 28 | 507037 | 522002 | 536809 | 551452 | 565927 | 580229 | 594355 | 608300 | 622059 | 635629 | 32 |
| 29 | 507288 | 522251 | 537054 | 551694 | 566166 | 580466 | 594589 | 608531 | 622287 | 635854 | 31 |
| 30 | 507538 | 522499 | 537300 | 551937 | 566406 | 580703 | 594823 | 608761 | 622515 | 636078 | 30 |
| 31 | 507789 | 522747 | 537545 | 552180 | 566646 | 580940 | 595057 | 608992 | 622742 | 636303 | 29 |
| 32 | 508040 | 522995 | 537790 | 552422 | 566886 | 581176 | 595290 | 609223 | 622970 | 636527 | 28 |
| 33 | 508290 | 523242 | 538035 | 552664 | 567125 | 581413 | 595524 | 609454 | 623197 | 636751 | 27 |
| 34 | 508541 | 523490 | 538281 | 552907 | 567365 | 581650 | 595758 | 609684 | 623425 | 636976 | 26 |
| 35 | 508791 | 523738 | 538526 | 553149 | 567604 | 581886 | 595991 | 609915 | 623652 | 637200 | 25 |
| 36 | 509041 | 523986 | 538771 | 553392 | 567844 | 582123 | 596225 | 610145 | 623880 | 637424 | 24 |
| 37 | 509292 | 524234 | 539016 | 553634 | 568083 | 582359 | 596458 | 610376 | 624107 | 637648 | 23 |
| 38 | 509542 | 524481 | 539261 | 553876 | 568323 | 582596 | 596692 | 610606 | 624334 | 637872 | 22 |
| 39 | 509792 | 524729 | 539506 | 554118 | 568562 | 582832 | 596925 | 610836 | 624561 | 638096 | 21 |
| 40 | 510043 | 524977 | 539751 | 554360 | 568801 | 583069 | 597159 | 611067 | 624789 | 638320 | 20 |
| 41 | 510293 | 525224 | 539996 | 554602 | 569040 | 583305 | 597392 | 611297 | 625016 | 638544 | 19 |
| 42 | 510543 | 525472 | 540240 | 554844 | 569280 | 583541 | 597625 | 611527 | 625243 | 638768 | 18 |
| 43 | 510793 | 525719 | 540485 | 555086 | 569519 | 583777 | 597858 | 611757 | 625470 | 638992 | 17 |
| 44 | 511043 | 525967 | 540730 | 555328 | 559758 | 584014 | 598092 | 611987 | 625697 | 639215 | 16 |
| 45 | 511293 | 526214 | 540974 | 555570 | 569997 | 584250 | 598325 | 612217 | 625923 | 639439 | 15 |
| 46 | 511543 | 526461 | 541219 | 555812 | 570236 | 584486 | 598558 | 612447 | 626150 | 639663 | 14 |
| 47 | 511793 | 526709 | 541464 | 556054 | 570475 | 584722 | 598791 | 612677 | 626377 | 639886 | 13 |
| 48 | 512043 | 526956 | 541708 | 556296 | 570714 | 584958 | 599024 | 612907 | 626604 | 640110 | 12 |
| 49 | 512293 | 527203 | 541953 | 556537 | 570952 | 585194 | 599256 | 613137 | 626830 | 640333 | 11 |
| 50 | 512543 | 527450 | 542197 | 556779 | 571191 | 585429 | 599489 | 613367 | 627057 | 640557 | 10 |
| 51 | 512792 | 527697 | 542442 | 557021 | 571430 | 585665 | 599722 | 613596 | 627284 | 640780 | 9 |
| 52 | 513042 | 527944 | 542686 | 557262 | 571669 | 585901 | 599955 | 613826 | 627510 | 641003 | 8 |
| 53 | 513292 | 528191 | 542930 | 557504 | 571907 | 586137 | 600188 | 614056 | 627737 | 641226 | 7 |
| 54 | 513541 | 528438 | 543174 | 557745 | 572146 | 586372 | 600420 | 614285 | 627963 | 641450 | 6 |
| 55 | 513791 | 528685 | 543419 | 557987 | 572384 | 586608 | 600653 | 614515 | 628189 | 641673 | 5 |
| 56 | 514040 | 528932 | 543663 | 558228 | 572623 | 586844 | 600885 | 614744 | 628416 | 641896 | 4 |
| 57 | 514290 | 529179 | 543907 | 558469 | 572861 | 587079 | 601118 | 614974 | 628642 | 642119 | 3 |
| 58 | 514539 | 529426 | 544151 | 558710 | 573100 | 587314 | 601350 | 615203 | 628868 | 642342 | 2 |
| 59 | 514789 | 529673 | 544395 | 558952 | 573338 | 587550 | 601583 | 615432 | 629094 | 642565 | 1 |
| 60 | 515038 | 529919 | 544639 | 559193 | 573576 | 587785 | 601815 | 615661 | 629320 | 642788 | 0 |
| M | 59° | 58° | 57° | 56° | 55° | 54° | 53° | 52° | 51° | 50° | M |

Natural Co-sines.

# NATURAL SINES.

| M | 40° | 41° | 42° | 43° | 44° | 45° | 46° | 47° | 48° | 49° | M |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 642788 | 656059 | 669131 | 681998 | 694658 | 707107 | 719340 | 731354 | 743145 | 754710 | 60 |
| 1 | 643010 | 656279 | 669347 | 682211 | 694868 | 707312 | 719542 | 731552 | 743339 | 754900 | 59 |
| 2 | 643233 | 656498 | 669563 | 682424 | 695077 | 707518 | 719744 | 731750 | 743534 | 755091 | 58 |
| 3 | 643456 | 656717 | 669779 | 682636 | 695286 | 707723 | 719946 | 731949 | 743728 | 755282 | 57 |
| 4 | 643679 | 656937 | 669995 | 682849 | 695495 | 707929 | 720148 | 732147 | 743923 | 755472 | 56 |
| 5 | 643901 | 657156 | 670211 | 683061 | 695704 | 708134 | 720349 | 732345 | 744117 | 755663 | 55 |
| 6 | 644124 | 657375 | 670427 | 683274 | 695913 | 708340 | 720551 | 732543 | 744312 | 755853 | 54 |
| 7 | 644346 | 657594 | 670642 | 683486 | 696122 | 708545 | 720753 | 732741 | 744506 | 756044 | 53 |
| 8 | 644569 | 657814 | 670858 | 683698 | 696330 | 708750 | 720954 | 732939 | 744700 | 756234 | 52 |
| 9 | 644791 | 658033 | 671074 | 683911 | 696539 | 708956 | 721156 | 733137 | 744894 | 756425 | 51 |
| 10 | 645013 | 658252 | 671289 | 684123 | 696748 | 709161 | 721357 | 733334 | 745088 | 756615 | 50 |
| 11 | 645236 | 658471 | 671505 | 684335 | 696957 | 709366 | 721559 | 733532 | 745282 | 756805 | 49 |
| 12 | 645458 | 658689 | 671721 | 684547 | 697165 | 709571 | 721760 | 733730 | 745476 | 756995 | 48 |
| 13 | 645680 | 658908 | 671936 | 684759 | 697374 | 709776 | 721962 | 733927 | 745670 | 757185 | 47 |
| 14 | 645902 | 659127 | 672151 | 684971 | 697582 | 709981 | 722163 | 734125 | 745864 | 757375 | 46 |
| 15 | 646124 | 659346 | 672367 | 685183 | 697790 | 710185 | 722364 | 734323 | 746057 | 757565 | 45 |
| 16 | 646346 | 659565 | 672582 | 685395 | 697999 | 710390 | 722565 | 734520 | 746251 | 757755 | 44 |
| 17 | 646568 | 659783 | 672797 | 685607 | 698207 | 710595 | 722766 | 734717 | 746445 | 757945 | 43 |
| 18 | 646790 | 660002 | 673013 | 685818 | 698415 | 710799 | 722967 | 734915 | 746638 | 758134 | 42 |
| 19 | 647012 | 660220 | 673228 | 686030 | 698623 | 711004 | 723168 | 735112 | 746832 | 758324 | 41 |
| 20 | 647233 | 660439 | 673443 | 686242 | 698832 | 711209 | 723369 | 735309 | 747025 | 758514 | 40 |
| 21 | 647455 | 660657 | 673658 | 686453 | 699040 | 711413 | 723570 | 735506 | 747218 | 758703 | 39 |
| 22 | 647677 | 660875 | 673873 | 686665 | 699248 | 711617 | 723771 | 735703 | 747412 | 758893 | 38 |
| 23 | 647898 | 661094 | 674088 | 686876 | 699455 | 711822 | 723971 | 735900 | 747605 | 759082 | 37 |
| 24 | 648120 | 661312 | 674302 | 687088 | 699663 | 712026 | 724172 | 736097 | 747798 | 759271 | 36 |
| 25 | 648341 | 661530 | 674517 | 687299 | 699871 | 712230 | 724372 | 736294 | 747991 | 759461 | 35 |
| 26 | 648563 | 661748 | 674732 | 687510 | 700079 | 712434 | 724573 | 736491 | 748184 | 759650 | 34 |
| 27 | 648784 | 661966 | 674947 | 687721 | 700287 | 712639 | 724773 | 736687 | 748377 | 759839 | 33 |
| 28 | 649006 | 662184 | 675161 | 687932 | 700494 | 712843 | 724974 | 736884 | 748570 | 760028 | 32 |
| 29 | 649227 | 662402 | 675376 | 688144 | 700702 | 713047 | 725174 | 737081 | 748763 | 760217 | 31 |
| 30 | 649448 | 662620 | 675590 | 688355 | 700909 | 713250 | 725374 | 737277 | 748956 | 760406 | 30 |
| 31 | 649669 | 662838 | 675805 | 688566 | 701117 | 713454 | 725575 | 737474 | 749148 | 760595 | 29 |
| 32 | 649890 | 663056 | 676019 | 688776 | 701324 | 713658 | 725775 | 737670 | 749341 | 760784 | 28 |
| 33 | 650111 | 663273 | 676233 | 688987 | 701531 | 713862 | 725975 | 737867 | 749534 | 760972 | 27 |
| 34 | 650332 | 663491 | 676448 | 689198 | 701739 | 714066 | 726175 | 738063 | 749726 | 761161 | 26 |
| 35 | 650553 | 663709 | 676662 | 689409 | 701946 | 714269 | 726375 | 738259 | 749919 | 761350 | 25 |
| 36 | 650774 | 663926 | 676876 | 689620 | 702153 | 714473 | 726575 | 738455 | 750111 | 761538 | 24 |
| 37 | 650995 | 664144 | 677090 | 689830 | 702360 | 714676 | 726775 | 738651 | 750303 | 761727 | 23 |
| 38 | 651216 | 664361 | 677304 | 690041 | 702567 | 714880 | 726975 | 738848 | 750496 | 761915 | 22 |
| 39 | 651437 | 664579 | 677518 | 690251 | 702774 | 715083 | 727174 | 739043 | 750688 | 762104 | 21 |
| 40 | 651657 | 664796 | 677732 | 690462 | 702981 | 715286 | 727374 | 739239 | 750880 | 762292 | 20 |
| 41 | 651878 | 665013 | 677946 | 690672 | 703188 | 715490 | 727573 | 739435 | 751072 | 762480 | 19 |
| 42 | 652098 | 665230 | 678160 | 690882 | 703395 | 715693 | 727773 | 739631 | 751264 | 762668 | 18 |
| 43 | 652319 | 665448 | 678373 | 691093 | 703601 | 715896 | 727972 | 739827 | 751456 | 762856 | 17 |
| 44 | 652539 | 665665 | 678587 | 691303 | 703808 | 716099 | 728172 | 740023 | 751648 | 763044 | 16 |
| 45 | 652760 | 665882 | 678801 | 691513 | 704015 | 716302 | 728371 | 740218 | 751840 | 763232 | 15 |
| 46 | 652980 | 666099 | 679014 | 691723 | 704221 | 716505 | 728570 | 740414 | 752032 | 763420 | 14 |
| 47 | 653200 | 666316 | 679228 | 691933 | 704428 | 716708 | 728769 | 740609 | 752223 | 763608 | 13 |
| 48 | 653421 | 666532 | 679441 | 692143 | 704634 | 716911 | 728969 | 740805 | 752415 | 763796 | 12 |
| 49 | 653641 | 666749 | 679654 | 692353 | 704841 | 717113 | 729168 | 741000 | 752606 | 763984 | 11 |
| 50 | 653861 | 666966 | 679868 | 692563 | 705047 | 717316 | 729367 | 741195 | 752798 | 764171 | 10 |
| 51 | 654081 | 667183 | 680081 | 692773 | 705253 | 717519 | 729566 | 741391 | 752989 | 764359 | 9 |
| 52 | 654301 | 667399 | 680295 | 692983 | 705459 | 717721 | 729765 | 741586 | 753181 | 764547 | 8 |
| 53 | 654521 | 667616 | 680508 | 693192 | 705665 | 717924 | 729963 | 741781 | 753372 | 764734 | 7 |
| 54 | 654741 | 667833 | 680721 | 693402 | 705872 | 718126 | 730162 | 741976 | 753563 | 764921 | 6 |
| 55 | 654961 | 668049 | 680934 | 693611 | 706078 | 718329 | 730361 | 742171 | 753755 | 765109 | 5 |
| 56 | 655180 | 668265 | 681147 | 693821 | 706284 | 718531 | 730560 | 742366 | 753946 | 765296 | 4 |
| 57 | 655400 | 668482 | 681360 | 694030 | 706489 | 718733 | 730758 | 742561 | 754137 | 765483 | 3 |
| 58 | 655620 | 668698 | 681573 | 694240 | 706695 | 718936 | 730957 | 742755 | 754328 | 765670 | 2 |
| 59 | 655839 | 668914 | 681786 | 694449 | 706901 | 719138 | 731155 | 742950 | 754519 | 765857 | 1 |
| 60 | 656059 | 669131 | 681998 | 694658 | 707107 | 719340 | 731354 | 743145 | 754710 | 766044 | 0 |
| M | 49° | 48° | 47° | 46° | 45° | 44° | 43° | 42° | 41° | 40° | M |

Natural Co-sines.

| M | 50° | 51° | 52° | 53° | 54° | 55° | 56° | 57° | 58° | 59° | M |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 766044 | 777146 | 788011 | 798636 | 809017 | 819152 | 829038 | 838671 | 848048 | 857167 | 60 |
| 1 | 766231 | 777329 | 788190 | 798811 | 809188 | 819319 | 829200 | 838829 | 848202 | 857317 | 59 |
| 2 | 766418 | 777512 | 788369 | 798985 | 809359 | 819486 | 829363 | 838987 | 848356 | 857467 | 58 |
| 3 | 766605 | 777695 | 788548 | 799160 | 809530 | 819652 | 829525 | 839146 | 848510 | 857616 | 57 |
| 4 | 766792 | 777778 | 788727 | 799335 | 809700 | 819819 | 829688 | 839304 | 848664 | 857766 | 56 |
| 5 | 766979 | 778060 | 788905 | 799510 | 809871 | 819985 | 829850 | 839462 | 848818 | 857915 | 55 |
| 6 | 767165 | 778243 | 789084 | 799685 | 810042 | 820152 | 830012 | 839620 | 848972 | 858065 | 54 |
| 7 | 767352 | 778426 | 789263 | 799859 | 810212 | 820318 | 830174 | 839778 | 849125 | 858214 | 53 |
| 8 | 767538 | 778608 | 789441 | 800034 | 810383 | 820485 | 830337 | 839936 | 849279 | 858364 | 52 |
| 9 | 767725 | 778791 | 789620 | 800208 | 810553 | 820651 | 830499 | 840094 | 849433 | 858513 | 51 |
| 10 | 767911 | 778973 | 789798 | 800383 | 810723 | 820817 | 830661 | 840251 | 849586 | 858662 | 50 |
| 11 | 768097 | 779156 | 789977 | 800557 | 810894 | 820983 | 830823 | 840409 | 849739 | 858811 | 49 |
| 12 | 768284 | 779338 | 790155 | 800731 | 811064 | 821149 | 830984 | 840567 | 849893 | 858960 | 48 |
| 13 | 768470 | 779520 | 790333 | 800906 | 811234 | 821315 | 831146 | 840724 | 850046 | 859109 | 47 |
| 14 | 768656 | 779702 | 790511 | 801080 | 811404 | 821481 | 831308 | 840882 | 850199 | 859258 | 46 |
| 15 | 768842 | 779884 | 790690 | 801254 | 811574 | 821647 | 831470 | 841039 | 850352 | 859406 | 45 |
| 16 | 769028 | 780067 | 790868 | 801428 | 811744 | 821813 | 831631 | 841196 | 850505 | 859555 | 44 |
| 17 | 769214 | 780249 | 791046 | 801602 | 811914 | 821978 | 831793 | 841354 | 850658 | 859704 | 43 |
| 18 | 769400 | 780430 | 791224 | 801776 | 812084 | 822144 | 831954 | 841511 | 850811 | 859852 | 42 |
| 19 | 769585 | 780612 | 791401 | 801949 | 812253 | 822310 | 832115 | 841668 | 850964 | 860001 | 41 |
| 20 | 769771 | 780794 | 791579 | 802123 | 812423 | 822475 | 832277 | 841825 | 851117 | 860149 | 40 |
| 21 | 769957 | 780976 | 791757 | 802297 | 812592 | 822641 | 832438 | 841982 | 851269 | 860297 | 39 |
| 22 | 770142 | 781157 | 791935 | 802470 | 812762 | 822806 | 832599 | 842139 | 851422 | 860446 | 38 |
| 23 | 770328 | 781339 | 792112 | 802644 | 812931 | 822971 | 832760 | 842296 | 851575 | 860594 | 37 |
| 24 | 770513 | 781520 | 792290 | 802817 | 813101 | 823136 | 832921 | 842452 | 851727 | 860742 | 36 |
| 25 | 770699 | 781702 | 792467 | 802991 | 813270 | 823302 | 833082 | 842609 | 851879 | 860890 | 35 |
| 26 | 770884 | 781883 | 792644 | 803164 | 813439 | 823467 | 833243 | 842766 | 852032 | 861038 | 34 |
| 27 | 771069 | 782065 | 792822 | 803337 | 813608 | 823632 | 833404 | 842922 | 852184 | 861186 | 33 |
| 28 | 771254 | 782246 | 792999 | 803511 | 813778 | 823797 | 833565 | 843079 | 852336 | 861334 | 32 |
| 29 | 771440 | 782427 | 793176 | 803684 | 813947 | 823961 | 833725 | 843235 | 852488 | 861481 | 31 |
| 30 | 771625 | 782608 | 793353 | 803857 | 814116 | 824126 | 833886 | 843391 | 852640 | 861629 | 30 |
| 31 | 771810 | 782789 | 793530 | 804030 | 814284 | 824291 | 834046 | 843548 | 852792 | 861777 | 29 |
| 32 | 771995 | 782970 | 793707 | 804203 | 814453 | 824456 | 834207 | 843704 | 852944 | 861924 | 28 |
| 33 | 772179 | 783151 | 793884 | 804376 | 814622 | 824620 | 834367 | 843860 | 853096 | 862072 | 27 |
| 34 | 772364 | 783332 | 794061 | 804548 | 814791 | 824785 | 834527 | 844016 | 853248 | 862219 | 26 |
| 35 | 772549 | 783513 | 794238 | 804721 | 814959 | 824949 | 834688 | 844172 | 853399 | 862366 | 25 |
| 36 | 772734 | 783693 | 794415 | 804894 | 815128 | 825113 | 834848 | 844328 | 853551 | 862514 | 24 |
| 37 | 772918 | 783874 | 794591 | 805066 | 815296 | 825278 | 835008 | 844484 | 853702 | 862661 | 23 |
| 38 | 773103 | 784055 | 794768 | 805239 | 815465 | 825442 | 835168 | 844640 | 853854 | 862808 | 22 |
| 39 | 773287 | 784235 | 794944 | 805411 | 815633 | 825606 | 835328 | 844795 | 854005 | 862955 | 21 |
| 40 | 773472 | 784416 | 795121 | 805584 | 815801 | 825770 | 835488 | 844951 | 854156 | 863102 | 20 |
| 41 | 773656 | 784596 | 795297 | 805756 | 815969 | 825934 | 835648 | 845106 | 854308 | 863249 | 19 |
| 42 | 773840 | 784776 | 795473 | 805928 | 816138 | 826098 | 835807 | 845262 | 854459 | 863396 | 18 |
| 43 | 774024 | 784957 | 795650 | 806100 | 816306 | 826262 | 835967 | 845417 | 854610 | 863542 | 17 |
| 44 | 774209 | 785137 | 795826 | 806273 | 816474 | 826426 | 836127 | 845573 | 854761 | 863689 | 16 |
| 45 | 774393 | 785317 | 796002 | 806445 | 816642 | 826590 | 836286 | 845728 | 854912 | 863836 | 15 |
| 46 | 774577 | 785497 | 796178 | 806617 | 816809 | 826753 | 836416 | 845883 | 855063 | 863982 | 14 |
| 47 | 774761 | 785677 | 796354 | 806788 | 816977 | 826917 | 836605 | 846038 | 855214 | 864128 | 13 |
| 48 | 774944 | 785857 | 796530 | 806960 | 817145 | 827081 | 836764 | 846193 | 855364 | 864275 | 12 |
| 49 | 775128 | 786037 | 796706 | 807132 | 817313 | 827244 | 836924 | 846348 | 855515 | 864421 | 11 |
| 50 | 775312 | 786217 | 796882 | 807304 | 817480 | 827407 | 837083 | 846503 | 855665 | 864567 | 10 |
| 51 | 775496 | 786396 | 797057 | 807475 | 817648 | 827571 | 837242 | 846658 | 855816 | 864713 | 9 |
| 52 | 775679 | 786576 | 797233 | 807647 | 817815 | 827734 | 837401 | 846813 | 855966 | 864860 | 8 |
| 53 | 775863 | 786756 | 797408 | 807818 | 817982 | 827897 | 837560 | 846967 | 856117 | 865006 | 7 |
| 54 | 776046 | 786935 | 797584 | 807990 | 818150 | 828060 | 837719 | 847122 | 856267 | 865151 | 6 |
| 55 | 776230 | 787114 | 797759 | 808161 | 818317 | 828223 | 837878 | 847277 | 856417 | 865297 | 5 |
| 56 | 776413 | 787294 | 797935 | 808333 | 818484 | 828386 | 838036 | 847431 | 856567 | 865443 | 4 |
| 57 | 776596 | 787473 | 798110 | 808504 | 818651 | 828549 | 838195 | 847585 | 856718 | 865589 | 3 |
| 58 | 776780 | 787652 | 798285 | 808675 | 818818 | 828712 | 838354 | 847740 | 856868 | 865734 | 2 |
| 59 | 776963 | 787832 | 798460 | 808846 | 818985 | 828875 | 838512 | 847894 | 857017 | 865880 | 1 |
| 60 | 777146 | 788011 | 798636 | 809017 | 819152 | 829038 | 838671 | 848048 | 857167 | 866025 | 0 |
| M | 39° | 38° | 37° | 36° | 35° | 34° | 33° | 32° | 31° | 30° | M |

Natural Co-sines.

# NATURAL SINES.

| M | 60° | 61° | 62° | 63° | 64° | 65° | 66° | 67° | 68° | 69° | M |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 866025 | 874620 | 882948 | 891007 | 898794 | 906308 | 913545 | 920505 | 927184 | 933580 | 60 |
| 1 | 866171 | 874761 | 883084 | 891139 | 898922 | 906431 | 913664 | 920618 | 927293 | 933685 | 59 |
| 2 | 866316 | 874902 | 883221 | 891270 | 899049 | 906554 | 913782 | 920732 | 927402 | 933789 | 58 |
| 3 | 866461 | 875042 | 883357 | 891402 | 899176 | 906676 | 913900 | 920846 | 927510 | 933893 | 57 |
| 4 | 866607 | 875183 | 883493 | 891534 | 899304 | 906799 | 914018 | 920959 | 927619 | 933997 | 56 |
| 5 | 866752 | 875324 | 883629 | 891666 | 899431 | 906922 | 914136 | 921072 | 927728 | 934101 | 55 |
| 6 | 866897 | 875465 | 883766 | 891798 | 899558 | 907044 | 914254 | 921185 | 927836 | 934204 | 54 |
| 7 | 867042 | 875605 | 883902 | 891929 | 899685 | 907166 | 914372 | 921299 | 927945 | 934308 | 53 |
| 8 | 867187 | 875746 | 884038 | 892061 | 899812 | 907289 | 914490 | 921412 | 928053 | 934412 | 52 |
| 9 | 867331 | 875886 | 884174 | 892192 | 899939 | 907411 | 914607 | 921525 | 928161 | 934515 | 51 |
| 10 | 867476 | 876026 | 884309 | 892323 | 900065 | 907533 | 914725 | 921638 | 928270 | 934619 | 50 |
| 11 | 867621 | 876167 | 884445 | 892455 | 900192 | 907655 | 914842 | 921750 | 928378 | 934722 | 49 |
| 12 | 867765 | 876307 | 884581 | 892586 | 900319 | 907777 | 914960 | 921863 | 928486 | 934826 | 48 |
| 13 | 867910 | 876447 | 884717 | 892717 | 900445 | 907899 | 915077 | 921976 | 928594 | 934929 | 47 |
| 14 | 868054 | 876587 | 884852 | 892848 | 900572 | 908021 | 915194 | 922088 | 928702 | 935031 | 46 |
| 15 | 868199 | 876727 | 884988 | 892979 | 900698 | 908143 | 915311 | 922201 | 928810 | 935135 | 45 |
| 16 | 868343 | 876867 | 885123 | 893110 | 900825 | 908265 | 915429 | 922313 | 928917 | 935238 | 44 |
| 17 | 868487 | 877006 | 885258 | 893241 | 900951 | 908387 | 915546 | 922426 | 929025 | 935341 | 43 |
| 18 | 868632 | 877146 | 885394 | 893371 | 901077 | 908508 | 915663 | 922538 | 929133 | 935444 | 42 |
| 19 | 868776 | 877286 | 885529 | 893502 | 901203 | 908630 | 915779 | 922650 | 929240 | 935547 | 41 |
| 20 | 868920 | 877425 | 885664 | 893633 | 901329 | 908751 | 915896 | 922762 | 929348 | 935650 | 40 |
| 21 | 869064 | 877565 | 885799 | 893763 | 901455 | 908872 | 916013 | 922874 | 929455 | 935752 | 39 |
| 22 | 869207 | 877704 | 885934 | 893894 | 901581 | 908994 | 916130 | 922986 | 929562 | 935855 | 38 |
| 23 | 869351 | 877844 | 886069 | 894024 | 901707 | 909115 | 916246 | 923098 | 929669 | 935957 | 37 |
| 24 | 869495 | 877983 | 886204 | 894154 | 901833 | 909236 | 916363 | 923210 | 929776 | 936060 | 36 |
| 25 | 869639 | 878122 | 886338 | 894284 | 901958 | 909357 | 916479 | 923322 | 929884 | 936162 | 35 |
| 26 | 869782 | 878261 | 886473 | 894415 | 902084 | 909478 | 916595 | 923434 | 929990 | 936264 | 34 |
| 27 | 869926 | 878400 | 886608 | 894545 | 902209 | 909599 | 916712 | 923545 | 930097 | 936366 | 33 |
| 28 | 870069 | 878539 | 886742 | 894675 | 902335 | 909720 | 916828 | 923657 | 930204 | 936468 | 32 |
| 29 | 870212 | 878678 | 886876 | 894805 | 902460 | 909841 | 916944 | 923768 | 930311 | 936570 | 31 |
| 30 | 870356 | 878817 | 887011 | 894934 | 902585 | 909961 | 917060 | 923880 | 930418 | 936672 | 30 |
| 31 | 870499 | 878956 | 887145 | 895064 | 902710 | 910082 | 917176 | 923991 | 930524 | 936774 | 29 |
| 32 | 870642 | 879095 | 887279 | 895194 | 902836 | 910202 | 917292 | 924102 | 930631 | 936876 | 28 |
| 33 | 870785 | 879233 | 887413 | 895323 | 902961 | 910323 | 917408 | 924213 | 930737 | 936977 | 27 |
| 34 | 870928 | 879372 | 887548 | 895453 | 903086 | 910443 | 917523 | 924324 | 930843 | 937079 | 26 |
| 35 | 871071 | 879510 | 887681 | 895582 | 903210 | 910563 | 917639 | 924435 | 930950 | 937181 | 25 |
| 36 | 871214 | 879649 | 887815 | 895712 | 903335 | 910684 | 917755 | 924546 | 931056 | 937282 | 24 |
| 37 | 871357 | 879787 | 887949 | 895841 | 903460 | 910804 | 917870 | 924657 | 931162 | 937383 | 23 |
| 38 | 871499 | 879925 | 888083 | 895970 | 903585 | 910924 | 917986 | 924768 | 931268 | 937485 | 22 |
| 39 | 871642 | 886063 | 888217 | 896099 | 903709 | 911044 | 918101 | 924878 | 931374 | 937586 | 21 |
| 40 | 871784 | 880201 | 888350 | 896229 | 903834 | 911164 | 918216 | 924989 | 931480 | 937687 | 20 |
| 41 | 871927 | 880339 | 888484 | 896358 | 903958 | 911284 | 918331 | 925099 | 931586 | 937788 | 19 |
| 42 | 872069 | 880477 | 888617 | 896486 | 904083 | 911403 | 918446 | 925210 | 931691 | 937889 | 18 |
| 43 | 872212 | 880615 | 888751 | 896615 | 904207 | 911523 | 918561 | 925320 | 931797 | 937990 | 17 |
| 44 | 872354 | 880753 | 888884 | 896744 | 904331 | 911643 | 918676 | 925430 | 931902 | 938091 | 16 |
| 45 | 872496 | 880891 | 889017 | 896873 | 904455 | 911762 | 918791 | 925541 | 932008 | 938191 | 15 |
| 46 | 872638 | 881028 | 889150 | 897001 | 904579 | 911881 | 918906 | 925651 | 932113 | 938292 | 14 |
| 47 | 872780 | 881166 | 889283 | 897130 | 904703 | 912001 | 919021 | 925761 | 932219 | 938393 | 13 |
| 48 | 872922 | 881303 | 889416 | 897258 | 904827 | 912120 | 919135 | 925871 | 932324 | 938493 | 12 |
| 49 | 873064 | 881441 | 889549 | 897387 | 904951 | 912239 | 919250 | 925980 | 932429 | 938593 | 11 |
| 50 | 873206 | 881578 | 889682 | 897515 | 905075 | 912358 | 919364 | 926090 | 932534 | 938694 | 10 |
| 51 | 873347 | 881716 | 889815 | 897643 | 905198 | 912477 | 919479 | 926200 | 932639 | 938794 | 9 |
| 52 | 873489 | 881853 | 889948 | 897771 | 905322 | 912596 | 919593 | 926310 | 932744 | 938894 | 8 |
| 53 | 873631 | 881990 | 890080 | 897900 | 905445 | 912715 | 919707 | 926419 | 932849 | 938994 | 7 |
| 54 | 873772 | 882157 | 890213 | 898028 | 905569 | 912834 | 919821 | 926529 | 932954 | 939094 | 6 |
| 55 | 873914 | 882264 | 890345 | 898156 | 905692 | 912953 | 919936 | 926638 | 933058 | 939194 | 5 |
| 56 | 874055 | 882401 | 890478 | 898283 | 905815 | 913072 | 920050 | 926747 | 933163 | 939294 | 4 |
| 57 | 874196 | 882538 | 890610 | 898411 | 905939 | 913190 | 920164 | 926857 | 933267 | 939394 | 3 |
| 58 | 874338 | 882674 | 890742 | 898539 | 906062 | 913309 | 920277 | 926966 | 933372 | 939493 | 2 |
| 59 | 874479 | 882811 | 890874 | 898666 | 906185 | 913427 | 920391 | 927075 | 933476 | 939593 | 1 |
| 60 | 874620 | 882948 | 891007 | 898794 | 906308 | 913545 | 920505 | 927184 | 933580 | 939693 | 0 |
| M | 29° | 28° | 27° | 26° | 25° | 24° | 23° | 22° | 21° | 20° | M |

Natural Co-sines.

# NATURAL SINES.

| M | 70° | 71° | 72° | 73° | 74° | 75° | 76° | 77° | 78° | 79° | M |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 939693 | 945519 | 951057 | 956305 | 961262 | 965926 | 970296 | 974370 | 978148 | 981627 | 60 |
| 1 | 939792 | 945613 | 951146 | 956390 | 961342 | 966001 | 970366 | 974435 | 978208 | 981683 | 59 |
| 2 | 939891 | 945708 | 951236 | 956475 | 961422 | 966076 | 970436 | 974501 | 978268 | 981738 | 58 |
| 3 | 939991 | 945802 | 951326 | 956560 | 961502 | 966151 | 970506 | 974566 | 978329 | 981793 | 57 |
| 4 | 940090 | 945897 | 951415 | 956644 | 961582 | 966226 | 970577 | 974631 | 978389 | 981849 | 56 |
| 5 | 940189 | 945991 | 951505 | 956729 | 861662 | 966301 | 970647 | 974696 | 978449 | 981904 | 55 |
| 6 | 940288 | 946085 | 951594 | 956814 | 961741 | 966376 | 970716 | 974761 | 978509 | 981959 | 54 |
| 7 | 940387 | 946180 | 951684 | 956898 | 961821 | 966451 | 970786 | 974826 | 978569 | 982014 | 53 |
| 8 | 940486 | 946274 | 951773 | 956983 | 961901 | 966526 | 970856 | 974891 | 978629 | 982069 | 52 |
| 9 | 940585 | 946368 | 951862 | 957067 | 961980 | 966600 | 970926 | 974956 | 978689 | 982123 | 51 |
| 10 | 940684 | 946462 | 951951 | 957151 | 962059 | 966675 | 970995 | 975020 | 978748 | 982178 | 50 |
| 11 | 940782 | 946555 | 952040 | 957235 | 962139 | 966749 | 971065 | 975085 | 978808 | 982233 | 49 |
| 12 | 940881 | 946649 | 952129 | 957319 | 962218 | 966823 | 971134 | 975149 | 978867 | 982287 | 48 |
| 13 | 940979 | 946743 | 952218 | 957404 | 962297 | 966898 | 971204 | 975214 | 978927 | 982342 | 47 |
| 14 | 941078 | 946837 | 952307 | 957487 | 962376 | 966972 | 971273 | 975278 | 978986 | 982396 | 46 |
| 15 | 941176 | 946930 | 952396 | 957571 | 962455 | 967046 | 971342 | 975342 | 979046 | 982450 | 45 |
| 16 | 941274 | 947024 | 952484 | 957655 | 962534 | 967120 | 971411 | 975406 | 979105 | 982505 | 44 |
| 17 | 941372 | 947117 | 952573 | 957739 | 962613 | 967194 | 971480 | 975471 | 979164 | 982559 | 43 |
| 18 | 941471 | 947210 | 952661 | 957822 | 962692 | 967268 | 971549 | 975535 | 979223 | 982613 | 42 |
| 19 | 941569 | 947304 | 952750 | 957906 | 962770 | 967342 | 971618 | 975598 | 979282 | 982667 | 41 |
| 20 | 941666 | 947397 | 952838 | 957990 | 962849 | 967415 | 971687 | 975662 | 979341 | 982721 | 40 |
| 21 | 941764 | 947490 | 952926 | 958073 | 962928 | 967489 | 971755 | 975726 | 979399 | 982774 | 39 |
| 22 | 941862 | 947583 | 953015 | 958156 | 963006 | 967562 | 971824 | 975790 | 979458 | 982828 | 38 |
| 23 | 941960 | 947676 | 953103 | 958239 | 963084 | 967636 | 971893 | 975853 | 979517 | 982882 | 37 |
| 24 | 942057 | 947768 | 953191 | 958323 | 963163 | 967709 | 971961 | 975917 | 979575 | 982935 | 36 |
| 25 | 942155 | 947861 | 953279 | 958406 | 963241 | 967782 | 972030 | 975980 | 979634 | 982980 | 35 |
| 26 | 942252 | 947954 | 953366 | 958489 | 963319 | 967856 | 972098 | 976044 | 979692 | 983042 | 34 |
| 27 | 942350 | 948046 | 953454 | 958572 | 963397 | 967929 | 972166 | 976107 | 979750 | 983096 | 33 |
| 28 | 942447 | 948139 | 953542 | 958654 | 963475 | 968002 | 972234 | 976170 | 979809 | 983149 | 32 |
| 29 | 942544 | 948231 | 953629 | 958737 | 963553 | 968075 | 972302 | 976233 | 979867 | 983202 | 31 |
| 30 | 942641 | 948324 | 953717 | 958820 | 963630 | 968148 | 972370 | 976296 | 979925 | 983255 | 30 |
| 31 | 942739 | 948416 | 953804 | 958902 | 963708 | 968220 | 972438 | 976359 | 979983 | 983308 | 29 |
| 32 | 942836 | 948508 | 953892 | 958985 | 963786 | 968293 | 972506 | 976422 | 980041 | 983361 | 28 |
| 33 | 942932 | 948600 | 953979 | 959067 | 963863 | 968366 | 972573 | 976485 | 980098 | 983414 | 27 |
| 34 | 943029 | 948692 | 954066 | 959150 | 963941 | 968438 | 972641 | 976547 | 980156 | 983466 | 26 |
| 35 | 943126 | 948784 | 954153 | 959232 | 964018 | 968511 | 972708 | 976610 | 980214 | 983519 | 25 |
| 36 | 943223 | 948876 | 954240 | 959314 | 964095 | 968583 | 972776 | 976672 | 980271 | 983571 | 24 |
| 37 | 943319 | 948968 | 954327 | 959396 | 964173 | 968656 | 972843 | 976735 | 980329 | 983624 | 23 |
| 38 | 943416 | 949059 | 954414 | 959478 | 964250 | 968728 | 972911 | 976797 | 980386 | 983676 | 22 |
| 39 | 943512 | 949151 | 954501 | 959560 | 964327 | 968800 | 972978 | 976850 | 980443 | 983729 | 21 |
| 40 | 943609 | 949243 | 954588 | 959642 | 964404 | 968872 | 973045 | 976921 | 980500 | 983781 | 20 |
| 41 | 943705 | 949334 | 954674 | 959724 | 964481 | 968944 | 973112 | 976984 | 980558 | 983383 | 19 |
| 42 | 943801 | 949425 | 954761 | 959805 | 964557 | 969016 | 973179 | 977046 | 980615 | 983885 | 18 |
| 43 | 943897 | 949517 | 954847 | 959887 | 964634 | 969088 | 973246 | 977108 | 980672 | 983937 | 17 |
| 44 | 943993 | 949608 | 954934 | 959968 | 964711 | 969159 | 973313 | 977169 | 980728 | 983989 | 16 |
| 45 | 944089 | 949699 | 955020 | 960050 | 964787 | 969231 | 973379 | 977231 | 980785 | 984041 | 15 |
| 46 | 944185 | 949790 | 955106 | 960131 | 964864 | 969302 | 973446 | 977293 | 980842 | 984092 | 14 |
| 47 | 944281 | 949881 | 955192 | 960212 | 964940 | 969374 | 973512 | 977354 | 980899 | 984144 | 13 |
| 48 | 944376 | 949972 | 955278 | 960294 | 965016 | 969445 | 973579 | 977416 | 980955 | 984196 | 12 |
| 49 | 944472 | 950063 | 955364 | 960875 | 965093 | 969517 | 973645 | 977477 | 981012 | 984247 | 11 |
| 50 | 944568 | 950154 | 955450 | 960456 | 965169 | 969588 | 973712 | 977539 | 981068 | 984298 | 10 |
| 51 | 944663 | 950244 | 955536 | 960537 | 965245 | 969659 | 973778 | 977600 | 981124 | 984350 | 9 |
| 52 | 944758 | 950335 | 955622 | 960618 | 965321 | 969730 | 973844 | 977661 | 981181 | 984401 | 8 |
| 53 | 944854 | 950425 | 955707 | 960698 | 965397 | 969801 | 973910 | 977722 | 981237 | 984452 | 7 |
| 54 | 944949 | 950516 | 955793 | 960779 | 965473 | 969872 | 973976 | 977783 | 981293 | 984503 | 6 |
| 55 | 945044 | 950606 | 955879 | 960860 | 965548 | 969943 | 974042 | 977844 | 981349 | 984554 | 5 |
| 56 | 945139 | 950696 | 955964 | 960940 | 965624 | 970014 | 974108 | 977905 | 981405 | 984605 | 4 |
| 57 | 945234 | 950786 | 956049 | 961021 | 965700 | 970084 | 974173 | 977966 | 981460 | 984656 | 3 |
| 58 | 945329 | 950877 | 956134 | 961101 | 965775 | 970155 | 974239 | 978026 | 981516 | 984707 | 2 |
| 59 | 945424 | 950967 | 956220 | 961181 | 965850 | 970225 | 974305 | 978087 | 981572 | 984757 | 1 |
| 60 | 945519 | 951057 | 956305 | 961262 | 965926 | 970296 | 974370 | 978148 | 981627 | 984808 | 0 |
| M | 19° | 18° | 17° | 16° | 15° | 14° | 13° | 12° | 11° | 10° | M |

Natural Co-sines.

# NATURAL SINES.

| M | 80° | 81° | 82° | 83° | 84° | 85° | 86° | 87° | 88° | 89° | M |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 984808 | 987688 | 990268 | 992546 | 994522 | 996195 | 997564 | 998630 | 999391 | 999848 | 60 |
| 1 | 984858 | 987734 | 990309 | 992582 | 994552 | 996220 | 997584 | 998645 | 999401 | 999853 | 59 |
| 2 | 984909 | 987779 | 990349 | 992617 | 994583 | 996245 | 997604 | 998660 | 999411 | 999858 | 58 |
| 3 | 984959 | 987824 | 990389 | 992652 | 994613 | 996270 | 997625 | 998675 | 999421 | 999863 | 57 |
| 4 | 985009 | 987870 | 990429 | 992687 | 994643 | 996295 | 997645 | 998690 | 999431 | 999867 | 56 |
| 5 | 985059 | 987915 | 990469 | 992722 | 994673 | 996320 | 997664 | 998705 | 999441 | 999872 | 55 |
| 6 | 985109 | 987960 | 990509 | 992757 | 994703 | 996345 | 997684 | 998719 | 999450 | 999877 | 54 |
| 7 | 985159 | 988005 | 990549 | 992792 | 994733 | 996370 | 997704 | 998734 | 999460 | 999881 | 53 |
| 8 | 985209 | 988050 | 990589 | 992827 | 994762 | 996395 | 997724 | 998749 | 999469 | 999886 | 52 |
| 9 | 985259 | 988094 | 990629 | 992862 | 994792 | 996419 | 997743 | 998763 | 999479 | 999890 | 51 |
| 10 | 985309 | 988139 | 990669 | 992896 | 994822 | 996444 | 997763 | 998778 | 999488 | 999894 | 50 |
| 11 | 985358 | 988184 | 990708 | 992931 | 994851 | 996468 | 997782 | 998792 | 999497 | 999898 | 49 |
| 12 | 985408 | 988228 | 990748 | 992966 | 994881 | 996493 | 997801 | 998806 | 999507 | 999903 | 48 |
| 13 | 985457 | 988273 | 990787 | 993000 | 994910 | 996517 | 997821 | 998820 | 999516 | 999907 | 47 |
| 14 | 985507 | 988317 | 990827 | 993034 | 994939 | 996541 | 997840 | 998834 | 999525 | 999910 | 46 |
| 15 | 985556 | 988362 | 990866 | 993068 | 994969 | 996566 | 997859 | 998848 | 999534 | 999914 | 45 |
| 16 | 985605 | 988406 | 990905 | 993103 | 994998 | 996589 | 997878 | 998862 | 999542 | 999918 | 44 |
| 17 | 985654 | 988450 | 990944 | 993137 | 995027 | 996614 | 997897 | 998876 | 999551 | 999922 | 43 |
| 18 | 985703 | 988494 | 990983 | 993171 | 995056 | 996637 | 997916 | 998890 | 999560 | 999925 | 42 |
| 19 | 985752 | 988538 | 991022 | 993205 | 995084 | 996661 | 997934 | 998904 | 999568 | 999929 | 41 |
| 20 | 985801 | 988582 | 991061 | 993238 | 995113 | 996685 | 997953 | 998917 | 999577 | 999932 | 40 |
| 21 | 985850 | 988626 | 991100 | 993272 | 995142 | 996709 | 997972 | 998931 | 999585 | 999936 | 39 |
| 22 | 985899 | 988669 | 991138 | 993306 | 995170 | 996732 | 997990 | 998944 | 999594 | 999939 | 38 |
| 23 | 985947 | 988713 | 991177 | 993339 | 995199 | 996756 | 998008 | 998957 | 999602 | 999942 | 37 |
| 24 | 985996 | 988756 | 991216 | 993373 | 995227 | 996779 | 998027 | 998971 | 999610 | 999945 | 36 |
| 25 | 986045 | 988800 | 991254 | 993406 | 995256 | 996802 | 998045 | 998984 | 999618 | 999948 | 35 |
| 26 | 986093 | 988843 | 981292 | 993439 | 995284 | 996825 | 998063 | 998997 | 999626 | 999951 | 34 |
| 27 | 986141 | 988886 | 991331 | 993473 | 995312 | 996848 | 998081 | 999010 | 999634 | 999954 | 33 |
| 28 | 986189 | 988930 | 991369 | 993506 | 995340 | 996872 | 998099 | 999023 | 999642 | 999957 | 32 |
| 29 | 986238 | 988973 | 991407 | 993539 | 995368 | 996894 | 998117 | 999035 | 999650 | 999959 | 31 |
| 30 | 986286 | 989016 | 991445 | 993572 | 995396 | 996917 | 998135 | 999048 | 999657 | 999962 | 30 |
| 31 | 986334 | 989059 | 991483 | 993605 | 995424 | 996940 | 998153 | 999061 | 999665 | 999964 | 29 |
| 32 | 986381 | 989102 | 991521 | 993638 | 995452 | 996963 | 998170 | 999073 | 999672 | 999967 | 28 |
| 33 | 986429 | 989145 | 991558 | 993670 | 995479 | 996985 | 998188 | 999086 | 999680 | 999969 | 27 |
| 34 | 986477 | 989187 | 991596 | 993703 | 995507 | 997008 | 998205 | 999098 | 999687 | 999971 | 26 |
| 35 | 986525 | 989230 | 991634 | 993735 | 995535 | 997030 | 998223 | 999111 | 999694 | 999974 | 25 |
| 36 | 986572 | 989272 | 991671 | 993768 | 995562 | 997053 | 998240 | 999123 | 999701 | 999976 | 24 |
| 37 | 986620 | 989315 | 991709 | 993800 | 995589 | 997075 | 998257 | 999135 | 999709 | 999978 | 23 |
| 38 | 986667 | 989357 | 991746 | 993833 | 995617 | 997097 | 998274 | 999147 | 999716 | 999980 | 22 |
| 39 | 986714 | 989399 | 991783 | 993865 | 995644 | 997119 | 998291 | 999159 | 999722 | 999981 | 21 |
| 40 | 986762 | 989442 | 991820 | 993897 | 995671 | 997141 | 998308 | 999171 | 999729 | 999983 | 20 |
| 41 | 986809 | 989484 | 991857 | 993929 | 995698 | 997163 | 998325 | 999183 | 999736 | 999985 | 19 |
| 42 | 986856 | 989526 | 991894 | 993961 | 995725 | 997185 | 998342 | 999194 | 999743 | 999986 | 18 |
| 43 | 986903 | 989568 | 991931 | 993993 | 995752 | 997207 | 998359 | 999206 | 999749 | 999988 | 17 |
| 44 | 986950 | 989610 | 991968 | 994025 | 995778 | 997229 | 998375 | 999218 | 999756 | 999989 | 16 |
| 45 | 986996 | 989651 | 992005 | 994056 | 995805 | 997250 | 998392 | 999229 | 999762 | 999990 | 15 |
| 46 | 987043 | 989693 | 992042 | 994088 | 995832 | 997272 | 998408 | 999240 | 999768 | 999992 | 14 |
| 47 | 987090 | 989735 | 992078 | 994120 | 995858 | 997293 | 998425 | 999252 | 999775 | 999993 | 13 |
| 48 | 987136 | 989776 | 992115 | 994151 | 995884 | 997314 | 998441 | 999263 | 999781 | 999994 | 12 |
| 49 | 987183 | 989818 | 992151 | 994182 | 995911 | 997336 | 998457 | 999274 | 999787 | 999995 | 11 |
| 50 | 987229 | 989859 | 992187 | 994214 | 995937 | 997357 | 998473 | 999285 | 999793 | 999996 | 10 |
| 51 | 987275 | 989900 | 992224 | 994245 | 995963 | 997378 | 998489 | 999296 | 999799 | 999997 | 9 |
| 52 | 987322 | 989942 | 992260 | 994276 | 995989 | 997399 | 998505 | 999307 | 999804 | 999997 | 8 |
| 53 | 987368 | 989983 | 992296 | 994307 | 996015 | 997420 | 998521 | 999318 | 999810 | 999998 | 7 |
| 54 | 987414 | 990024 | 992332 | 994338 | 996041 | 997441 | 998537 | 999328 | 999816 | 999998 | 6 |
| 55 | 987460 | 990065 | 992368 | 994369 | 996067 | 997462 | 998552 | 999339 | 999821 | 999999 | 5 |
| 56 | 987506 | 990105 | 992404 | 994400 | 996093 | 997482 | 998568 | 999350 | 999827 | 999999 | 4 |
| 57 | 987551 | 990146 | 992439 | 994430 | 996118 | 997503 | 998583 | 999360 | 999832 | 1000000 | 3 |
| 58 | 987597 | 990187 | 992475 | 994461 | 996144 | 997523 | 998599 | 999370 | 999837 | 1000000 | 2 |
| 59 | 987643 | 990228 | 992511 | 994491 | 996169 | 997544 | 998614 | 999381 | 999843 | 1000000 | 1 |
| 60 | 987688 | 990268 | 992546 | 994522 | 996105 | 997564 | 998630 | 999391 | 999848 | 1000000 | 0 |
| M | 9° | 8° | 7° | 6° | 5° | 4° | 3° | 2° | 1° | 0° | M |

Natural Co-sines.

# ERRATA.

BOOK I. In the diagram to Prop. II, a right line should be drawn from E to A.

In Prop VI, the parallel lines kK, should be drawn parallel to the axis EF.

In Scholium to Prop. VII, AB or CD, should be the axis of the parabola instead of AC or BD.

On page 31, eighth line from bottom, read $(dx-x^2)$ instead of $(d-x^2)$.

BOOK II. Page 60, second line from bottom, IN should be changed to LN.

Page 61, third line from the top, MH should be changed to GLMH.

BOOK III. On page 101, eleventh line from the bottom, for fluction, read fluxion.

On page 110, eight lines from the bottom, for $\frac{2}{3}rx$, read $\frac{3}{2}rs$, and for $\frac{3}{2}is$ read $3rs$.

Page 112, sixteenth line from bottom, for $\frac{2}{3}rs$ read $\frac{3}{2}rs$. And thirteenth line from bottom, supply 2 to $^3rs$, to make it read $\frac{3}{2}rs$.

Page 113 Formula 2, for $\frac{1}{4}s3x$, read $\frac{1}{4}sx$.; and in the next line for $\frac{2}{3}rs$ read $\frac{3}{2}rs$.

On pages 143 and 144, CQ should be changed to AQ CP. to AP, and fifth line from bottom, page 144, change C to A; also in the third line from top for $\sqrt{ax-x^2}$ read $\sqrt{2ax-x^2}$.

BOOK V. On 158 page Article 6th, fourth line, for $\underline{z}^z$ read $\underline{z}\bar{z}$; and on the fifth line, for $\underline{z}^z$ read $\underline{z}\underline{z}$; also in article 7th on the ninth line, for $\sqrt{(d\underline{x})\underline{x}}$, read $\sqrt{\overline{(d\underline{x})}x}$.

Page 158, tenth line from top, for AD, read BC; also in article 10th, supply the inferior dash to $z$, in first, second, third and fourth lines, so that they may read $\underline{z}^2$, $a\underline{z}^2$, $a\underline{z}$ and $a\underline{z}^2$; also on the sixth line, article 10th, for $a\sqrt{z}$, read $a\sqrt{\underline{z}}$.

Page 152, on tenth line, for $au$, read $au$.